1. Introduction

We should point out that the proof requires very lengthy (at least to us) calculations and in order to make lend some credence to the veracity of it in the general setting of \( n \geq 2 \), we usually conducted simultaneous calculations for the cases of type \( A_2 \) and \( A_3 \).

Some motivation for the study of toroidal Lie algebras comes from their possible applications to math and physics. With respect to representations of toroidal Lie algebras a fair amount interesting progress has been made. See for example [BB99], [BC94], [CP87], [CF01], [Cox02], [ER96], [ER01], [ER02], [EF94], [IUK97], [RMY92], [HKT90], [IT01], [Lar00], [ERM94], [VV96] and [Yam89]). For instance one of the cocycles that is used in the construction of the toroidal extended affine Lie algebra is also used in Y. Billig’s study of a magnetic hydrodynamics equation with asymmetric stress tensor (see [Bil07] and [Bil05]). Moreover in Billig and independently Iohara, Saito, and Wakimoto’s work (see [Bil99], and [ISW99]) they derive Hirota bilinear equations arising from both homogeneous and principal realizations of the vertex operator representations of 2-toroidal Lie algebras of type \( A_l, D_l, E_l \). They derive the hierarchy of Hirota equations and present their soliton-type solutions. In [KIT02], Kakei, Ikeda, and Takasaki construct the hierarchy associated to the \((2+1)\)-dimensional nonlinear Schrödinger (NLS) equation and show how the representation theory of toroidal \( \widehat{sl}_2 \) can be used to derive the Hirota-type equations for \( \tau \)-functions. On the somewhat more mathematical side, in interesting work of V. Ginzburg, M. Kapranov, and E. Vasserot on Langland’s reciprocity for algebraic surfaces, they construct Hecke operators for vector bundles on an algebraic surface. The main point of their paper is that under certain conditions the corresponding algebra of Hecke operators is the homomorphic image of a quantum toroidal algebra (see [GKV95]). One should also see some of the interesting work of Slodowy, Berman and Moody, Benkart and Zelmanov on generalized intersection matrix algebras and their relationship to toroidal Lie algebras (see [BM92], [BZ96] and [Slo86]).

Wakimoto’s free field realization of affine \( \widehat{sl}_2 \) and Feigin and Frenkel’s generalization to non-twisted affine algebras \( \widehat{g} \) plays a fundamental role in describing integral solutions to the Knizhnik-Zamolodchikov equations (see for example [Wak86], [FF90b], [Kur91], [EFK98], [SV90], [SV91] and [SV91]). One reason we are interested in Wakimoto type realizations of toroidal \( \widehat{sl}_{n+1} \) is that it might give us clues as to how 2-cocycles plays a role in the construction of free field realizations for universal central extensions of Lie algebras of the form \( g \otimes R \) where \( R \) is an algebra over the complex numbers. Another reason is that they often can provide, in the generic setting, realizations in terms of partial differential operators of imaginary type Verma modules for toroidal Lie algebras. We plan to see how the realizations in this paper are related to these modules in future work.

2. Definitions and Notation

All vector spaces are over the field of complex numbers \( \mathbb{C} \). We use multi-index notation, given an element \((m_0, m_1, m_2, \cdots, m_N) \in \mathbb{Z}^{N+1}\) we write \( \mathbf{m} = (m_0, m_1, m_2, \cdots, m_N) \). If \( z_i \) are indeterminates, let \( z^\mathbf{m} = z_0^{m_0}z_1^{m_1}z_2^{m_2} \cdots z_N^{m_N} \). Fix a decomposition of \( \mathbb{Z}^{p+1} = \mathbb{Z}_{p+1}^+ \cup \{0\} \cup \mathbb{Z}_{p+1}^- \) into three subsets.
such that $\mathbb{Z}_{+}^{p+1}$ are sets closed under vector addition i.e. for example, if $j, k \in \mathbb{Z}_{+}^{p+1}$, then $j+k \in \mathbb{Z}_{+}^{p+1}$. Declare $m > 0$ if $m \in \mathbb{Z}_{+}^{p+1}$ and $m < 0$ if $m \in \mathbb{Z}_{+}^{p+1}$. Define the function $\theta$ by

$$\theta(m) = \begin{cases} 1 & \text{if } m > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Let $A_n = (A_{ij})_{i,j=0}^n$ be the indecomposable Cartan matrix of affine type $A_n^1$ with $n \geq 2$. Let $\Pi = \{\alpha_0, \alpha_1, \ldots, \alpha_n\}$ denote the simple roots, a basis for the set of roots denoted $\Delta$. Let $Q$ be the root lattice, i.e. the free $\mathbb{Z}$-module with generators $\alpha_0, \alpha_1, \ldots, \alpha_n$. The matrix $A_n$ induces a symmetric bilinear form $(\cdot, \cdot)$ on $Q$ satisfying $(\alpha_i | \alpha_j) = A_{ij}$. For $0 \leq i \leq n$, we set $\hat{\alpha}_i := \alpha_i$.

### 2.1. The toroidal Lie algebra

We define the toroidal Lie algebra $\tau(A_n)$ by generators

$$K_{m,j}, \quad H_i(m), \quad E_i(m), \quad F_i(m), \quad 0 \leq i \leq n, \quad 0 \leq j \leq N, \quad m \in \mathbb{Z}^{N+1}$$

and relations

1. The $K_{m,j}, 0 \leq j \leq N$ are central;
2. $\sum_{i=0}^N m_i K_{m,i} = 0$;
3. $[H_i(m), H_j(n)] = A_{ij} \left( \sum_{l=0}^N m_l K_{m+n,l} \right)$ ($0 \leq i, j \leq n$);
4. $[H_i(m), E_j(n)] = A_{ij} E_j(m + n)$,
   $[H_i(m), F_j(n)] = -A_{ij} F_j(m + n)$;
5. $[E_i(m), F_j(n)] = -\delta_{i,j} \left( H_i(m + n) + \frac{2}{A_{ij}} \sum_{l=0}^N m_l K_{m+n,l} \right)$;
6. $[E_i(m), E_j(n)] = 0 = [F_i(m), F_j(n)]$;
7. ad $E_i(m)$ and ad $F_i(m)$
8. $\text{ad} E_i(m)^{-A_{ij}+1} E_j(n) = 0$ for $i \neq j$;
9. $\text{ad} F_i(m)^{-A_{ij}+1} F_j(n) = 0$ for $i \neq j$;

We also write generating functions for the generators of $\tau(A_n)$ $1 \leq i \leq n, 0 \leq s \leq N$:

$$K_s(z) = \sum_{m \in \mathbb{Z}^{N+1}} K_{m,s} z^{-m}, \quad H_i(z) = \sum_{m \in \mathbb{Z}^{N+1}} H_i(m) z^{-m},$$

$$E_i(z) = \sum_{m \in \mathbb{Z}^{N+1}} E_i(m) z^{-m}, \quad F_i(z) = \sum_{m \in \mathbb{Z}^{N+1}} F_i(m) z^{-m}$$

### 2.2. The toroidal Heisenberg algebra

Define the *toroidal Heisenberg algebra*, $\mathcal{B}$, as the Lie algebra with generators $b_i(r)$ ($1 \leq i \leq n$) and $K_{r,p}$ ($0 \leq p \leq N, \ r \in \mathbb{Z}^{n-1}$) which satisfy the following relation:

$$[b_i(r), b_j(s)] = A_{ij} \sum_{p=0}^N r_p K_{r+s,p}, \quad \text{and} \quad \sum_{p=0}^N r_p K_{r,p} = 0, \quad \forall r \in \mathbb{Z}^{N+1}.$$ 

Here $A_{ij}$ denotes the $i,j^{th}$ entry of the Cartan matrix $A_n$ where we have deleted the first row and column. This algebra has an anti-automorphism given by

$$\rho_0(b_i(m)) = b_i(-m), \quad \text{and} \quad \rho_0(K_{m,p}) = K_{-m,p}.$$
If we set $b_0(m) := -\sum_{i=1}^n b_i(m)$, then one can check that the first equality in (2.2) is satisfied also for $i = 0$ or $j = 0$.

2.3. Representation of the Heisenberg algebra $g$. We define polynomial rings over indeterminates indexed by $0 < i, j \leq n + 1$ and $m \in \mathbb{Z}^{N+1}$.

$$\mathbb{C}[x] := \mathbb{C}[x_{ij}(m)|0 < i < j \leq n + 1, m \in \mathbb{Z}^{N+1}]$$
$$\mathbb{C}[y] := \mathbb{C}[y_i(k)|k \in \mathbb{Z}^{N+1}, k > 0, 1 \leq i \leq n]$$

Using heuristic ideas about how the toroidal Lie algebra should act on sections of certain (not well defined) line bundles, one should be able to show that the following map $\Phi$ defines a representation into some type of a completion of this algebra. (For readers who are interested in this heuristic type of completion of $C$ for $1 \leq m$ we have $C[y_0(s) = 0 = y_{n+1}(s)$. Now this is not well defined on $\mathbb{C}[y]$ as it takes $\mathbb{C}[y]$ to some kind of completion of $\mathbb{C}[y]$. It seems to be possible to make this more precise, but instead we will conjugate by the the anti-automorphisms $p_0$ and $p_2$ to obtain the representation on $\mathbb{C}[y]$ given by

$$\Phi(b_i(r)) = \theta(r) \left( \sum_{s>0} y_{i-1}(s) \sum_{p=0}^N r_p \kappa_r + s_p - \sum_{s>0} y_i(s) \sum_{p=0}^N r_p \kappa_r + s_p \right)$$
$$+ \theta(-r) \left( \sum_{s>0} y_{i-1}(s) \sum_{p=0}^N r_p \kappa_r + s_p - 2 \sum_{s>0} y_i(s) \sum_{p=0}^N r_p \kappa_r + s_p + \sum_{s>0} y_{i+1}(s) \sum_{p=0}^N r_p \kappa_r + s_p \right)$$
$$- \theta(r) \partial_{y_i(r)} - \delta_{r,0} \lambda_i$$

where we set $y_0(s) = 0 = y_{n+1}(s)$. Now this is not well defined on $\mathbb{C}[y]$ as it takes $\mathbb{C}[y]$ to some kind of completion of $\mathbb{C}[y]$. It seems to be possible to make this more precise, but instead we will conjugate by the the anti-automorphisms $p_0$ and $p_2$ to obtain the representation on $\mathbb{C}[y]$ given by

$$\Phi(b_i(m)) = \theta(-m) \sum_{p=0}^N \sum_{s>0} \left( \partial_{y_{i-1}(s)} - \partial_{y_i(s)} \right) m_p \kappa_{-m} + s_p$$
$$+ \theta(m) \sum_{p=0}^N \sum_{s>0} \left( \partial_{y_{i-1}(s)} - 2 \partial_{y_i(s)} + \partial_{y_{i+1}(s)} \right) m_p \kappa_{-m} + s_p$$
$$+ \theta(-m) y_i(-m) - \delta_{m,0} \lambda_i$$

for $1 \leq i \leq n$. This agrees with our earlier paper [Cox04]. For $i = 0$ we have

$$\Phi(b_0(m)) = -\theta(-m) \sum_{p=0}^N \sum_{s>0} \left( \partial_{y_1(s)} - \partial_{y_{n+1}(s)} \right) m_p \kappa_{-m} + s_p$$
$$+ \theta(m) \sum_{p=0}^N \sum_{s>0} \left( \partial_{y_1(s)} + \partial_{y_{n+1}(s)} \right) m_p \kappa_{-m} + s_p$$
$$- \theta(-m) \sum_{i=1}^n y_i(-m) + \sum_{i=1}^n \delta_{m,0} \lambda_i$$
Proposition 2.1 (Realization of the Toroidal Heisenberg Algebra). Fix $\kappa_{m,p} \in \mathbb{C}$, $0 \leq p \leq N$ and $\lambda_i \in \mathbb{C}$ where $0 \leq i \leq n$. Assume

\begin{equation}
\sum_{p=0}^{N} m_p \kappa_{m,p} = 0, \quad \text{for all } m,
\end{equation}

and

\begin{equation}
m_p \kappa_{-m-n,p} = 0 \quad \text{for } m > 0 \text{ and } n > 0.
\end{equation}

Then

$$
\Phi(b_i(m)) = \theta(-m) \sum_{p=0}^{s>0} \left( \partial_{y_{i-1}(s)} - \partial_{y_i(s)} \right) m_p \kappa_{m-s,p} + \theta(m) \sum_{p=0}^{s>0} \left( \partial_{y_{i-1}(s)} - 2\partial_{y_i(s)} + \partial_{y_{i+1}(s)} \right) m_p \kappa_{m+s,p} + \theta(-m) \kappa_i(-m) - \delta_{m,0} \lambda_i
$$

\begin{equation}
\Phi(K_{m+n,p}) = -\kappa_{m-n,p}
\end{equation}

for $1 \leq i \leq n$, $m, n \in \mathbb{Z}^{N+1}$ defines a representation $\mathfrak{B}$ on $\mathbb{C}[y]$.

\textbf{Proof.} The proof is straightforward and is left to the reader. \hfill \square

3. Main Result, the representation of $\tau(A_n)$

The elements $x_{ij}(m)$ act via multiplication on the ring $\mathbb{C}[x]$, and hence on the ring $\mathbb{C}[x] \otimes \mathbb{C}[y]$ (as $x_{ij}(m) \otimes 1$). Define the following differential operators acting on the polynomial ring $\mathbb{C}[x] \otimes \mathbb{C}[y]$.

\begin{equation}
\begin{aligned}
a_{ij,m} &:= -x_{ij}(m), & \quad \partial_{x_{ij}(-m)} = a^*_{ij,m}, \\

\kappa_i(z) &= \sum_{m \in \mathbb{Z}^{N+1}} \kappa_{m,s} z^{-m},
\end{aligned}
\end{equation}

With corresponding generating functions:

$$
\begin{aligned}
a_{ij}(z) &= \sum_{m \in \mathbb{Z}^{N+1}} a_{ij,m} z^{-m}, & \quad a^*_{ij}(z) &= \sum_{m \in \mathbb{Z}^{N+1}} a^*_{ij,m} z^{-m}, \\

K(z) \cdot D &= \sum_{i=0}^{N} K_i(z) \frac{\partial}{\partial z_i}, & \quad K(z) &= \sum_{i=0}^{N} K_i(z) = \sum_{i=0}^{N} \sum_{m \in \mathbb{Z}^{N+1}} K_{m,i} z^{-m} z_i,
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
\kappa(z) \cdot D &= \sum_{i=0}^{N} \kappa_i(z) \frac{\partial}{\partial z_i}, & \quad \kappa(z) &= \sum_{i=0}^{N} \kappa_i(z) = \sum_{i=0}^{N} \sum_{m \in \mathbb{Z}^{N+1}} \kappa_{m,i} z^m z_i.
\end{aligned}
\end{equation}

Note that $K(z) \cdot D$ is a weighted version of Euler’s differential operator. We can then write (2.2) as

\begin{equation}
[b_i(z), b_j(w)] = A_{ij} K(w) \cdot D \delta(z/w).
\end{equation}
Theorem 3.1 (Realization). Let $\kappa_{m,l}$ be fixed complex numbers satisfying conditions (2.4) and (2.5) and fix $\lambda_i \in \mathbb{C}$ for $0 \leq i \leq n$. Then the generating functions given below (where we omit the variable $z$)

$$\rho(F_r) = a_{r,r+1} - \sum_{j=1}^{r-1} a_{j,r+1}a^*_j,$$

$$\rho(H_r) = 2a_{r,r+1}a^*_{r,r+1} + \sum_{i=1}^{r-1} (a_{i,r+1}a^*_{i,r+1} - a_{i,r}a^*_{i,r}) + \Phi(b_r),$$

$$\rho(E_r) = a_{r,r+1}a^*_{r,r+1}a^*_{r,r+1} + \sum_{j=r+2}^{n+1} a_{r+1,j}a^*_{r,j} + \sum_{j=1}^{r-1} a_{j,r}a^*_{j,r+1}$$

$$+ \sum_{j=r+2}^{n+1} (a_{r,j}a^*_{r,j} - a_{r+1,j}a^*_{r+1,j}) a^*_{r,r+1}$$

$$+ a^*_{r,r+1} \Phi(b_r) + \kappa \cdot Da^*_{r,r+1},$$

for $1 \leq r \leq n$, together with

$$\rho(E_0) = -a_{1,n+1},$$

$$\rho(H_0) = -\sum_{r=1}^{n} \rho(H_r) = -\sum_{r=1}^{n} a_{r,n+1}a^*_{r,n+1} - \sum_{r=2}^{n+1} a_{1,r}a^*_{1,r} + \Phi(b_0),$$

$$\rho(F_0) = \sum_{1 \leq r < j \leq n+1} -a_{rj} \sum_{q_i \geq q_{i-1}} \prod_{l=1}^{i-1} a^*_{q_i,q_{i+1}} a^*_{r,n+1}$$

$$- \sum_{1 \leq r < n+1} \sum_{q_i \geq q_{i-1}} \prod_{j=1}^{i} a^*_{q_i,q_{j+1}} \Phi(b_r)$$

$$- \sum_{1 \leq r < n+1} \sum_{q_i \geq q_{i-1}} \prod_{j=1}^{i-1} a^*_{q_i,q_{j+1}} \kappa \cdot Da^*_{r,n+1}$$

defines an action of the generators $E_r(m), F_r(m)$ and $H_r(m)$ on the Fock space $\mathbb{C}[x] \otimes \mathbb{C}[y]$ (notation given earlier). In the partitions above $1 = q_1 < q_2 < \cdots < q_i \leq q_{i+1} = n + 1$. In addition $K_{m,l}$ acts as left multiplication by $-\kappa_{m,l}$.

Note it appears that one should also have for $k < l$,

$$\rho(E_{lk}) = -a_{kl} + \sum_{j=1}^{k-1} a_{jl}a^*_{jk}$$

but we don’t seem to need this general formula, so we don’t determine whether it is always true.
3.1. Relation to other work. As in S. Eswara Rao and R. V. Moody, we define the toroidal Lie algebra \( t(A_t) \) to be the Lie algebra over \( \mathbb{C} \) with generators:

\[
\delta_r(s), \quad \delta_i(k), \quad X_k(\pm \alpha_i), \quad (i = 0, 1, \ldots, l, s, k, r \in \mathbb{Z}^{n-1})
\]

subject to the relations

(TA0)  
  i. The \( \delta_r(s) \) are central; 
  ii. \( \delta_r(s) + \delta_k(s) = \delta_{r+k}(s) \); 
  iii. \( \delta_r(r) = 0 \);

(TA1)  
  \[ [\delta_i(r), \delta_j(s)] = (\delta_i|\delta_j)\delta_r(r + s); \]

(TA2)  
  \[ [\delta_i(r), X_m(\pm \alpha_j)] = \pm(\delta_i|\delta_j)X_{m+r}(\pm \alpha_j) \]

(TA3)  
  \[ [X_m(\alpha_i), X_n(-\alpha_i)] = -\delta_{i-j}\left(\delta_i(m + n) + \frac{2}{(\delta_i|\delta_i)}\delta_i(m + n)\right); \]

(TA4)  
  i. \( [X_m(\pm \alpha_i), X_n(\pm \alpha_j)] = 0 \);
  ii. \( \text{ad} X_0(\alpha_i)^{-A_{i+1}}X_m(\alpha_j) = 0 \) for \( i \neq j \);
  iii. \( \text{ad} X_0(-\alpha_i)^{-A_{i+1}}X_m(-\alpha_j) = 0 \) for \( i \neq j \);

The algebra \( \tau(A_t) \) contains an isomorphic copy of \( t(A_k) \). Hence if we find a representation of \( \tau(A_k) \) it restricts to a representation of \( t(A_k) \). On the other hand it should be true that \( \tau(A_k) \cong t(A_k) \).

4. Proof of the main result

We begin by introducing a slightly modified form of V. Kac’s \( \lambda \)-bracket notation and Fourier transform (see [Kac98]). For any

\[
a(z, w) = \sum_{m,n} a_{m,n} z^m w^n
\]

we define the Fourier transform

\[
F^\lambda_{z,w}a(z, w) = \text{Res}_{z_0} \cdots \text{Res}_{z_N} e^{\sum_{i=0}^N \lambda_i (z_i - w_i)} a(z, w).
\]

If we set

\[
\delta(z/w) := \sum_{m \in \mathbb{Z}^{N+1}} z^m w^{-m}
\]

then

\[
\delta(z/w) = \prod_{i=0}^N \delta(z_i/w_i) \quad \text{where} \quad \delta(z_i/w_i) = \sum_{m \in \mathbb{Z}} z_i^m w_i^{-m}.
\]

For \( j = (j_0, \ldots, j_N) \in \mathbb{N}^{N+1} \), set \( j! = j_0!j_1! \cdots j_N! \), \( \partial_{w_i}^{(j_i)} = \frac{1}{j_i!} \partial_{w_i}^{j_i} \), and \( \partial^{(j)} = \prod_{j=0}^N \partial_{w_i}^{(j_i)} \). Moreover we get

\[
F^\lambda_{z,w} \partial^{(j)} \delta(z/w) = \lambda^{(j)} := \lambda_i^{j_i} = \prod_{i=0}^N \frac{\lambda_i^{j_i}}{j_i!}.
\]

due to three facts ([Kac98, Prop. 2.1]): For \( j > 0 \),

\[
\text{Res}_z \partial_{w_i}^{(j_i)} \delta(z/w) = 0,
\]

\[
(z - w) \partial_{w_i}^{(j_i+1)} \delta(z/w) = \partial_{w_i}^{(j_i)} \delta(z/w), \quad \text{and}
\]

\[
(z - w)^{j_i+1} \partial_{w_i}^{(j_i)} \delta(z/w) = 0.
\]
Indeed
\[ F^\lambda_{z,w}(z/w) = \text{Res}_{z_0} \cdots \text{Res}_{z_n} \sum_{i=0}^N \lambda_i (z_i - w_i) \partial^{(j)} \delta(z/w) \]
\[ = \text{Res}_{z_0} \cdots \text{Res}_{z_n} \left( \prod_{i=0}^N \left( \sum_{k_i=0}^\infty \frac{1}{k_i!} \lambda_i^{k_i} (z_i - w_i)^{k_i} \right) \prod_{i=0}^N \partial^{(j_i)} (z_i/w_i) \right) \]
\[ = \prod_{i=0}^N \text{Res}_{z_i} \left( \sum_{k_i=0}^\infty \frac{1}{k_i!} \lambda_i^{k_i} (z_i - w_i)^{k_i} \right) \partial^{(j_i)} \delta(z_i/w_i). \]

As a consequence if \(a(z), b(w)\) and \(c^{(j)}(w)\) are formal distributions satisfying
\[ [a(z), b(w)] = \sum_{j \in \mathbb{N}^{N+1}} c^{(j)}(w) \partial^{(j)} \delta(z/w), \]
then the \(\lambda\)-bracket is defined as
\[ [a, b] = \sum_{j \in \mathbb{N}^{N+1}} c^{(j)} \lambda^{(j)} \]
where
\[ F^\lambda_{z,w}[a(z), b(w)] = \sum_{j \in \mathbb{N}^{N+1}} c^{(j)}(w) \lambda^{(j)} \]

Let
\[ \Phi(b_r) := \Phi(b_r)(z) = \sum_m \Phi(b_r)(m) z^{-m} \]
then we can write the last calculation in the proof of Proposition 2.1 as
\[ [\Phi(b_r), \lambda] \Phi(b_s)] = A_{rs} \sum_{l=0}^N \rho(K_l) \lambda_l = -A_{rs} K \cdot \lambda. \]

Set
\[ K_l(w) := \left( \sum_r K_{r,l} w^{-r} \right) w_l. \]
The relations (TA1)-(TA4) will follow if the following \(\lambda\)-brackets are satisfied

- **(TA1)** \( [\rho(H_i), \rho(H_j)] = A_{ij} \sum_{l=0}^N \rho(K_l) \lambda_l \) (0 ≤ i, j ≤ n);
- **(TA2)** \( [\rho(H_i), \rho(E_j)] = A_{ij} \rho(E_j), \)
  \( [\rho(H_i), \rho(F_j)] = -A_{ij} \rho(F_j); \)
- **(TA3)** \( [\rho(E_i), \rho(F_j)] = -\delta_{i,j} \left( \rho(H_i) + \frac{2}{A_{ij}} \sum_{l=0}^N \rho(K_l) \lambda_l \right); \)
- **(TA4)** i. \( [\rho(E_i), \rho(E_j)] = 0 = [\rho(F_i), \rho(F_j)], \) if |i − j| ≠ 1.
ii. \( [\rho(E_i), \rho(F_i)] = 0 \) if i = j ± 1.
iii. \( [\rho(F_i), \rho(F_i)] = 0 \) if i = j ± 1.
Proof. We prove only the first one.

\[
\sum_{m,n} [H_i(m), H_j(n)] z^{-m} w^{-n} = A_{ij} \sum_{m,n} \left( \sum_{l=0}^{N} m_l K_{m+n,l} \right) z^{-m} w^{-n}
\]

\[
= A_{ij} \sum_{m,n} \left( \sum_{l=0}^{N} K_{m+n,l} w^{-m-n} \right) m_l (w^m z^n)
\]

\[
= A_{ij} \sum_{r} \left( \sum_{l=0}^{N} K_{r,l} w^{-r} \right) \sum_{m} m_l (w^m z^n)
\]

\[
= A_{ij} \sum_{r} \left( \sum_{l=0}^{N} K_{r,l} w^{-r} \right) w_l \partial_{w_l} \sum_{m} (w^m z^n)
\]

\[
= A_{ij} \sum_{l=0}^{N} K_l(w) \partial_{w_l} \delta(z/w)
\]

5. Preliminary Lemmas

As an example of a computation using the \(\lambda\)-bracket we have the following
Lemma 5.1 ([Cox04], Lemma 4.1).

(a) \[ [a_{ij} \lambda] = \delta_{i,k} \delta_{j,l}, \]
(b) \[ [a_{ij}] = 0, \]
(c) \[ [a_{ij} \kappa \cdot D a^*_k] = \delta_{i,k} \delta_{j,l} \sum_{p=0}^{N} \kappa_p \lambda_p = [\kappa \cdot D a^*_k \lambda a_{ij}], \]
(d) \[ \sum_{j=r+2}^{n+1} \sum_{k=1}^{s-1} \left[ a_{ks} a_{k,s+1}^* \lambda a_{r+1,j}^* a_{r+1,j}^* \right] = -\delta_{s,r+1} a_{r,r+1} a_{r+1,r+2}, \]
(e) \[ \sum_{j=r+2}^{n+1} \sum_{k=1}^{s+2} \left[ a_{ks} a_{k,s+1}^* \lambda a_{r+1,j}^* a_{r+1,j}^* \right] = 0, \]
(f) \[ \sum_{j=r+2}^{n+1} \sum_{k=1}^{s+1} \left[ a_{ks} a_{k,s+1}^* \lambda a_{r+1,j}^* a_{r+1,j}^* \right] = 0, \]
(g) \[ \sum_{j=r+2}^{n+1} \sum_{k=s+2}^{s+1} \left[ a_{ks} a_{k,s+1}^* \lambda \left( a_{r+1,j}^* - a_{r+1,j} a_{r+1,j}^* \right) \right] \]
\[ = -2\delta_{r,s} \sum_{j=r+2}^{n+1} a_{r+1,j}^* a_{r+1,j}^* + \delta_{r,s+1} \sum_{j=r+2}^{n+1} a_{r+1,j}^* a_{r+1,j}^* + \delta_{s,r+1} \sum_{j=r+2}^{n+1} a_{r+1,j}^* a_{r+1,j}^* \]
(h) \[ \sum_{j=1}^{n+1} a_{s+1}^* \lambda a_{r+1,j}^* a_{r+1,j}^* = -\delta_{r,s+1} a_{r+1,r+1}, \]
(i) \[ \sum_{j=r+2}^{n+1} \left[ a_{s+1}^* \lambda a_{r+1,j}^* a_{r+1,j}^* \right] = -\delta_{r+1,s} a_{r+1,r+2}. \]

Proof. Only statement (c), is new. So we prove this:

\[ [a_{ij}(z), \kappa(w) \cdot D a^*_k(w)] = \sum_{m=0}^{N} \sum_{n,q} \kappa_{n,p} [a_{ij}(m), a^*_k(q)] w^n w^p \frac{\partial}{\partial w^p} z^{-m} w^{-q}, \]
\[ = \delta_{i,k} \delta_{j,l} \sum_{p=0}^{N} \kappa_{p} w^n \sum_{m} m_p z^{-m} w^m, \]
\[ = \delta_{i,k} \delta_{j,l} \sum_{p=0}^{N} \kappa_{p} w^n \frac{\partial}{\partial w^p} \delta(z/w). \]
and

\[ [\kappa(z) \cdot D_{kl}^*(z), a_{ij}(w)] = \sum_{m} \sum_{p=0}^{N} \sum_{n,q} \kappa_{n,p}[a_{kl}^*(q), a_{ij}(m)]z^nw^{-m}z_p \frac{\partial}{\partial z_p}z^{-q} \]

\[ = -\delta_{i,k}\delta_{j,l} \sum_{p=0}^{N} \sum_{n} \kappa_{n,p}z^n \frac{\partial}{\partial w} \delta(z/w) \]

\[ = \delta_{i,k}\delta_{j,l} \sum_{p=0}^{N} \sum_{n} \kappa_{n,p}w^n \frac{\partial}{\partial w} \delta(z/w) + \delta_{i,k}\delta_{j,l} \sum_{p=0}^{N} \sum_{n} \kappa_{n,p}w^n \frac{\partial}{\partial w} \delta(z/w) \]

by the relation (TA0). Then when we take the Fourier transform of the above we obtain

\[ [a_{ij} \lambda \kappa \cdot D_{kl}^*] = [\kappa \cdot D_{kl}^* \lambda a_{ij}] = \delta_{i,k}\delta_{j,l} \sum_{p=0}^{N} \kappa_{p} \lambda_p = \delta_{i,k}\delta_{j,l} \kappa \cdot \lambda. \]

\[ \square \]

6. The Relations

Because of the definition of the \( a_{ij} \) and \( a_{ij}^* \) given in (3.1) there are no multiple contractions when computing out the operator product expansion for these terms. We can now check

Lemma 6.1 (R1).

(6.1) \[ [\rho(H_r) \lambda \rho(H_s)] = -A_{rs} \sum_{l=0}^{N} \kappa_l \lambda_l. \]

Proof. Initially we assume that \( r, s \neq 0 \). First we calculate for \( r = s \). We use Lemma 5.1 repeatedly in the following calculations:

\[ 2[a_{r,r+1}a_{r,r+1}^* \rho(H_r)] = 2[a_{r,r+1}a_{r,r+1}^* a_{r,r+1}a_{r,r+1}^*] \]

\[ = 0 \quad \text{by Lemma 5.1b.} \]
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\[
\sum_{i=1}^{r-1} \left[ (a_{i,r+1}a_{i,r+1}^* - a_{ir}a_{ir}^*) \rho(H_r) \right] = \sum_{i=1}^{r-1} [a_{i,r+1}a_{i,r+1}^* + a_{ir}a_{ir}^*] + \sum_{i=1}^{r-1} [a_{ir}a_{ir}^* - a_{i,r+1}a_{i,r+1}^*] = 0 \quad \text{by Lemma 5.1b.}
\]

and similarly

\[
\sum_{j=r+2}^{n+1} \left[ (a_{rj}a_{rj}^* - a_{r+1,j}a_{r+1,j}^*) \rho(H_r) \right] = 0.
\]

The last term in $\rho(H_r)$ contributes

\[
[\Phi(b_r)\lambda \rho(H_r)] = [\Phi(b_r)\lambda \Phi(b_r)] = A_{rr} \sum_{l=0}^{N} \rho(K_l) \lambda_l.
\]

These equations above combine to give the result for $r = s$.

Now we assume that $s = r + 1$.

\[
2[a_{r,r+1}a_{r,r+1}^* - a_{r+1,r+1}a_{r+1,r+1}^*] = 4[a_{r,r+1}a_{r,r+1}^* + a_{r+1,r+1}a_{r+1,r+1}^*] = 0,
\]

and similarly

\[
\frac{1}{r-1} \sum_{i=1}^{r-1} \left[ (a_{i,r+1}a_{i,r+1}^* - a_{ir}a_{ir}^*) \rho(H_r+1) \right] = \sum_{i=1}^{r-1} [a_{i,r+1}a_{i,r+1}^* + a_{ir}a_{ir}^*] = 0,
\]

and similarly

\[
\sum_{j=r+2}^{n+1} \left[ (a_{rj}a_{rj}^* - a_{r+1,j}a_{r+1,j}^*) \lambda \rho(H_r+1) \right] = 0.
\]

Last but not least we calculate the contribution due to the Heisenberg Lie subalgebra:

\[
[\rho(H_r)\lambda \rho(H_{r+1})] = [\Phi(b_r)\lambda \Phi(b_{r+1})] = A_{r,r+1} \sum_{l=0}^{N} \rho(K_l) \lambda_l.
\]

Note that the case $s = r - 1$ is obtained above by replacing $s$ and $r$ by $s'$ and $r'$ respectively in the above argument where $s' = r$ and $r' = s$.

Finally observe that if $|r - s| > 1$ then the indices of $a_{ij}(z)$ and $a_{kl}(w)$ appearing in $H_r$ and $H_s$, respectively, are disjoint and it is straightforward to see that the terms with $b_{k}(z)$ and $b_{l}(w)$ in them have bracket zero. Thus $[\rho(H_r)\lambda \rho(H_{s})] = 0$ in this last case.

If either $r = 0$ and/or $s = 0$, then since $\rho(H_0) = -\sum_{k=1}^{n} \rho(H_k)$, we get that (6.1) is satisfied. \[\square\]

**Lemma 6.2 (R2).**

\[
[\rho(H_r)\lambda \rho(E_s)] = A_{rs} \rho(E_s).
\]
Proof. Initially we take \(r, s \neq 0\) and we first assume \(r = s\). In this case \(\rho(E_r)(w)\) is equal to

\[
a_{r,r+1}^* a_{r,r+1}^* + \sum_{j=r+2}^{n+1} (a_{r,j} a_{r_j}^* - a_{r+1,j} a_{r+1,j}^*) a_{r,r+1}^* \\
+ \sum_{j=1}^{r-1} a_{j,r} a_{j,r+1}^* - \sum_{j=r+2}^{n+1} a_{r+1,j} a_{r+1,j}^* + a_{r,r+1}^* \Phi(b_r) + \kappa \cdot D a_{r,r+1}^*
\]

and \(\rho(H_r)(z)\) expands to

\[
2a_{r,r+1}^* a_{r,r+1}^* + \sum_{i=1}^{r-1} (a_i a_{i,r+1}^* - a_i a_{i,r+1}^*) + \sum_{j=r+2}^{n+1} (a_{r,j} a_{r,j}^* - a_{r+1,j} a_{r+1,j}^*) + \Phi(b_r).
\]

Now

\[
2[a_{r,r+1}^* a_{r,r+1}^* \chi \rho(E_r)] = 2a_{r,r+1}^* a_{r,r+1}^* + \sum_{j=r+2}^{n+1} (a_{r,j} a_{r,j}^* - a_{r+1,j} a_{r+1,j}^*) a_{r,r+1}^*
\]

\[
+ 2a_{r,r+1}^* \Phi(b_r) + 2\kappa \cdot D a_{r,r+1}^* + 2a_{r,r+1}^* \sum_{l=0}^{N} \kappa l \lambda_l.
\]

The second summation in \(\rho(H_r)(z)\) contributes

\[
\sum_{i=1}^{r-1} \left[ (a_{i,r+1} a_{i,r+1}^* - a_{i,r} a_{i,r}^*) \chi \rho(E_r) \right] = \sum_{i=1}^{r-1} \left[ (a_{i,r+1} a_{i,r+1}^* - a_{i,r} a_{i,r}^*) \chi a_{i,r} a_{i,r+1}^* \right]
\]

\[
= 2 \sum_{i=1}^{r-1} a_{i,r} a_{i,r+1}^*.
\]

Now the third summation in \(\rho(H_r)\) contains indices where at least one index is greater than or equal to \(r + 2\) and thus

\[
\sum_{j=r+2}^{n+1} \left[ (a_{r,j} a_{r,j}^* - a_{r+1,j} a_{r+1,j}^*) \chi \rho(E_r) \right] = \sum_{j=r+2}^{n+1} \left[ a_{r,j} a_{r,j}^* a_{r,j} a_{r,j}^* \right] + \sum_{j=r+2}^{n+1} \left[ a_{r+1,j} a_{r+1,j}^* a_{r+1,j} a_{r+1,j}^* \right]
\]

\[
- \sum_{i=r+1}^{n+1} \sum_{j=r+2}^{n+1} \left[ (a_{i,r+1}^* - a_{i+1,j} a_{i+1,j}^*) \lambda a_{r+1,j} a_{r+1,j}^* \right]
\]

\[
= -2 \sum_{j=r+2}^{n+1} a_{r+1,j} a_{r+1,j}^*.
\]

by Lemma 5.1 b and g. The last term in \(\rho(H_r)(w)\) contribute

\[
[\Phi(b_r) \chi \rho(E_r)] = [\Phi(b_r) \chi a_{r,r+1}^* \Phi(b_r)] = -2a_{r,r+1}^* \sum_{p=0}^{N} \kappa p \lambda_p.
\]

The previous four calculations sum up to give us the desired result \([\rho(H_r) \chi \rho(E_r)] = 2\rho(E_r)\).
Now suppose \( s = r + 1 \) so that \( \rho(E_{r+1})(w) \) is equal to
\[
\begin{align*}
ar_{r+1,r+2}(a_{r+1,r+2}^*)^2 & + \sum_{j=r+3}^{n+1} \left( a_{r+1,j}a_{r+1,j}^* - a_{r+2,j}a_{r+2,j}^* \right) a_{r+1,r+2}^* \\
& + \sum_{i=1}^{r} a_{i,r+1}a_{i,r+2}^* - \sum_{j=r+3}^{n+1} a_{r+2,j}a_{r+1,j}^* + a_{r+1,r+2}^* \Phi(b_{r+1}) + \kappa \cdot Da_{r+1,r+2}^*.
\end{align*}
\]
Then the first summand in \( H_r(z) \) contributes
\[
2[a_{r,r+1}a_{r,r+1}^* \rho(E_{r+1})] = -2a_{r,r+1}a_{r,r+2}^*.
\]
The second summation in \( H_r(z) \) contributes
\[
\sum_{i=1}^{r-1} \left( a_{i,r+1}a_{i,r+1}^* - a_{ir}a_{ir}^* \right) \rho(E_{r+1}) = \sum_{i=1}^{r-1} \left( a_{i,r+1}a_{i,r+1}^* - a_{ir}a_{ir}^* \right) a_{i,r+1}a_{i,r+1}^*
\]
\[
= - \sum_{i=1}^{r-1} a_{i,r+1}a_{i,r+2}^*.
\]
The third summand contributes by Lemma 5.1
\[
\sum_{j=r+2}^{n+1} \left( a_{rj}a_{rj}^* - a_{r+1,j}a_{r+1,j}^* \right) \rho(E_{r+1})
\]
\[
= -a_{r+1,r+2}^*a_{r+1,r+2}^* - \sum_{j=r+3}^{n+1} \left( a_{r+1,j}a_{r+1,j}^* - a_{r+2,j}a_{r+2,j}^* \right) a_{r+1,r+2}^*
\]
\[
= -a_{r+1,r+2}^*a_{r+1,r+2}^* - \sum_{j=r+3}^{n+1} a_{r+2,j}a_{r+1,j}^*
\]
\[
- a_{r+1,r+2}^*a_{r+1,r+2}^* + 2\Phi(b_{r+1}) - \kappa \cdot Da_{r+1,r+2}^* - a_{r+1,r+2}^* \sum_{l=0}^{N} \kappa_l \lambda_i.
\]
The last summand in \( \rho(H_r)(z) \) has \( \lambda \)-bracket with \( \rho(E_{r+1})(w) \) equal to
\[
\Phi(b_r)\lambda \rho(E_{r+1}) = a_{r+1,r+2}^* [\rho(b_r)\lambda \rho(b_{r+1})] = a_{r+1,r+2}^* \sum_{p=0}^{N} \kappa_p \lambda_p.
\]
Adding the previous four equations up we get \( [\rho(H_r)\lambda \rho(E_{r+1})] = -\rho(E_{r+1}) \).

The final nontrivial case to consider is when \( s = r - 1 \) (and \( rs \neq 0 \)) so that \( \rho(E_{r-1})(w) \) is equal to
\[
\begin{align*}
ar_{r-1,r}(a_{r-1,r}^*)^2 & + \sum_{j=r+1}^{n+1} \left( a_{r-1,j}a_{r-1,j}^* - a_{rj}a_{rj}^* \right) a_{r-1,r}^* \\
& + \sum_{j=1}^{r-2} a_{j,r-1}a_{j,r}^* - \sum_{j=r+1}^{n+1} a_{rj}a_{r-1,j}^* + a_{r-1,r}^* \Phi(b_{r-1}) + \kappa \cdot Da_{r-1,r}^*.
\end{align*}
\]
Then
\[
2[a_{r,r+1}a_{r,r+1}^* \rho(E_{r-1})] = -2a_{r-1,r+1}a_{r,r+1}.
\]
The second summation in $\rho(H_r)(z)$ contributes by Lemma 5.1
\[
\sum_{i=1}^{r-1} \left( a_{i,r+1}a_{i,r+1}^* - a_{i,r}a_{i,r}^* \right) \rho(E_{r-1}) = -a_{r-1,r}a_{r-1,r}^*,
\]
\[
- \sum_{j=r+1}^{n+1} (a_{r-1,j}a_{r-1,j}^* - a_{r,j}a_{r,j}^*) a_{r-1,r},
\]
\[
- \sum_{j=1}^{r-2} a_{j,r-1}a_{j,r}^* + a_{r,r+1}a_{r-1,r}^* + 1
\]
\[
- a_{r-1,r}^* \Phi(b_{r-1}) - \kappa \cdot Da_{r-1,r} - \sum_{p=0}^{N} \kappa_p \lambda_p.
\]
The third summand contributes
\[
\sum_{j=r+2}^{n+1} \left( a_{r,j}a_{r,j}^* - a_{r+1,j}a_{r+1,j}^* \right) \lambda \rho(E_{r-1}) = \sum_{j=r+2}^{n+1} a_{r,j}a_{r-1,j}^*.
\]
The last summation in $\rho(H_r)(z)$ has commutator with $\rho(E_{r-1})(w)$ equal to
\[
\Phi(b_r)\lambda \rho(E_{r-1}) = -\sum_{p=0}^{N} \kappa_p \lambda_p.
\]
Adding the previous four equations up we get $[\rho(H_r)(z), \rho(E_{r-1})(w)] = -\rho(E_{r-1})(w)\delta(w/z)$.
Lastly if $|r-s| > 1$ then observe the indices of $a_{ij}$ and $a_{ij}^*$ in $\rho(H_r)(z)$ and $\rho(E_s)(w)$ are disjoint and thus contribute nothing to the commutator $[\rho(H_r)(z), \rho(E_s)(w)]$. The remaining terms coming from the $b_j$ have trivial commutator and thus $[\rho(H_r)(z), \rho(E_s)(w)] = 0$.
Lastly if $|r-s| > 1$ with $rs \neq 0$, then observe the indices of $a_{ij}$ and $a_{ij}^*$ in $\rho(H_r)(z)$ and $\rho(E_s)(w)$ are disjoint and thus contribute nothing to the commutator $[\rho(H_r)(z), \rho(E_s)(w)]$. The remaining terms coming from the $b_j$ have trivial commutator and thus $[\rho(H_r)(z), \rho(E_s)(w)] = 0$.
We now consider the case of $s = 0$ and $r \neq 0$: Then $\rho(E_s)(w) = \rho(E_0)(w) = -a_{1,n+1}$ and hence
\[
2[a_{r,r+1}a_{r+1,n+1}^* \rho(E_0)] = 0.
\]
The second summation in $\rho(H_r)(z)$ contributes
\[
- \sum_{i=1}^{r-1} \left( a_{i,r+1}a_{i,r+1}^* - a_{i,r}a_{i,r}^* \right) a_{1,n+1} = \delta_{r,n}a_{1,n+1}.
\]
The third summand contributes
\[
- \sum_{j=r+2}^{n+1} \left( a_{r,j}a_{r,j}^* - a_{r+1,j}a_{r+1,j}^* \right) \lambda a_{1,n+1} = \delta_{r,n}a_{1,n+1}.
\]
The last summation in $\rho(H_r)(z)$ has commutator with $\rho(E_0)(w)$ equal to $0$. Adding the previous three equations up we get $[\rho(H_r)(z), \rho(E_0)(w)] = -\rho(E_0)(w)\delta(w/z)$. 
If $r = 0$, then since $\rho(H_0) = -\sum_{r=1}^{n} \rho(H_r)$, we get

$$\rho(H_0)(z), \rho(E_s)(w) = -\sum_{r=1}^{n} [\rho(H_r)(z), \rho(E_s)(w)]$$

$$= -\sum_{r=1}^{n} A_{r,s} \rho(E_s)(w) \delta(z/w) = A_{0,s} \rho(E_s)(w) \delta(z/w)$$

which holds for any $s$. This completes the proof of the Lemma. \qed

**Lemma 6.3** (R2).

$$[\rho(H_r), \rho(F_s)] = -C_{r,s} \rho(F_s)(z) \delta(z/w)$$

**Proof.** First we consider $r \neq 0$ and $s \neq 0$. First we calculate for $r = s$. Recall the expansion for $\rho(F_r)(w)$ is

$$-a_{r,r+1}(w) + \sum_{j=1}^{r-1} a_{j,r+1}(w) a_{r,j}^*(w).$$

The first summand in $\rho(H_r)(z)$ gives us

$$2[a_{r,r+1}(z) a_{r,r+1}^*(z), \rho(F_r)(w)] = 2a_{r,r+1}(z) \delta(z/w)$$

and the second

$$\sum_{i=1}^{r-1} [a_{i,r+1}(z) a_{i,r+1}^*(z) - a_{ir}(z) a_{ir}^*(z), \rho(F_r)(w)]$$

$$= \sum_{i=1}^{r-1} [a_{i,r+1}(z) a_{i,r+1}^*(z), a_{ir}(z) a_{ir}^*(w)]$$

$$- \sum_{i=1}^{r-1} [a_{ir}(z) a_{ir}^*(z), a_{i,r+1}(w) a_{ir}^*(w)]$$

$$= -2 \sum_{i=1}^{r-1} a_{i,r+1}(z) a_{ir}^*(w) \delta(z/w).$$

Since the second indices $j$ of $a_{ij}(w)$ and $a_{ij}^*(w)$ in $\rho(F_r)(w)$ are less than or equal to $r + 1$ we get

$$\sum_{j=r+2}^{n+1} [a_{rj}(z) a_{rj}^*(z) - a_{r+1,j}(z) a_{r+1,j}^*(z), \rho(F_r)(w)] = 0.$$

The last term in $\rho(H_r)(z)$ contributes

$$[-\gamma b_r(z) + \gamma 2(b_{r-1}^+(z) + b_{r+1}^+(z)), \rho(F_r)(w)] = 0.$$

These equations above combine to give the result for $r = s$.

Now we assume that $s = r + 1$. The expansion for $\rho(F_{r+1})(w)$ is

$$-a_{r+1,r+2}(w) + \sum_{j=1}^{r} a_{j,r+2}(w) a_{j,r+1}^*(w).$$
The first summand in \( \rho(H_r)(z) \) gives us

\[
2[a_{r,r+1}(z)a_{r,r+1}^*(z), \rho(F_{r+1})(w)] = 2[a_{r,r+1}(z)a_{r,r+1}^*(z), a_{r,r+2}(w)a_{r,r+1}^*(w)] = 2a_{r,r+2}(w)a_{r,r+1}^*(z)\delta(w/z),
\]

Next using the observation that the second indices in \( \rho(F_{r+1})(w) \) are either an \( r+1 \) or an \( r+2 \) we get

\[
\sum_{i=1}^{r-1} [a_{i,r+1}(z)a_{i,r+1}^*(z) - a_{i,r}(z)a_{i,r}^*(z), \rho(F_{r+1})(w)] = \sum_{i=1}^{r-1} [a_{i,r+1}(z)a_{i,r+1}^*(z), \rho(F_{r+1})(w)] = \sum_{i=1}^{r-1} \sum_{j=1}^{r-1} [a_{i,r+1}(z)a_{i,r+1}^*(z), a_{j,r+2}(w)a_{j,r+1}^*(w)] = \sum_{i=1}^{r-1} a_{i,r+2}(w)a_{i,r+1}^*(z)\delta(z/w)
\]

The second summation in \( \rho(H_r)(z) \) contributes

\[
\sum_{j=r+2}^{n+1} [a_{rj}(z)a_{rj}^*(z) - a_{r+1,j}(z)a_{r+1,j}^*(z), \rho(F_{r+1})(w)] = \sum_{j=r+2}^{n+1} \sum_{k=1}^{r} [a_{rj}(z)a_{rj}^*(z), a_{k,r+2}(w)a_{k,r+1}^*(w)] + \sum_{j=r+2}^{n+1} [a_{r+1,j}(z)a_{r+1,j}^*(z), a_{r+1,r+2}(w)\delta(z/w) - a_{r+1,r+2}(z)a_{r+1,r+1}^*(w)]
\]

The last summands in \( \rho(H_r)(z) \) contributes zero to the commutator:

\[
[-\gamma b_r(z) + \frac{\gamma}{2} (h_{r-1}^+(z) + h_{r+1}^+(z)), \rho(F_{r+1})(w)] = 0.
\]

These four calculations above combine to give

\[
[\rho(H_r)(z), \rho(F_{r+1})(w)] = -a_{r+1,r+2}(z) + \sum_{i=1}^{r} a_{i,r+2}(w)a_{i,r+1}^*(z) \delta(z/w)
\]

Next we consider \( s = r - 1 \). The expansion for \( \rho(F_{r-1})(w) \) is

\[
-a_{r-1,r}(w) + \sum_{j=1}^{r-2} a_{j,r}(w)a_{j,r-1}^*(w).
\]
The first summand in $\rho(H_r)(z)$ gives us
\[ 2[a_{r,r+1}(z)a^*_{r,r+1}(z), \rho(F_{r-1})(w)] = 0 \]

The second summation in $\rho(H_r)(z)$ gives
\[
\sum_{i=1}^{r-1} [a_{i,r+1}(z)a^*_{i,r+1}(z) - a_{ir}(z)a^*_{ir}(z), \rho(F_{r-1})(w)] \\
= [a_{r-1,r}(z)a^*_{r-1,r}(z), a_{r-1,r}(w)] \\
- \sum_{i=1}^{r-2} a_{ir}(z)[a^*_{ir}(z), a_{ir}(w)]a^*_{ir-1}(w) \\
= \left( -a_{r-1,r}(z) + \sum_{i=1}^{r-2} a_{ir}(z)a^*_{ir-1}(w) \right) \delta(z/w)
\]

Since the first indices $j$ of $a_{ij}(w)$ and $a^*_{ij}(w)$ in $\rho(F_{r-1})(w)$ are less than or equal to $r - 1$ we get
\[
\sum_{j=r+2}^{n+1} [a_{rj}(z)a^*_{rj}(z) - a_{r+1,j}(z)a^*_{r+1,j}(z), \rho(F_r)(w)] = 0.
\]

As above the last term in $\rho(H_r)(z)$ contributes zero to the commutator. These equations above combine to give the result for $s = r - 1$.

Finally observe that if $|r - s| > 1$ then the indices of $a_{ij}(w)$ and $a^*_{ij}(w)$ in $\rho(F_{r-1})(w)$ are disjoint and thus $[\rho(H_r)(z), \rho(F_s)(w)] = 0$ in this last case. \(\square\)

**Lemma 6.4** (R3). $[\rho(E_s), \lambda \rho(F_r)] = -\delta_{r,s} \left( \rho(H_r) + \frac{2}{A_{rs}} \sum_{l=0}^{N} \rho(K_l) \lambda_l \right)$.

**Proof.** For $r \neq 0$ and $s \neq 0$, the proof is nearly the same as those in [Cox02, Lemma 3.4] where
\[-\gamma b_r(z) - \frac{1}{2} (b^+_r(z) + b^+_r(z)) \]
is replaced by $\Phi(b_r)$ and
\[-\frac{\gamma}{2} a^*_{r,r+1}(z) \]
is replace by $\kappa \cdot Da^*_{r,r+1}$. We refer the interested reader to that paper for the proof.

It is also straightforward to check that
\[ [\rho(E_0), \lambda \rho(F_r)] = -\delta_{0,r} \left( \rho(H_r) + \frac{2}{A_{r0}} \sum_{l=0}^{N} \rho(K_l) \lambda_l \right). \]
Now we consider $[\rho(E_s) \lambda \rho(F_0)]$ with $s > 0$: We break this up into pieces:

$$-[a_{s,s+1}(a_{s,s+1}^*)^2 \lambda \sum_{1 \leq r < j \leq n+1} a_{rj} \sum_{q,j=q_i,r \geq q_{i-1}}^{i-1} a_{q_jq_{i+1}} a_{r,n+1}^*]$$

$$= -\delta_{s,n} a_{n,n+1}(a_{n,n+1}^*)^2 \sum_{q,n+1=q_i}^{i-1} a_{q_jq_{i+1}}$$

$$- \sum_{1 \leq r < j \leq n+1} a_{rj} \sum_{q,j=q_i,r \geq q_{i-1}}^{i-1} a_{q_jq_{i+1}} (a_{r,q_{j+1}}^*)^2 a_{r,n+1}^*$$

$$+ 2a_{s,s+1} a_{s,s+1}^* \sum_{q,s+1=q_i,s \geq q_{i-1}}^{i-1} a_{s,n+1}^*$$

$$-[a_{s,s+1}(a_{s,s+1}^*)^2 \lambda \sum_{1 \leq r < n+1} \sum_{q,r \geq q_i} a_{q_jq_{i+1}} \Phi(b_r)] = -\sum_{1 \leq r < n+1} \sum_{q,r \geq q_i}^{i-1} a_{q_jq_{i+1}} (a_{s,s+1}^*)^2 \Phi(b_r),$$

$$-[a_{s,s+1}(a_{s,s+1}^*)^2 \lambda \sum_{1 \leq r < n+1} \sum_{q,r \geq q_i}^{i-1} a_{q_jq_{i+1}} \Phi(b_r)] = -\sum_{1 \leq r < n+1} \sum_{q,r \geq q_i}^{i-1} a_{q_jq_{i+1}} (a_{s,s+1}^*)^2 \Phi(b_r),$$

The second summand in $\rho(E_s)$ contributes the following:

$$[\sum_{k=s+2}^{n+1} a_{s+1,k} a_{sk}^* \lambda \sum_{1 \leq r < j \leq n+1} a_{rj} \sum_{q,j=q_i,r \geq q_{i-1}}^{i-1} a_{q_jq_{i+1}} \Phi(b_r)] = -\sum_{k=s+2}^{n+1} a_{s+1,k} \sum_{q,k=q_i,r \geq q_{i-1}}^{i-1} a_{q_jq_{i+1}} a_{r,n+1}^*$$

$$+ \sum_{k=s+2}^{n+1} \sum_{1 \leq r < j \leq n+1} a_{rj} \sum_{q,j=q_i,r \geq q_{i-1}}^{i-1} a_{q_jq_{i+1}} (a_{s,s+1}^*)^2 a_{s,k} a_{r,n+1}^*$$

$$+ \sum_{s+1 \leq j \leq n+1} \sum_{q,j=q_i,s+1 \leq q_{i-1}}^{i-1} a_{s+1,j} \sum_{q,j=q_i,s+1 \leq q_{i-1}}^{i-1} a_{q_jq_{i+1}} a_{s,n+1}^*$$
A WAKIMOTO TYPE REALIZATION OF TOROIDAL $sl_{n+1}$

\[
\begin{align*}
&\left[ \sum_{k=s+2}^{n+1} a_{s+1,k}a_{sk}\lambda \sum_{1 \leq r < n+1} \sum \prod_{q_i=q_j, j=1}^{i-1} a_{q_j q_{j+1}}^* \cdot Da_{r,n+1}^* \right] \\
&= \sum_{k=s+2}^{n+1} \sum_{1 \leq r < n+1} \sum_{q_i=q_j, j=1}^{i-1} a_{q_j q_{j+1}}^* a_{sk}\lambda \cdot Da_{r,n+1}^* \\
&+ (1 - \delta_{s,n}) \sum_{q_i=q_j, j=1}^{i-1} a_{q_j q_{j+1}}^* \cdot (\lambda + D)a_{s,n+1}^*.
\end{align*}
\]

Next we consider the third summation in $\rho(E_s)$:

\[
\begin{align*}
&- \sum_{k=1}^{s-1} a_{ks} a_{k,s+1}^* \lambda \sum_{1 \leq r < j \leq n+1} a_{rj} \sum_{q_i=q_j, r \geq q_i-1} \left( \prod_{l=1}^{i-1} a_{q_l q_{l+1}}^* \right) a_{r,n+1}^* \\
&= \sum_{k=1}^{s-1} a_{ks} \sum_{q_i=q_j, k \geq q_i-1} \left( \prod_{l=1}^{i-1} a_{q_l q_{l+1}}^* \right) a_{k,n+1}^* \\
&- \sum_{k=1}^{s-1} \sum_{1 \leq r < j \leq n+1} a_{rj} \sum_{q_i=q_j, r \geq q_i-1} \left( \prod_{l=1}^{i-1} a_{q_l q_{l+1}}^* \right) a_{k,s+1}^* a_{r,n+1}^*.
\end{align*}
\]

\[
\begin{align*}
&- \sum_{k=1}^{s-1} a_{ks} a_{k,s+1}^* \lambda \sum_{1 \leq r < n+1} \sum_{q_i=q_j, j=1}^{i-1} \prod_{q_i=q_j} a_{q_i q_{j+1}}^* \Phi(b_r) = - \sum_{k=1}^{s-1} \sum_{1 \leq r < n+1} \sum_{q_i=q_j, r \geq q_i-1} \left( \prod_{j=1, j \neq t}^{i-1} a_{q_j q_{j+1}}^* \right) a_{k,s+1}^* \Phi(b_r),
\end{align*}
\]

\[
\begin{align*}
&- \sum_{k=1}^{s-1} a_{ks} a_{k,s+1}^* \lambda \sum_{1 \leq r < n+1} \sum_{q_i=q_j, j=1}^{i-1} a_{q_i q_{j+1}}^* \cdot Da_{r,n+1}^* \\
&= - \sum_{k=1}^{s-1} \sum_{1 \leq r < n+1} \sum_{q_i=q_j, j=1}^{i-1} \left( \prod_{j=1, j \neq t}^{i-1} a_{q_j q_{j+1}}^* \right) a_{k,s+1}^* \cdot Da_{r,n+1}^*.
\end{align*}
\]
The forth summation in $\rho(E_s)$ λ-brackets with the summands of $\rho(F_0)$ as follows:

\[- \sum_{k=s+2}^{n+1} \left( (a_{sk}a_{sk}^* - a_{s+1,k}a_{s+1,k}^*) a_{s,s+1}^* \lambda \right) \sum_{a_{rj} \text{ s.t. } (q_r,q_{r+1})=(s,k)} \prod_{l=1}^{i-1} a_{q_{l+1}}^* a_{r,l+n+1}^* \]

\[= \sum_{j=s+2}^{n+1} a_{sj} a_{s,s+1}^* \prod_{q_j=q_j \geq q_{j-1}}^{i-1} a_{q_{j+1}}^* a_{s,n+1}^* \]

\[- \sum_{j=s+2}^{n+1} a_{s+1,j} a_{s,s+1}^* \prod_{q_j=q_j \geq q_{j-1}}^{i-1} a_{q_{j+1}}^* a_{s+1,n+1}^* \]

\[+ \sum_{k=s+2}^{n+1} \left( (a_{sk}a_{sk}^* - a_{s+1,k}a_{s+1,k}^*) \prod_{q_k=q_k \geq q_{k-1}}^{i-1} a_{q_{k+1}}^* a_{s,n+1}^* \right) \]

\[= (1-\delta_{s,n}) \sum_{s<j}^{n+1} a_{sj} \prod_{q_j=q_j \geq q_{j-1}}^{i-1} a_{q_{j+1}}^* a_{s,n+1}^* a_{s,s+1}^* \]

\[+ \sum_{s+1<j}^{n+1} a_{s+1,j} \prod_{q_j=q_j \geq q_{j-1}}^{i-1} a_{q_{j+1}}^* a_{s+1,n+1}^* a_{s,s+1}^* \]

\[- \sum_{k=s+2}^{n+1} \left( (a_{sk}a_{sk}^* - a_{s+1,k}a_{s+1,k}^*) a_{s,s+1}^* \lambda \right) \sum_{1 \leq r<n+1} \prod_{q_r=q_r \geq q_{r-1}}^{i} a_{q_{r+1}}^* a_{s,s+1}^* \Phi(b_r) \]

\[= - \sum_{k=s+2}^{n+1} \left( (a_{sk}a_{sk}^* - a_{s+1,k}a_{s+1,k}^*) \sum_{1 \leq r<n+1} \prod_{q_r=q_r \geq q_{r-1}}^{i} a_{q_{r+1}}^* a_{s,s+1}^* \Phi(b_r) \right) \]

\[+ \sum_{k=s+2}^{n+1} \left( (a_{sk}a_{sk}^* - a_{s+1,k}a_{s+1,k}^*) \sum_{1 \leq r<n+1} \prod_{q_r=q_r \geq q_{r-1}}^{i} a_{q_{r+1}}^* a_{s,s+1}^* \Phi(b_r) \right) .\]
and

\[- \sum_{k=s+2}^{n+1} \left( (a_{sk}a_{sk} - a_{s+1,k}a_{s+1,k}) a_{s,s+1} \right) \sum_{1 \leq r < n+1} \sum_{q:r=q_i}^{i-1} a_{q_j,q_{j+1}}^* \kappa \cdot Da_{r,n+1}^* \]

\[= - \sum_{s \leq r < n+1} \sum_{k=s+2}^{n+1} a_{rj} \sum_{q:r=q_i}^{i-1} a_{q_j,q_{j+1}}^* \kappa \cdot Da_{r,n+1}^* \]

Now we consider the last two summands in \( \rho(\mathcal{E}_s) \):

\[- [(a_{s,s+1}^* \Phi(b_s) + \kappa \cdot Da_{s,s+1}^*) \lambda \sum_{1 \leq r < j \leq n+1} a_{rj} \sum_{q:r=q_i}^{i-1} a_{q_j,q_{j+1}}^* \kappa \cdot \lambda] \]

and

\[- [(a_{s,s+1}^* \Phi(b_s) + \kappa \cdot Da_{s,s+1}^*) \lambda \sum_{1 \leq r < n+1} \sum_{q:r=q_i}^{i-1} a_{q_j,q_{j+1}}^* \Phi(b_r)] \]

\[= \sum_{1 \leq r < n+1} \sum_{q:r=q_i}^{i-1} a_{q_j,q_{j+1}}^* A_{sr} \lambda \sum_{1 \leq r < n+1} \sum_{q:r=q_i}^{i-1} A_{sr} \kappa \cdot (\lambda + D) a_{s,s+1}^*. \]
Summing up the terms with $\Phi(b_k)$’s in them we get

$$- \sum_{1 \leq r < n+1} \sum_{q_r \geq q_i} \prod_{j=1, j \neq t}^{i} a_{q_{r+1}}^* (a_{s,s+1}^*)^{2} \Phi(b_r),$$

$$+ \sum_{k=s+2}^{n+1} \sum_{1 \leq r < n+1} \sum_{q_r \geq q_i} \prod_{j=1, j \neq t}^{i} a_{q_{r+1}}^* a_{s,k}^* \Phi(b_r)$$

$$- \sum_{k=1}^{s-1} \sum_{1 \leq r < n+1} \sum_{q_r \geq q_i} \left( \prod_{j=1, j \neq t}^{i} a_{q_{r+1}}^* \right) a_{k,s+1}^* \Phi(b_r)$$

$$- \sum_{k=s+2}^{n+1} \sum_{1 \leq r < n+1} \sum_{q_r \geq q_i} \prod_{j=1}^{i} a_{q_{r+1}}^* a_{s,s+1}^* \Phi(b_r)$$

$$+ \sum_{k=s+2}^{n+1} \sum_{1 \leq r < n+1} \sum_{q_r \geq q_i} \prod_{j=1}^{i} a_{q_{r+1}}^* a_{s,s+1}^* \Phi(b_r)$$

$$+ \sum_{q_{s+1}=q_s \geq q_{i-1}} \left( \prod_{l=1}^{i-1} a_{q_{l+1}}^* \right) a_{s,n+1}^* \Phi(b_s)$$

$$= - \sum_{r=s+1}^{n} \sum_{q_r \geq q_i} \prod_{j=1}^{i} a_{q_{r+1}}^* a_{s,s+1}^* \Phi(b_r),$$

$$+ \sum_{k=s+2}^{n+1} \sum_{r=s+1}^{n} \sum_{q_r \geq q_i} \prod_{j=1, j \neq t}^{i} a_{q_{r+1}}^* a_{s,k}^* \Phi(b_r)$$

$$- \sum_{k=1}^{s-1} \sum_{r=s+1}^{n} \sum_{q_r \geq q_i} \left( \prod_{j=1, j \neq t}^{i} a_{q_{r+1}}^* \right) a_{k,s+1}^* \Phi(b_r)$$

$$- \sum_{k=1}^{s-1} \sum_{q_{s+1}=q_s \geq q_{i-1}} \left( \prod_{l=1}^{i-1} a_{q_{l+1}}^* \right) a_{k,s+1}^* \Phi(b_s)$$

$$- \sum_{q_{s+1}=q_s \geq q_{i-1}} \prod_{j=1}^{i} a_{q_{j+1}}^* a_{s,s+1}^* \Phi(b_s)$$

$$= \prod_{j=1}^{i} a_{s,s+1}^* \Phi(b_s)$$
\[
\begin{align*}
&= - \sum_{r=s+1}^{n} \sum_{q_i \geq q_i} \prod_{j=1}^{i} a_{q_i q_j + 1}^* \Phi(b_r), \\
&\quad + \sum_{k=s+2}^{n} \sum_{r=s+1}^{n} \sum_{q_i \geq q_i} \prod_{j=1}^{i} a_{q_i q_j + 1}^* a_{s, s+1}^* \Phi(b_r) \\
&\quad - \sum_{k=1}^{n} \sum_{q_i \geq q_i} \prod_{j=1}^{i} a_{q_i q_j + 1}^* a_{k, s+1}^* \Phi(b_r) \\
&\quad + \sum_{k=s+2}^{n} \sum_{r=s+1}^{n} \sum_{q_i \geq q_i} \prod_{j=1}^{i} a_{q_i q_j + 1}^* a_{s, s+1}^* \Phi(b_r) \\
&\quad - \sum_{k=1}^{n} \sum_{q_i \geq q_i} \prod_{j=1}^{i} a_{q_i q_j + 1}^* a_{k, s+1}^* \Phi(b_r) \\
&\quad + \sum_{k=s+1}^{n} \sum_{q_i \geq q_i} \prod_{j=1}^{i} a_{q_i q_j + 1}^* a_{s, s+1}^* \Phi(b_r) \\
&= - \sum_{r=s+1}^{n} \sum_{q_i \geq q_i} \prod_{j=1}^{i} a_{1, q_1}^* \cdots a_{q_i - 1, s}^* a_{s, s+1}^* a_{q_i, n+1}^* a_{s, s+1}^* \Phi(b_r), \\
&\quad + \sum_{r=s+1}^{n} \sum_{k=s+2}^{n} \sum_{q_i \geq q_i} \prod_{j=1}^{i} a_{1, q_1}^* \cdots a_{q_i - 1, s}^* a_{k, q_i + 2}^* \cdots a_{q_i, n+1}^* a_{s, k}^* \Phi(b_r) \\
&\quad - \sum_{r=s+1}^{n} \sum_{k=1}^{n} \sum_{q_i \geq q_i} \prod_{j=1}^{i} a_{1, q_1}^* \cdots a_{q_i - 1, k}^* a_{s, q_i + 2}^* \cdots a_{q_i, n+1}^* a_{k, s+1}^* \Phi(b_r) \\
&\quad - \sum_{r=s+1}^{n} \sum_{k=s+2}^{n} \sum_{q_i \geq q_i} \prod_{j=1}^{i} a_{1, q_1}^* \cdots a_{s, k}^* \cdots a_{q_i, n+1}^* a_{s, s+1}^* \Phi(b_r) \\
&\quad + \sum_{r=s+1}^{n} \sum_{k=s+2}^{n} \sum_{q_i \geq q_i} \prod_{j=1}^{i} a_{1, q_1}^* \cdots a_{s, s+1}^* a_{q_i, n+1}^* a_{s, s+1}^* \Phi(b_r)
\end{align*}
\]
\[
\begin{align*}
&= \sum_{r=s+1}^{n} \sum_{k=s+2}^{n+1} a_{r,q_1}^* a_{q_1-s+1}^* a_{k,q_1+2}^* \cdots a_{q_1,n+1}^* a_{s,k}^* \Phi(b_r) \\
&\quad - \sum_{r=s+1}^{n} \sum_{k=1}^{n} a_{r,q_1}^* a_{q_1-k}^* a_{s,k}^* \cdots a_{q_1,n+1}^* a_{k,s+1}^* \Phi(b_r) \\
&\quad - \sum_{r=s+1}^{n} \sum_{k=s+2}^{n} a_{r,q_1}^* a_{s,k}^* a_{q_1-n+1}^* a_{s,s+1}^* \Phi(b_r) \\
&\quad + \sum_{r=s+1}^{n} \sum_{k=1}^{n} a_{r,q_1}^* a_{q_1-k}^* a_{k,s+1}^* \cdots a_{q_1,n+1}^* a_{s,s+1}^* \Phi(b_r) \\
\end{align*}
\]

\[
\begin{align*}
&= \sum_{r=s+1}^{n} \sum_{k=s+2}^{n+1} a_{r,q_1}^* a_{q_1-s+1}^* a_{k,q_1+2}^* \cdots a_{q_1,n+1}^* a_{s,k}^* \Phi(b_r) \\
&\quad - \sum_{r=s+1}^{n} \sum_{k=1}^{n} a_{r,q_1}^* a_{q_1-k}^* a_{s,k}^* \cdots a_{q_1,n+1}^* a_{k,s+1}^* \Phi(b_r) \\
&\quad - \sum_{r=s+1}^{n} \sum_{k=s+2}^{n} a_{r,q_1}^* a_{s,k}^* a_{q_1-n+1}^* a_{s,s+1}^* \Phi(b_r) \\
&\quad + \sum_{r=s+1}^{n} \sum_{k=1}^{n} a_{r,q_1}^* a_{q_1-k}^* a_{k,s+1}^* \cdots a_{q_1,n+1}^* a_{s,s+1}^* \Phi(b_r) \\
\end{align*}
\]

\[
\begin{align*}
&= \sum_{r=s+1}^{n} \sum_{k=s+2}^{n+1} a_{r,q_1}^* a_{s,s+1}^* a_{k,q_1+2}^* \cdots a_{q_1,n+1}^* a_{s,k}^* \Phi(b_r) \\
&\quad + \sum_{r=s+1}^{n} \sum_{k=s+2}^{n} \sum_{l=1}^{s+1} a_{r,q_1}^* a_{s,l}^* a_{k,q_1+2}^* \cdots a_{q_1,n+1}^* a_{s,k}^* \Phi(b_r) \\
&\quad - \sum_{r=s+1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{s+1} a_{r,q_1}^* a_{q_1-k}^* a_{s,l}^* \cdots a_{q_1,n+1}^* a_{k,s+1}^* \Phi(b_r) \\
&\quad - \sum_{r=s+1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{s+1} a_{r,q_1}^* a_{s,k}^* a_{q_1-n+1}^* a_{s,s+1}^* \Phi(b_r) \\
\end{align*}
\]

\[
\begin{align*}
&= \sum_{r=s+1}^{n} \sum_{k=s+2}^{n+1} a_{r,q_1}^* a_{s,s+1}^* a_{k,q_1+2}^* \cdots a_{q_1,n+1}^* a_{s,k}^* \Phi(b_r) \\
&\quad + \sum_{r=s+1}^{n} \sum_{k=s+2}^{n} \sum_{l=1}^{s+1} a_{r,q_1}^* a_{s,l}^* a_{k,q_1+2}^* \cdots a_{q_1,n+1}^* a_{s,k}^* \Phi(b_r) \\
&\quad - \sum_{r=s+1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{s+1} a_{r,q_1}^* a_{q_1-k}^* a_{s,l}^* \cdots a_{q_1,n+1}^* a_{k,s+1}^* \Phi(b_r) \\
&\quad - \sum_{r=s+1}^{n} \sum_{k=1}^{n} \sum_{l=1}^{s+1} a_{r,q_1}^* a_{s,k}^* a_{q_1-n+1}^* a_{s,s+1}^* \Phi(b_r) \\
&= 0.
\end{align*}
\]
Next we consider the terms with the $\kappa \cdot \lambda$'s and $D$'s in them (here $A_{1,0} := 0 = A_{n+1,n+2} := 0$):

$$\sum_{s+1 \leq r < n+1} \sum_{q_i, r = q_i} \prod_{j=1}^{i-1} a_{q_j q_{j+1}} a_{s,s+1}^2 \kappa \cdot Da_{r,n+1}$$

$$- \delta_{s,n} \sum_{q_i, r = q_i} \prod_{j=1}^{i-1} a_{q_j q_{j+1}} \kappa \cdot (\lambda + D)(a_{s,n+1})^2$$

$$+ \sum_{s+1 \leq r < n+1} \sum_{q_i, r = q_i} \prod_{j=1}^{i-1} a_{q_j q_{j+1}} a_{s,s+1}^{*} \kappa \cdot Da_{r,n+1}$$

$$+ (1 - \delta_{s,n}) \sum_{q_i, r = q_i} \prod_{j=1}^{i-1} a_{q_j q_{j+1}} \kappa \cdot (\lambda + D)a_{s,n+1}$$

$$- \sum_{s \leq r < n+1} \sum_{k=1}^{n-1} \sum_{q_i, r = q_i} \prod_{j=1}^{i-1} a_{q_j q_{j+1}} a_{s,s+1}^{*} \kappa \cdot Da_{r,n+1}$$

$$+ \sum_{s \leq r < n+1} \sum_{k=1}^{n-1} \sum_{q_i, r = q_i} \prod_{j=1}^{i-1} a_{q_j q_{j+1}} a_{s,s+1}^{*} \kappa \cdot Da_{r,n+1}$$

$$+ (1 - \delta_{s,n}) \sum_{q_i, r = q_i} \prod_{j=1}^{i-1} a_{q_j q_{j+1}} \kappa \cdot (\lambda + D)(a_{s,n+1} a_{s,s+1})$$

$$+ (1 - \delta_{s,n}) \sum_{q_i, r = q_i} \prod_{j=1}^{i-1} a_{q_j q_{j+1}} \kappa \cdot (\lambda + D)(a_{s+1,n+1} a_{s,s+1})$$

$$- \sum_{q_i, r = q_i} \prod_{j=1}^{i-1} a_{q_j q_{j+1}} a_{s,n+1}^{*} \kappa \cdot \lambda$$

$$+ \sum_{q_i, r = q_i} \prod_{j=1}^{i} a_{q_j q_{j+1}} A_{s,s-1} \kappa \cdot (\lambda + D)a_{s,s+1}$$

$$+ 2 \sum_{q_i, r = q_i} \prod_{j=1}^{i} a_{q_j q_{j+1}} \kappa \cdot (\lambda + D)a_{s,s+1}$$

$$+ \sum_{s+1 \leq r < n+1} \sum_{q_i, r = q_i} \prod_{j=1}^{i} a_{q_j q_{j+1}} A_{s,s+1} \kappa \cdot (\lambda + D)a_{s,s+1}.$$
\[= - \sum_{q_i+1=q_j} a_{q_i}^* \cdots a_{q_{s-1}}^* (a_{s,s+1}^*)^2 \kappa \cdot Da_{s+1,n+1}^* \]
\[\quad - \sum_{s+2 \leq r < n+1} \sum_{q_i=r}^{q_j} a_{q_i}^* \cdots a_{q_{s-1}}^* (a_{s,s+1}^*)^2 \kappa \cdot Da_{r,n+1}^* \]
\[- \delta_{s,n} \sum_{q_{s+1}=q_i} \prod_{q_j=q_i} (\lambda + D)(a_{s,n+1}^*)^2 \]
\[+ \sum_{s+2 \leq r < n+1} \sum_{k=s+2}^{r} \sum_{q_i=r}^{q_j} a_{q_i}^* \cdots a_{q_{s+1}}^* a_{k,k}^* \cdots a_{r-1,r}^* (a_{s,s+1}^*)^2 \kappa \cdot Da_{r,n+1}^* \]
\[\quad + (1 - \delta_{s,n}) \sum_{q_{s+1}=q_i} \prod_{q_j=q_i} (\lambda + D)(a_{s,n+1}^*) \]
\[- \sum_{s+2 \leq r < n+1} \sum_{k=s+2}^{r} \sum_{q_i=r}^{q_j} \left( \prod_{q_j=q_i} (a_{q_j}^*) \right) a_{s+1}^*(a_{s+1}^* - 1) \kappa \cdot Da_{s+1,n+1}^* \]
\[- \sum_{k=1}^{s-1} \sum_{q_i=q_j} (a_{q_i}^*)^2 \kappa \cdot Da_{s,n+1}^* \]
\[- \sum_{k=1}^{s-1} \sum_{q_i=q_j} (a_{q_i}^*) \kappa \cdot Da_{s+1,n+1}^* \]
\[- \sum_{s+2 \leq r < n+1} \sum_{k=s+2}^{r} \sum_{q_i=r}^{q_j} (a_{q_i}^*)^2 \kappa \cdot Da_{r,n+1}^* \]
\[+ (1 - \delta_{s,n}) \sum_{q_{s+1}=q_i} \prod_{q_j=q_i} (\lambda + D)(a_{s,n+1}^*) \]
\[- (1 - \delta_{s,n}) \sum_{q_i=q_j} \prod_{q_j=q_i} (\lambda + D)(a_{s,n+1}^*) - (1 - \delta_{s,n}) \sum_{q_i=q_j} \prod_{q_j=q_i} (a_{s,n+1}^*) \]
\[+ (1 - \delta_{s,n}) \sum_{q_i=q_j} \prod_{q_j=q_i} (\lambda + D)(a_{s,n+1}^*) + (1 - \delta_{s,n}) \sum_{q_i=q_j} \prod_{q_j=q_i} (a_{s,n+1}^*) \]
\[- \sum_{q_i=q_j} \prod_{q_j=q_i} (a_{q_i}^*) \kappa \cdot D(a_{s,n+1}^*) \]
\[+ 2 \sum_{q_i=q_j} \prod_{q_j=q_i} (\lambda + D)(a_{s,n+1}^*) + \sum_{s+1 \leq q_i} \prod_{q_j=q_i} (\lambda + D)(a_{s,n+1}^*) \]
We break the above into three summations; summing over $\kappa \cdot Da_{s+1,n+1}$, over $\kappa \cdot Da_{s,n+1}$, over $\kappa \cdot Da_{r,n+1}$, $r \geq s+2$ and over $\kappa \cdot Da_{s,s+1}$ (there is some overlap with $\kappa \cdot Da_{s,n+1}$ and $\kappa \cdot Da_{s,s+1}$ when $s = n$):

\[
- \sum_{q_i \leq q_1} a_{q_1,q_2}^* \cdots a_{q_{i-2},s}^* (a_{s,s+1}^*)^2 \kappa \cdot Da_{s+1,n+1}^* - \sum_{s+1 \leq q_i} \sum_{k=1}^{s-1} a_{q_i,q_{i+1}}^* \cdots a_{q_{i-1},s}^* a_{s,s+1}^* \kappa \cdot Da_{s+1,n+1}^* \\
+ \sum_{q_i = q_i} a_{q_i}^* \cdots a_{q_{i-1},s+1}^* a_{s,s+1}^* \kappa \cdot D(a_{s+1,n+1}^*) = 0.
\]

Next we have summands with $(\lambda + D)a_{s,n+1}^*$ in them:

\[
(1 - \delta_{s,n}) \sum \prod_{q_{s+1} = q_i}^{i-1} a_{q_i,q_{i+1}}^* \kappa \cdot (\lambda + D)a_{s,n+1}^* - \sum_{k=1}^{s-1} \sum_{q_i \leq q_j} a_{q_i,q_{j+1}}^* \cdots a_{q_{j-1},s}^* a_{s,s+1}^* \kappa \cdot Da_{s,n+1}^*
\]

\[
- (1 - \delta_{s,n}) \sum \prod_{q_i = q_i}^{i-1} a_{q_i,q_{i+1}}^* a_{s,s+1}^* \kappa \cdot D(a_{s,n+1}^*) - \sum_{q_i \leq q_i} \left( \prod_{l=1}^{i-1} a_{q_l,q_{l+1}}^* \right) a_{s,n+1}^* \kappa \cdot \lambda
\]

\[
= \sum a_{q_i}^* \cdots a_{q_{i-1},s+1}^* \kappa \cdot (\lambda + D)a_{s,n+1}^* - \sum_{k=1}^{s-1} \sum_{q_i \leq q_j} a_{q_i,q_{j+1}}^* \cdots a_{q_{j-1},s}^* a_{s,s+1}^* \kappa \cdot Da_{s,n+1}^*
\]

\[
- \sum_{q_i = q_i} a_{q_i}^* \cdots a_{q_{i-1},s}^* a_{s,s+1}^* \kappa \cdot D(a_{s,n+1}^*) - \sum_{q_i \leq q_i} a_{q_i}^* \cdots a_{q_{i-1},s+1}^* a_{s,s+1}^* \kappa \cdot \lambda
\]

\[
- \delta_{s,n} \left( \sum \prod_{q_i \leq q_j}^{i-1} a_{q_i,q_{j+1}}^* \kappa \cdot (\lambda + D)a_{s,n+1}^* - \sum_{q_i \leq q_i} \left( \prod_{l=1}^{i-1} a_{q_l,q_{l+1}}^* \right) a_{s,s+1}^* \kappa \cdot D(a_{s,n+1}^*) \right)
\]

\[
= -\delta_{s,n} \left( \sum \prod_{q_i \leq q_j}^{i-1} a_{q_i,q_{j+1}}^* \kappa \cdot (\lambda + D)a_{s,n+1}^* - \sum_{q_i = q_i} \left( \prod_{l=1}^{i-1} a_{q_l,q_{l+1}}^* \right) a_{s,n+1}^* \kappa \cdot D(a_{s,n+1}^*) \right).
\]
Moreover the summands with $(\lambda+D)a_{s,s+1}^*$ sum up to

\[
- \delta_{s,n} \sum_{q:s+1=q_i} \prod_{j=1}^{i-1} a_{q_j,q_{j+1}}^* \cdot (\lambda + D)(a_{n,n+1}^*)^2 - (1 - \delta_{s,n}) \sum_{q:s=q_i} a_{q_1,q_2}^* \cdots a_{q_{i-1},s}^* a_{s,s+1}^* \cdot (\lambda + D)(a_{s,s+1}^*) + (1 - \delta_{s,n}) \sum_{q:s+1=q_i} a_{q_1,q_2}^* \cdots a_{q_{i-1},s}^* a_{s,s+1}^* \cdot (\lambda + D)(a_{s,s+1}^*) - \sum_{q:s-1=q_i} a_{q_1,q_2}^* \cdots a_{q_{i-1},s}^* a_{s,s+1}^* \cdot (\lambda + D)(a_{s,s+1}^*) + 2 \sum_{q:s \geq q_i} a_{q_1,q_2}^* \cdots a_{q_{i-1},s+1}^* a_{s,s+1}^* \cdot (\lambda + D)(a_{s,s+1}^*) + (\delta_{s,n} - 1) \sum_{s+1 \geq q_i} a_{q_1,q_2}^* \cdots a_{q_{i-1},s+1}^* a_{s,s+1}^* \cdot (\lambda + D)(a_{s,s+1}^*) \\
= - \sum_{q:s=q_i} a_{q_1,q_2}^* \cdots a_{q_{i-1},s}^* a_{s,s+1}^* \cdot (\lambda + D)(a_{s,s+1}^*) + \sum_{q:s+1=q_i} a_{q_1,q_2}^* \cdots a_{q_{i-1},s+1}^* a_{s,s+1}^* \cdot (\lambda + D)(a_{s,s+1}^*) \\
- \sum_{q:s-1=q_i} a_{q_1,q_2}^* \cdots a_{q_{i-1},s}^* a_{s,s+1}^* \cdot (\lambda + D)(a_{s,s+1}^*) + \sum_{q:s \geq q_i} a_{q_1,q_2}^* \cdots a_{q_{i-1},s}^* a_{s,s+1}^* \cdot (\lambda + D)(a_{s,s+1}^*) \\
- \delta_{s,n} \sum_{q:s+1=q_i} \prod_{j=1}^{i-1} a_{q_j,q_{j+1}}^* \cdot (\lambda + D)(a_{n,n+1}^*)^2 + \delta_{s,n} \sum_{q:s=q_i} a_{q_1,q_2}^* \cdots a_{q_{i-1},s}^* a_{s,s+1}^* \cdot (\lambda + D)(a_{s,s+1}^*) \\
- \delta_{s,n} \sum_{q:s+1=q_i} a_{q_1,q_2}^* \cdots a_{q_{i-1},s}^* a_{s,s+1}^* a_{s+1}^* \cdot (\lambda + D)(a_{s+1}^*) + \delta_{s,n} \sum_{s+1 \geq q_i} a_{q_1,q_2}^* \cdots a_{q_{i-1},s}^* a_{s+1}^* \cdot (\lambda + D)(a_{s+1}^*) \\
= - \delta_{s,n} \sum_{q:n=q_i} \prod_{j=1}^{i-1} a_{q_j,q_{j+1}}^* a_{n,n+1}^* \cdot (\lambda + D)(a_{n,n+1}^*) - \delta_{s,n} \sum_{q:n=q_i} \prod_{j=1}^{i-1} a_{q_j,q_{j+1}}^* a_{n,n+1}^* \cdot D(a_{n,n+1}^*) + \delta_{s,n} \sum_{q:n=q_i} a_{q_1,q_2}^* \cdots a_{q_{i-1},n}^* a_{n,n+1}^* \cdot (\lambda + D)(a_{n,n+1}^*) \\
+ \delta_{s,n} \sum_{q:n=q_i} a_{q_1,q_2}^* \cdots a_{q_{i-1},n}^* a_{n,n+1}^* \cdot (\lambda + D)(a_{n,n+1}^*) \\
+ \delta_{s,n} \sum_{n+1 \geq q_i} a_{q_1,q_2}^* \cdots a_{q_{i-1},n+1}^* \cdot (\lambda + D)(a_{n,n+1}^*) \\
= - \delta_{s,n} \sum_{q:n=q_i} \prod_{j=1}^{i-1} a_{q_j,q_{j+1}}^* a_{n,n+1}^* \cdot D(a_{n,n+1}^*) + \delta_{s,n} \sum_{n+1 \geq q_i} a_{q_1,q_2}^* \cdots a_{q_{i-1},n+1}^* \cdot (\lambda + D)(a_{n,n+1}^*). \]

Hence the summands with $(\lambda + D)a_{s,s+1}^*$ and $(\lambda + D)a_{s,n+1}^*$ sum to zero.
Next we have for the summands with $\kappa \cdot Da^*_{r,n+1}$ in them contribute

\[
- \sum_{s+2 \leq r < n+1} \sum_{\substack{q,r = q_i, \\ r \neq (q_i, q_{i+1}) \implies (s,s+1)}} a_{1,q_2}^* \cdots a_{q_i-2,s}^* a_{s,s+1}^* \cdots a_{q_i-1,r}^* a_{s,s+1}^* \kappa \cdot Da^*_{r,n+1}
+ \sum_{s+2 \leq r < n+1} \sum_{s+1 = 1}^r \sum_{q,r = q_i, \\ r \neq (q_i, q_{i+1}) \implies (s,s+1)} a_{1,q_2}^* \cdots a_{q_i-1,r}^* a_{s,s+1}^* \cdots a_{q_i-1,r}^* a_{s,s+1}^* \kappa \cdot Da^*_{r,n+1}
+ \sum_{s+2 \leq r < n+1} \sum_{s+2 = 1}^r \sum_{q,r = q_i} a_{1,q_2}^* \cdots a_{q_i-1,r}^* a_{s,s+1}^* \cdots a_{q_i-1,r}^* a_{s,s+1}^* \kappa \cdot Da^*_{r,n+1}
- \sum_{s+2 \leq r < n+1} \sum_{s+1 = 1}^r \sum_{q,r = q_i} a_{1,q_2}^* \cdots a_{q_i-1,r}^* a_{s,s+1}^* \cdots a_{q_i-1,r}^* a_{s,s+1}^* \kappa \cdot Da^*_{r,n+1}
+ \sum_{s+2 \leq r < n+1} \sum_{s+1 = 1}^r a_{1,q_2}^* \cdots a_{s+1,k}^* a_{k,q_{i+1}}^* \cdots a_{s,s+1}^* \kappa \cdot Da^*_{r,n+1}
\]

\[
= - \sum_{s+2 \leq r < n+1} \sum_{\substack{q,r = q_i, \\ r \neq (q_i, q_{i+1}) \implies (s,s+1)}} a_{1,q_2}^* \cdots a_{q_i-2,s}^* a_{s,s+1}^* \cdots a_{q_i-1,r}^* a_{s,s+1}^* \kappa \cdot Da^*_{r,n+1}
+ \sum_{s+2 \leq r < n+1} \sum_{s+1 = 1}^r \sum_{q,r = q_i, \\ r \neq (q_i, q_{i+1}) \implies (s,s+1)} a_{1,q_2}^* \cdots a_{q_i-1,r}^* a_{s,s+1}^* \cdots a_{q_i-1,r}^* a_{s,s+1}^* \kappa \cdot Da^*_{r,n+1}
- \sum_{s+2 \leq r < n+1} \sum_{s+1 = 1}^r \sum_{q,r = q_i} a_{1,q_2}^* \cdots a_{q_i-1,r}^* a_{s,s+1}^* \cdots a_{q_i-1,r}^* a_{s,s+1}^* \kappa \cdot Da^*_{r,n+1}
+ \sum_{s+2 \leq r < n+1} a_{1,q_2}^* \cdots a_{s+1,k}^* a_{k,q_{i+1}}^* \cdots a_{s,s+1}^* \kappa \cdot Da^*_{r,n+1}
\]
\[
\begin{align*}
&= - \sum_{s+2 \leq r \leq n+1} \sum_{k=s+2}^{r} \sum_{q_{r+1} = q_{k}}^{\text{ex}} a_{1,q_{2}} \cdots a_{q_{r-1},-} a_{s,s+1}^* \cdots a_{q_{r-1},-}^* a_{s,s+1}^\ast - 1 \cdot Da_{r,n+1}^* \\
&+ \sum_{s+2 \leq r \leq n+1} \sum_{k=s+2}^{r} \sum_{q_{r+1} = q_{k}}^{\text{ex}} a_{1,q_{2}}^* \cdots a_{q_{r-1},-}^* a_{s,k,q_{r+2}}^* \cdots a_{q_{r-1},-}^* a_{s,k+1}^* Da_{n+1}^* \\
&- \sum_{s+2 \leq r \leq n+1} \sum_{k=s+2}^{r} \sum_{q_{r+1} = q_{k}}^{\text{ex}} a_{1,q_{2}}^* \cdots a_{q_{r-1},-}^* a_{s,q_{r+2}}^* \cdots a_{q_{r-1},-}^* a_{s,q_{r+2}+1}^* Da_{r,n+1}^* \\
&+ \sum_{s+2 \leq r \leq n+1} \sum_{k=s+2}^{r} \sum_{q_{r+1} = q_{k}}^{\text{ex}} a_{1,q_{2}}^* \cdots a_{s,q_{r+1}}^* a_{s,s+1}^* \cdot Da_{r,n+1}^* \\
&+ \sum_{s+2 \leq r \leq n+1} \sum_{k=s+2}^{r} \sum_{q_{r+1} = q_{k}}^{\text{ex}} a_{1,q_{2}}^* \cdots a_{s,s+1}^* a_{s,s+1}^* \cdots a_{q_{r-1},-}^* a_{s,s+1}^* Da_{r,n+1}^* \\
&- \sum_{s+2 \leq r \leq n+1} \sum_{k=s+2}^{r} \sum_{q_{r+1} = q_{k}}^{\text{ex}} a_{1,q_{2}}^* \cdots a_{q_{r-1},-}^* a_{s,q_{r+2}}^* \cdots a_{q_{r-1},-}^* a_{s,q_{r+2}+1}^* Da_{r,n+1}^* \\
&- \sum_{s+2 \leq r \leq n+1} \sum_{k=s+2}^{r} \sum_{q_{r+1} = q_{k}}^{\text{ex}} a_{1,q_{2}}^* \cdots a_{q_{r-1},-}^* a_{s,q_{r+2}}^* \cdots a_{q_{r-1},-}^* a_{s,q_{r+2}+1}^* Da_{r,n+1}^* \\
&+ \sum_{s+2 \leq r \leq n+1} \sum_{k=s+2}^{r} \sum_{q_{r+1} = q_{k}}^{\text{ex}} a_{1,q_{2}}^* \cdots a_{s,s+1}^* a_{s,s+1}^* \cdots a_{q_{r-1},-}^* a_{s,s+1}^* Da_{r,n+1}^* \\
&= 0.
\end{align*}
\]

Now for the terms without any $\kappa \cdot D$ or $b_l$’s in them.
First we pick off the summands with $a_{k,j}$ terms with $k \leq s - 1$:

\[
\sum_{k=1}^{s-1} a_{ks} \sum_{q:s+1=q_k:k \geq q_{k-1}} a_{1,q_2}^* \cdots a_{q_{i-1},s+1}^* a_{k,n+1}^* \\
- \sum_{k=1}^{s-1} \sum_{1 \leq r < j \leq n+1} a_{rj} \sum_{q:q_j=\min(q_k,q_{k-1}) \geq q_{i-1}} \sum_{s \leq r \leq s+1} a_{1,q_2}^* \cdots a_{q_{i-1},r}^* a_{k,s+1}^* a_{r,n+1}^* \\
= \sum_{k=1}^{s-1} a_{ks} \sum_{q:s+1=q_k:k \geq q_{k-1}} a_{1,q_2}^* \cdots a_{q_{i-1},s+1}^* a_{k,n+1}^* \\
- \sum_{k=1}^{s-1} \sum_{s \leq r < j \leq n+1} a_{rj} \sum_{q:q_j=\min(q_k,q_{k-1}) \geq q_{i-1}} \sum_{s \leq r \leq s+1} a_{1,q_2}^* \cdots a_{q_{i-1},r}^* a_{k,s+1}^* a_{r,n+1}^* \\
= -\sum_{k=1}^{s-1} \sum_{s \leq r < j \leq n+1} a_{rj} \sum_{q:q_j=\min(q_k,q_{k-1}) \geq q_{i-1}} \sum_{s \leq r \leq s+1} a_{1,q_2}^* \cdots a_{q_{i-1},r}^* a_{k,s+1}^* a_{r,n+1}^*
\]

This last summation splits up into three summations which contribute to later summations: The first summation has a factor of $a_{sj}$, the second a factor of $a_{s+1,j}$ and the last summation has factors of the form $a_{rj}$ with $s + 1 < r < j$. We deal with these consecutively in the following:
We consider the summands with a factor of $a_{s,j}$:

$$- \delta_{s,n} a_{n+1} \left( a_{n+1} \right)^2 \sum_{q} a_{1,q_2} \cdots a_{q_{n-1},n+1}^*$$

$$- \sum_{s < j \leq n+1} a_{s,j} \sum_{q_1 \geq j} a_{1,q_2} \cdots a_{q_{n-1},s} a_{s,n+1}^* \cdots a_{q_{n-1},j} a_{s,n+1}^* a_{s,n+1}^*$$

$$+ 2a_{s,s+1} a_{s,n+1}^* \sum_{q:s+1=q_i} a_{1,q_2} \cdots a_{q_{n-1},s}^* a_{s,n+1}^*$$

$$- \sum_{k=1}^{n+1} a_{s,j} \sum_{q_1, j \geq q_{n-1}} a_{1,q_2} \cdots a_{q_{n-1},k} a_{s,n+1}^* \cdots a_{q_{n-1},j} a_{k,s,n+1}^* a_{s,n+1}^*$$

$$+ \sum_{j=s+2}^{n+1} a_{s,j} a_{s,n+1}^* \sum_{q_1, j \geq q_{n-1}} a_{1,q_2} \cdots a_{q_{n-1},j} a_{s,n+1}^*$$

$$+ \sum_{k=s+2}^{n+1} a_{s,k} a_{s,n+1}^* \sum_{q_1 \geq j} a_{1,q_2} \cdots a_{q_{n-1},s} a_{s,n+1}^*$$

$$- \sum_{k=s+2}^{n+1} a_{s,j} \sum_{s < j \leq n+1} a_{1,q_2} \cdots a_{q_{n-1},s} a_{s,n+1}^* \cdots a_{q_{n-1},j} a_{s,n+1}^* a_{s,n+1}^*$$

$$= - a_{s,s+1} \sum_{q:s+1=q_i} a_{1,q_2} \cdots a_{q_{n-2},s} a_{s,n+1}^* a_{s,n+1}^*$$

$$+ a_{s,s+1} a_{s,n+1}^* \sum_{q:s+1=q_i} a_{1,q_2} \cdots a_{q_{n-1},s} a_{s,n+1}^*$$

$$- \sum_{k=1}^{n+1} a_{s,j} \sum_{q_1 \geq q_{n-1}} a_{1,q_2} \cdots a_{q_{n-1},k} a_{s,n+1}^* a_{s,n+1}^*$$

$$+ \sum_{k=s+2}^{n+1} a_{s,k} a_{s,n+1}^* \sum_{q_1 \geq j} a_{1,q_2} \cdots a_{q_{n-1},s} a_{s,n+1}^*$$

$$- \sum_{k=s+2}^{n+1} a_{s,j} \sum_{s < j \leq n+1} a_{1,q_2} \cdots a_{q_{n-1},s} a_{s,n+1}^* \cdots a_{q_{n-1},j} a_{s,n+1}^* a_{s,n+1}^*$$

$$= a_{s,s+1} a_{s,n+1}^* \sum_{q_1 \leq s-1} a_{1,q_2} \cdots a_{q_{n-1},s} a_{s,n+1}^* - \sum_{k=1}^{n+1} a_{s,j} \sum_{q_1 \geq q_{n-1}} a_{1,q_2} \cdots a_{q_{n-1},k} a_{s,j} a_{k,s,n+1}^* a_{s,n+1}^*$$

$$+ \sum_{k=s+2}^{n+1} a_{s,k} a_{s,n+1}^* \sum_{q_1 \geq j} a_{1,q_2} \cdots a_{q_{n-1},s} a_{s,n+1}^* - \sum_{k=s+2}^{n+1} a_{s,k} a_{s,n+1}^* \sum_{q_1 \geq j} a_{1,q_2} \cdots a_{q_{n-1},s} a_{s,n+1}^* a_{s,n+1}^*$$
Next we calculate the terms with $a_{s+1,j}$ in them:

\[
- \sum_{s+1<j \leq n+1} a_{s+1,j} \sum_{q_j=q_i} a_1^{q_2} \cdots a_{q_{i-1},j} a_{q_{i-2},s} a_{s+1} a_{s+1,n+1} + \sum_{k=s+2}^{n+1} a_{s+1,k} \sum_{q_k=q_i} a_1^{q_2} \cdots a_{q_{i-1},k} a_{s+1,n+1} \\
- \sum_{k=s+2}^{n+1} a_{s+1,k} \sum_{q_k=q_i} a_1^{q_2} \cdots a_{q_{i-1},k} a_{s+1,n+1} \\
+ \sum_{s+1<j \leq n+1} a_{s+1,j} \sum_{q_j=q_i} a_1^{q_2} \cdots a_{q_{i-1},j} a_{s+1,n+1} \\
- \sum_{k=s+2}^{n+1} a_{s+1,k} \sum_{q_k=q_i} a_1^{q_2} \cdots a_{q_{i-1},k} a_{s+1,n+1} \\
+ \sum_{s+1<j \leq n+1} a_{s+1,j} \sum_{q_j=q_i} a_1^{q_2} \cdots a_{q_{i-1},j} a_{s+1,n+1} \\
- \sum_{k=s+2}^{n+1} a_{s+1,k} \sum_{q_k=q_i} a_1^{q_2} \cdots a_{q_{i-1},k} a_{s+1,n+1} \\
- \sum_{s+1<j \leq n+1} a_{s+1,j} \sum_{q_j=q_i} a_1^{q_2} \cdots a_{q_{i-1},j} a_{s+1,n+1} \\
\]

Next we calculate the terms with $a_{s+1,j}$ in them:

\[
\begin{align*}
&= - \sum_{j=s+2}^{n+1} a_{s+1,j} \sum_{q_j=q_i} a_1^{q_2} \cdots a_{q_{i-1},j} a_{s+1} a_{s+1,n+1} + \sum_{k=s+2}^{n+1} a_{s+1,k} \sum_{q_k=q_i} a_1^{q_2} \cdots a_{q_{i-1},k} a_{s+1,n+1} \\
&= 0.
\end{align*}
\]
\[
\begin{align*}
&= \sum_{k=s+2}^{n+1} \sum_{l=1}^{s-1} a_{s+1,k} \sum_{q:k=q_1:s+1 \geq q_1-1} \sum_{q_1 \cdots q_{t-1}, s+1, k} a_{s+1,k} a_{s,s+1,k} a_{s+1,n+1} \\
&+ \sum_{k=s+2}^{n+1} a_{s+1,k} a_{s,k} \sum_{q:k=q_1:s+1 = q_1-1, \exists: (q_1, q_{t-1}) \cdots (s+1, k)} a_{s+1,k} a_{s+1,n+1} \\
&- \sum_{k=s+2}^{n+1} a_{s+1,k} \sum_{q:k=q_1:s+1 = q_1-1} a_{s+1,k} a_{s,s+1,k} a_{s+1,n+1} \\
&- \sum_{k=s+2}^{n+1} \sum_{j=1}^{s-1} a_{s+1,k} \sum_{q:k=q_1:s+1 \geq q_1-1, \exists: (q_1, q_{t-1}) = (j, s)} a_{s+1,k} a_{s,s+1,n+1} \\
&+ \sum_{k=s+2}^{n+1} a_{s+1,k} \sum_{q:k=q_1:s+1 \geq q_1-1} a_{s+1,k} a_{s+1,n+1} a_{s+1,s+1,n+1} \\
&= \sum_{k=s+2}^{n+1} \sum_{l=1}^{s-1} a_{s+1,k} \sum_{q:k=q_1:s+1 \geq q_1-1} \sum_{q_1 \cdots q_{t-1}, s+1, k} a_{s+1,k} a_{s,s+1,k} a_{s+1,n+1} \\
&+ \sum_{k=s+2}^{n+1} a_{s+1,k} a_{s,k} \sum_{q:k=q_1:s+1 = q_1-1, \exists: (q_1, q_{t-1}) \cdots (s+1, k)} a_{s+1,k} a_{s+1,n+1} \\
&- \sum_{k=s+2}^{n+1} a_{s+1,k} \sum_{q:k=q_1:s+1 = q_1-1} a_{s+1,k} a_{s,s+1,k} a_{s+1,n+1} \\
&- \sum_{k=s+2}^{n+1} \sum_{j=1}^{s-1} a_{s+1,k} \sum_{q:k=q_1:s+1 \geq q_1-1, \exists: (q_1, q_{t-1}) = (j, s)} a_{s+1,k} a_{s,s+1,n+1} \\
&+ \sum_{k=s+2}^{n+1} a_{s+1,k} \sum_{q:k=q_1:s+1 \geq q_1-1} a_{s+1,k} a_{s+1,n+1} a_{s+1,s+1,n+1}
\end{align*}
\]
\[
\begin{align*}
&= \sum_{k=s+2}^{n+1} \sum_{l=1}^{s-1} a_{s+1,k} \sum_{q:k=q,s+1 \geq q_i-1} a_{1,q_2}^* \cdots a_{l,s+1,k}^* a_{s,s+1}^* a_{s+1,n+1}^* \\
&+ \sum_{k=s+2}^{n+1} a_{s+1,k} a_{s,k}^* \sum_{q:k=q,s+1=q_i-1, \exists t: (q_t,q_{t+1})=(s+1,k)} a_{1,q_2}^* \cdots a_{q_{t-1},s+1}^* a_{s+1,n+1}^* \\
&- \sum_{k=s+2}^{n+1} \sum_{j=1}^{s-1} a_{s+1,k} a_{s,k}^* a_{1,q_2}^* \cdots a_{q_{t-1},s+1}^* a_{q_{t-1},s+1}^* a_{s,s+1}^* a_{s+1,n+1}^* \\
&- \sum_{k=s+2}^{n+1} a_{s+1,k} a_{s,k}^* \sum_{q:k=q,s+1=q_i-1, \exists t: (q_t,q_{t+1})=(s,s)} a_{1,q_2}^* \cdots a_{q_{t-1},s+1}^* a_{s,s+1}^* a_{s+1,n+1}^* \\
&= \sum_{k=s+2}^{n+1} \sum_{l=1}^{s-1} a_{s+1,k} \sum_{q:k=q,s+1 \geq q_i-1} a_{1,q_2}^* \cdots a_{l,s+1,k}^* a_{s,s+1}^* a_{s+1,n+1}^* \\
&+ \sum_{k=s+2}^{n+1} \sum_{l=1}^{s-1} a_{s+1,k} a_{s,k}^* \sum_{q:k=q,s+1=q_i-1, \exists t: (q_t,q_{t+1})=(s+1,k)} a_{1,q_2}^* \cdots a_{l,s+1}^* a_{s+1,n+1}^* \\
&- \sum_{k=s+2}^{n+1} \sum_{j=1}^{s-1} a_{s+1,k} a_{s,k}^* a_{1,q_2}^* \cdots a_{q_{t-1},s+1}^* a_{s,s+1}^* a_{s+1,n+1}^* \\
&- \sum_{k=s+2}^{n+1} a_{s+1,k} \sum_{q:k=q,s+1=q_i-1} a_{1,q_2}^* \cdots a_{q_{t-1},s+1}^* a_{s,s+1}^* a_{s+1,n+1}^* \\
&= 0.
\end{align*}
\]
Finally we calculate the last summations with \( a_{r,j} r \geq s + 2 \):

\[
- \sum_{s+2 \leq r < j \leq n+1} a_{r,j} \sum_{q_1, q_2 \geq q_1 - 1} a_{q_1, q_2} \cdots a_{q_{r-1}, s}^{s} a_{s, s+1}^{s} \cdots a_{q_{r-1}, j}^{s} a_{r, s+1}^{s} a_{r, n+1}^{s} \\
+ \sum_{k=s+2}^{n+1} \sum_{s+2 \leq r < j \leq n+1} a_{r,j} \sum_{q_1, q_2 \geq q_1 - 1} a_{q_1, q_2} \cdots a_{q_{r-1}, s}^{s} a_{s+1}^{s} a_{k, q_{r+2}}^{s} \cdots a_{q_{r-1}, j}^{s} a_{r, s+1}^{s} a_{r, n+1}^{s} \\
- \sum_{k=s+2}^{n+1} \sum_{s+2 \leq r < j \leq n+1} a_{r,j} \sum_{q_1, q_2 \geq q_1 - 1} a_{q_1, q_2} \cdots a_{q_{r-1}, s}^{s} a_{s, k}^{s} \cdots a_{q_{r-1}, j}^{s} a_{r, s+1}^{s} a_{r, n+1}^{s} \\
+ \sum_{k=s+2}^{n+1} \sum_{s+2 \leq r < j \leq n+1} a_{r,j} \sum_{q_1, q_2 \geq q_1 - 1} a_{q_1, q_2} \cdots a_{q_{r-1}, s}^{s} a_{s+1}^{s} a_{k, q_{r+2}}^{s} \cdots a_{q_{r-1}, j}^{s} a_{r, s+1}^{s} a_{r, n+1}^{s} \\
= \sum_{k=s+2}^{n+1} \sum_{s+2 \leq r < j \leq n+1} a_{r,j} \sum_{q_1, q_2 \geq q_1 - 1} a_{q_1, q_2} \cdots a_{q_{r-1}, s}^{s} a_{s+1}^{s} a_{k, q_{r+2}}^{s} \cdots a_{q_{r-1}, j}^{s} a_{r, s+1}^{s} a_{r, n+1}^{s} \\
- \sum_{k=s+2}^{n+1} \sum_{s+2 \leq r < j \leq n+1} a_{r,j} \sum_{q_1, q_2 \geq q_1 - 1} a_{q_1, q_2} \cdots a_{q_{r-1}, s}^{s} a_{s, k}^{s} \cdots a_{q_{r-1}, j}^{s} a_{r, s+1}^{s} a_{r, n+1}^{s} \\
- \sum_{s+2 \leq r < j \leq n+1} a_{r,j} \sum_{q_1, q_2 \geq q_1 - 1} a_{q_1, q_2} \cdots a_{q_{r-1}, s}^{s} a_{s, s+1}^{s} \cdots a_{q_{r-1}, j}^{s} a_{r, s+1}^{s} a_{r, n+1}^{s} \\
+ \sum_{s+2 \leq r < j \leq n+1} a_{r,j} \sum_{q_1, q_2 \geq q_1 - 1} a_{q_1, q_2} \cdots a_{q_{r-1}, s}^{s} a_{s+1}^{s} a_{k, q_{r+2}}^{s} \cdots a_{q_{r-1}, j}^{s} a_{r, s+1}^{s} a_{r, n+1}^{s} \\
+ \sum_{s+2 \leq r < j \leq n+1} a_{r,j} \sum_{q_1, q_2 \geq q_1 - 1} a_{q_1, q_2} \cdots a_{q_{r-1}, s}^{s} a_{s+1}^{s} a_{k, q_{r+2}}^{s} \cdots a_{q_{r-1}, j}^{s} a_{r, s+1}^{s} a_{r, n+1}^{s} 
\]
\[
= \sum_{k=s+2}^{n+1} \sum_{s+2 \leq r < j \leq n+1} a_{rj} \sum_{q_j = q_l}^{q_j = q_{l-1}} \sum_{r \geq q_{l-1} \wedge (q_l, q_{l+1}) = (+1, k)} a_{q_1, q_2} \cdots a_{q_{l-1}, j} a_{s, k} a_{r, n+1}^r \\
- \sum_{k=1}^{s-1} \sum_{s+2 \leq r < j \leq n+1} a_{rj} \sum_{q_j = q_l \wedge (q, q_{l+1}) = (k, s)} a_{q_1, q_2} \cdots a_{q_{l-1}, j} a_{s, k} a_{k, s} a_{r, n+1}^s \\
+ \sum_{s+2 \leq r < j \leq n+1} a_{rj} \sum_{q_j = q_l \wedge (q, q_{l+1}) = (k, s)} a_{q_1, q_2} \cdots a_{q_{l-1}, j} a_{s, s} a_{s, s} a_{r, n+1}^s \\
= \sum_{k=s+2}^{n+1} \sum_{s+2 \leq r < j \leq n+1} a_{rj} \sum_{q_j = q_l}^{q_j = q_{l-1}} \sum_{r \geq q_{l-1} \wedge (q_l, q_{l+1}) = (+1, k)} a_{q_1, q_2} \cdots a_{q_{l-1}, j} a_{s, k} a_{r, n+1}^r \\
- \sum_{k=1}^{s-1} \sum_{s+2 \leq r < j \leq n+1} a_{rj} \sum_{q_j = q_l \wedge (q, q_{l+1}) = (k, s)} a_{q_1, q_2} \cdots a_{q_{l-1}, j} a_{s, l} a_{s, s} a_{r, n+1}^s \\
- \sum_{k=1}^{s-1} \sum_{s+2 \leq r < j \leq n+1} a_{rj} \sum_{q_j = q_l \wedge (q, q_{l+1}) = (k, s)} a_{q_1, q_2} \cdots a_{q_{l-1}, j} a_{s, s} a_{s, s} a_{r, n+1}^s \\
- \sum_{s+2 \leq r < j \leq n+1} a_{rj} \sum_{q_j = q_l \wedge (q, q_{l+1}) = (k, s)} a_{q_1, q_2} \cdots a_{q_{l-1}, j} a_{s, s} a_{s, s} a_{r, n+1}^s \\
+ \sum_{s+2 \leq r < j \leq n+1} a_{rj} \sum_{q_j = q_l \wedge (q, q_{l+1}) = (k, s)} a_{q_1, q_2} \cdots a_{q_{l-1}, j} a_{s, s} a_{s, s} a_{r, n+1}^s 
\]
We are now left with the Serre relations:

**Lemma 6.5** (R5/R6).

\[
[r(F_r), r(F_s)] = [r(E_{r}), r(E_s)] = 0 \quad \text{if} \quad A_{rs} \neq -1 \\
[r(F_r), [r(F_r), r(F_s)]] = [r(E_{r}), [r(E_r), r(E_s)]] = 0 \quad \text{if} \quad A_{rs} = -1
\]
Proof. As in the previous lemmas we first assume \( rs \neq 0 \). In this case the proof is exactly the same as in [Cox02, Lemma 3.5] with the exception of a sign change in the formulation of the \( \rho(F_r) \). Let us check the relations for \( \rho(F_r) \). Now suppose \( r = 0 \), so that for \( s \neq 0 \) we get

\[
(6.2) \quad \rho(F_0) = \rho(F_s)
\]

\[
= \left[ \left( - \sum_{1 \leq r < j \leq n+1} a_{rj} \sum_{q_j = q_1, r \geq q_{i-1}}^{i-1} \prod_{l=1} \rho_{q_0, q_{i+1}} a_{r,n+1}^* - \sum_{1 \leq r < n+1} \sum_{q_r \geq q_i} a_{q,r+1}^* \Phi b_r \right) \right]
\]

\[
= \left[ \left( \sum_{1 \leq r < n+1} a_{rj} \sum_{q_j = q_1, r \geq q_{i-1}}^{i-1} \prod_{l=1} \rho_{q_0, q_{i+1}} a_{r,n+1}^* \right) \lambda(a_{s,s+1}) \right]
\]

\[
- \left[ \left( \sum_{1 \leq r < n+1} \sum_{q_r \geq q_i} a_{q,r+1}^* \Phi b_r \right) \lambda(a_{s,s+1}) \right]
\]

\[
- \left( \sum_{1 \leq r < n+1} \sum_{q_r \geq q_i} \prod_{j=1} a_{q,j+r+1}^* \cdot Da_{r,n+1}^* \right) \lambda(a_{s,s+1}) \right]
\]

\[
= \left[ \left( - \sum_{1 \leq r < j \leq n+1} a_{rj} \sum_{q_j = q_1, r \geq q_{i-1}}^{i-1} \prod_{l=1} \rho_{q_0, q_{i+1}} a_{r,n+1}^* \right) \lambda(a_{s,s+1}) \right]
\]

\[
- \left[ \left( \sum_{1 \leq r < n+1} \sum_{q_r \geq q_i} a_{q,r+1}^* \Phi b_r \right) \lambda(a_{s,s+1}) \right]
\]

\[
- \left( \sum_{1 \leq r < n+1} \sum_{q_r \geq q_i} \prod_{j=1} a_{q,j+r+1}^* \cdot Da_{r,n+1}^* \right) \lambda(a_{s,s+1}) \right]
\]

(In the above we use the notation \( q = (q_1, q_2, \ldots, q_i) \) where we sum over \( 1 = q_1 < \cdots < q_i < q_{i+1} = n + 1 \). We also some times let \( l(q) \) denote the second to last index in the multi-indexed element \( q \) if its last index is \( n + 1 \), otherwise we let \( l(q) \) denote the last index. For example if \( n = 5 \) and \( q = (1, 3, 5) \) then \( l(q) = 2 \) and if \( q = (1, 2, 4) \), then \( l(q) = 3 \). Now we break this up into the following calculations simplifying the six summations above: The first summmand becomes

\[
- \left[ \sum_{1 \leq r < j \leq n+1} a_{rj} \sum_{q_j = q_1, r \geq q_{i-1}}^{i-1} \prod_{l=1} a_{q_0, q_{i+1}}^{*} \right] a_{r,n+1} \lambda(a_{s,s+1})
\]

\[
= \delta_{s,n} a_{m,n+1} \sum_{q_{n+1} = q_i}^{i-1} \prod_{k=1} a_{q_k, q_{k+1}}^{*} \lambda(s_{s,+1})
\]

\[
= \delta_{s,n} a_{m,n+1} \sum_{q_{n+1} = q_i}^{i-1} \prod_{k=1} a_{q_k, q_{k+1}}^{*} + \sum_{1 \leq r < j \leq n+1} a_{rj} \sum_{q_j = q_1, r \geq q_{i-1}}^{i-1} \prod_{l=1} a_{q_0, q_{i+1}}^{*} \lambda(s_{s,+1})
\]

\[
= \delta_{s,n} a_{m,n+1} \prod_{k=1} a_{q_k, q_{k+1}}^{*} + \sum_{1 \leq r < j \leq n+1} a_{rj} \sum_{q_j = q_1, r \geq q_{i-1}}^{i-1} \prod_{l=1} a_{q_0, q_{i+1}}^{*} \lambda(s_{s,+1})
\]
The second and third summands in (6.2) simplify to

$$
\sum_{r \leq n} \left[ \left( \sum_{q_i \leq r} \prod_{j=1}^{i} a_{q_i,q_{j+1}} \Phi b_r \right) \lambda(a_{s,s+1}) \right] = - \sum_{r \leq n} \sum_{q_i \geq r} \prod_{j=1}^{i} a_{q_i,q_{j+1}} \Phi b_r
$$

and

$$
\sum_{1 \leq r \leq n} \left[ \left( \sum_{q_i=r}^{i-1} \prod_{j=1}^{i-1} a_{q_i,q_{j+1}} \kappa \cdot Da_{r,n+1}^* \right) \lambda(a_{s,s+1}) \right] = -(1 - \delta_{s,n}) \sum_{1 \leq r \leq n} \sum_{q_i \geq r} \prod_{j=1}^{i-1} a_{q_i,q_{j+1}} \kappa \cdot Da_{r,n+1}^*
$$

$$
+ \delta_{s,n} \sum_{q_i=n} \kappa \cdot (D + \lambda) \prod_{j=1}^{i-1} a_{q_i,q_{j+1}}.
$$

The last three summands in (6.2) are

$$
\sum_{1 \leq r < n+1} \left[ \left( \sum_{q_i=r}^{i-1} \prod_{j=1}^{i-1} a_{q_i,q_{j+1}} a_{r,n+1} \right) \lambda \left( \sum_{l=1}^{s-1} a_l a_{l+1} a_{s+1} \right) \right] = \sum_{l=1}^{s-1} \sum_{1 \leq r < j \leq n+1} a_{r,j} \sum_{q_i=q_j r \geq q_{j-1}} \prod_{l=1}^{i-1} a_{q_i q_{l+1}} \lambda a_{l,s+1} a_{l+1} a_{s+1}
$$

$$
+ \sum_{l=1}^{s-1} \sum_{1 \leq r < j \leq n+1} a_{r,j} \sum_{q_i=q_j r \geq q_{j-1}} \prod_{l=1}^{i-1} a_{q_i q_{l+1}} a_{r,n+1} a_{s+1}
$$

$$
+ \sum_{l=1}^{s-1} \sum_{1 \leq r < j \leq n+1} a_{r,j} \sum_{q_i=q_j r \geq q_{j-1}} \prod_{l=1}^{i-1} a_{q_i q_{l+1}} a_{l,s+1} a_{r,n+1}
$$

$$
= -\delta_{s,n} \sum_{l=1}^{s-1} a_{l,s+1} a_{l+1} a_{s+1} + \sum_{l=1}^{s-1} \sum_{a_{1,l} \geq q_i \geq a_{1,l+1}} \prod_{l=1}^{i-1} a_{q_i q_{l+1}} a_{1,n+1}
$$

$$
- \sum_{l=1}^{s-1} \sum_{a_{r,j} \geq q_i \geq a_{r,j-1}} \sum_{a_{1,l} \geq q_i \geq a_{1,l+1}} \prod_{l=1}^{i-1} a_{q_i q_{l+1}} a_{r,n+1} a_{1,n+1}
$$

$$
\sum_{1 \leq r < n+1} \sum_{r \geq q_i} \sum_{i=1}^{i-1} \left[ \prod_{j=1}^{i} a_{q_{j+1}} \Phi b_r \right] \lambda a_{l,s+1} a_{l+1} a_{s+1} = \sum_{i=1}^{i-1} \sum_{1 \leq r < n+1} \sum_{r \geq q_i} \sum_{a_{r,j} \geq q_i \geq a_{r,j-1}} \prod_{l=1}^{i} a_{q_{l+1}} \Phi b_r a_{r,n+1} a_{l,s+1} a_{l+1} a_{s+1}
$$
\[
\sum_{1 \leq r < n+1} \left( \sum_{q; q_i = r} \left( \prod_{j=1}^{i-1} a_{q_j, q_{j+1}}^* \kappa \cdot Da_{r,n+1}^* \right) \lambda \left( \sum_{l=1}^{s-1} a_{l,s+1}^* a_{l,n}^* \right) \right)
\]

\[
= -(1 - \delta_{s,n}) \sum_{1 \leq r < n+1} \sum_{l=1}^{s-1} \sum_{q; q_i = r, \exists a, (q_k, q_{k+1}) = (l, s+1)} \left( \prod_{k=1, k \neq a}^{i-1} a_{q_k, q_{k+1}}^* \right) a_{l,s}^* \kappa \cdot Da_{r,n+1}^*
\]

\[
+ \delta_{s,n} \sum_{l=1}^{n-1} \sum_{q; q_i = l} \kappa \cdot (D + \lambda) \left( \prod_{j=1}^{i-1} a_{q_j, q_{j+1}}^* \right) a_{l,n}^* .
\]
Putting it all together we get

\[(6.3)\]

\[
[\rho(F_0)\rho(F_s)] = \delta_{s,n} a_{n,n+1} \sum_{s+1 \geq s_i} \prod_{k=1}^{i-1} a^*_{q_k q_{k+1}} \\
+ \sum_{s \leq r < j \leq n+1} a_{rj} \sum_{q: j=q; 3 \leq k \leq i; (q_k q_{k+1}) = (s, s+1)} \prod_{j=1, j \neq k}^{i-1} a^*_{q_j q_{j+1}} a^*_{r, n+1} \\
+ \sum_{r \geq s} \sum_{q: j=q; j \geq s} \prod_{j=1, j \neq k}^{i} a^*_{q_j q_{j+1}} \Phi b_r \\
+ (1 - \delta_{s,n}) \sum_{1 \leq r \leq n; 3 \leq k \leq i; (q_k q_{k+1}) = (s, s+1)} \prod_{j=1, j \neq k}^{i-1} a^*_{q_j q_{j+1}} a^*_{k, n+1} \\
- \delta_{s,n} \sum_{q: q_i = n} \kappa \cdot (D + \lambda) \prod_{j=1}^{i-1} a^*_{q_j q_{j+1}} \\
- \delta_{s,n} \sum_{l=1}^{s-1} \sum_{l < j \leq n+1} a_{lj} \sum_{q: j=q; j \geq s-1} \prod_{l=1}^{i-1} a^*_{q_l q_{l+1}} a^*_{l, n+1} \\
+ \sum_{l=1}^{s-1} a_{l, n+1} \sum_{q: q_i = r} \prod_{l=1}^{i-1} a^*_{q_l q_{l+1}} a^*_{l, n+1} \\
- \sum_{l=1}^{s-1} \sum_{r < q} a_{rj} \sum_{q: j=q; j \geq s-1} \prod_{l=1}^{i-1} a^*_{q_l q_{l+1}} a^*_{l, n+1} \\
- \sum_{l=1}^{s-1} r \sum_{r \geq q, q \geq q + 1} \prod_{l=1}^{i-1} a^*_{q_l q_{l+1}} a^*_{l, n+1} \\
- (1 - \delta_{s,n}) \sum_{1 \leq r \leq n+1, l=1} a_{q_l q_{l+1}} \prod_{j=1, j \neq k}^{i-1} a^*_{q_j q_{j+1}} a^*_{l, n+1} \\
+ \delta_{s,n} \sum_{l=1}^{n-1} \sum_{q: q_i = l} \kappa \cdot (D + \lambda) \prod_{j=1}^{i-1} a^*_{q_j q_{j+1}} a^*_{l, n+1}
\]

Case I: \(1 < s < n\): If \(s > 1\), then the third summation with \(\Phi(b_r)\) in it, sums over all partitions \(q\) where there exists some \(1 \leq k \leq i\) and \(q_k \leq r\), such that

\[(1, q_2, \ldots, q_k, q_{k+1}, \ldots, q_i, n + 1) = (1, q_2, \ldots, s, s+1, \ldots, q_i, n+1)\]

where the \(s\) and \(s+1\) are in the 4th, respectively \(k+1\)-st entry. Note \(k > 1\) as otherwise \(s = 1\). Thus if \(s \leq n-1\), then products appearing in this third summation look like

\[
\prod_{j=1, j \neq k}^{i} a^*_{q_j q_{j+1}} \Phi(b_r) = a^*_{q_1 q_2} \cdots a^*_{q_{k-1}, q_k} a^*_{q_k, q_{k+1}} \cdots a^*_{q_{i-2}, q_{i-1}} a^*_{q_{i-1}, q_i} a^*_{q_i, n+1} \Phi(b_r)
\]
In the ninth summation with $\Phi(b_r)$ in it, the sum is over all partitions $q$ where there exists some $1 \leq a \leq i - 1$ and $q_i \leq r$ such that

$$(1, q_2, \ldots, q_s, q_{a+1}, \ldots, q_{i-1}, q_i, n+1) = (1, q_2, \ldots, l, s+1, \ldots, q_i, n+1)$$

where the $l$ and $s+1$ are in the $a$th, respectively $a+1$st entry. Note $1 < a \leq n$ as otherwise $s = 1$. Thus if $s \leq n - 1$, products appearing in this ninth summation look like

$$i \prod_{k=1, k \neq a}^{i-1} a^*_{q_k q_{k+1}} a_i^s \Phi(b_r) = a^*_i q_2 \cdots a^*_{q_{i-1}, l} a^*_{a+1, q_k+2} \cdots a^*_{q_{i-1}, q_i} a^*_{q_i, n+1} a_i^s \Phi(b_r)$$

where $l \leq s - 1$ and $r \geq q_i \geq l$. Thus the third and ninth summations are equal but of opposite sign and they cancel when $s \leq n - 1$.

If $1 < s < n$, then the fourth summation with $\kappa \cdot Da^*_{q_i, n+1}$ in it, sums over all partitions $q$ where there exists some $1 \leq k \leq i - 1$ such that

$$(1, q_2, \ldots, q_k, q_{k+1}, \ldots, q_{i-1}, q_i, n+1) = (1, q_2, \ldots, s, s+1, \ldots, q_{i-1}, r, n+1)$$

where the $s$ and $s+1$ are in the $k$th, respectively $k+1$st entry. Note $k > 1$ as otherwise $s = 1$. The products appearing in this fourth summation look like

$$i-1 \prod_{j=1, j \neq k}^{i-1} a^*_{q_j q_{j+1}} \kappa \cdot Da^*_{q_i, n+1} = a^*_i q_2 \cdots a^*_{q_{i-1}, s} a^*_{a+1, q_k+2} \cdots a^*_{q_{i-1}, q_i} a^*_{q_i, n+1} \kappa \cdot Da^*_{r, n+1}$$

In the tenth summation with $\kappa \cdot Da^*_{q_i, n+1}$ in it, the sum is over all partitions $q$ where there exists some $1 \leq a \leq i - 1$ such that

$$(1, q_2, \ldots, q_s, q_{a+1}, \ldots, q_{i-1}, q_i, n+1) = (1, q_2, \ldots, l, s+1, \ldots, q_{i-1}, r, n+1)$$

where the $l$ and $s+1$ are in the $a$th, respectively $a+1$st entry. Note $a > 1$ as otherwise $s = 1$. The products appearing in this tenth summation look like (after a change of indices $k \mapsto j$ and $a \mapsto k$)

$$i \prod_{j=1, j \neq k}^{i-1} a^*_{q_j q_{j+1}} a_i^s \kappa \cdot Da^*_{q_i, n+1} = a^*_i q_2 \cdots a^*_{q_{i-1}, l} a^*_{a+1, q_k+2} \cdots a^*_{q_{i-1}, q_i} a_i^s \kappa \cdot Da^*_{r, n+1}$$

where $l \leq s - 1$ and $q_i = r < n + 1$. Thus the forth and the tenth summations are equal but of opposite sign and they cancel. Moreover the fifth and the last summations are zero as $s < n$.

Case II: $s = n$:

If $s = n$, then the indices in the third summation are $n = s = q_k$, $k = i$, $s+1 = q_{i+1} = n+1$ and $r = q_i = n$. Then in this case the third summation consists of products of the form

$$i-1 \prod_{j=1}^{l-1} a^*_{q_j q_{j+1}} \Phi(b_n) = a^*_i q_2 \cdots a^*_{q_{i-1}, n} \Phi(b_n)$$

as

$$(1, q_2, \ldots, q_k, q_{k+1}) = (1, q_2, \ldots, q_{i-1}, n, n+1),$$

whereas the ninth summation consists of products of the form

$$i \prod_{k=1, k \neq i-1}^{i} a^*_{q_k q_{k+1}} a_i^s \Phi(b_r) = a^*_i q_2 \cdots a^*_{q_{i-1}, l} a_i^s \Phi(b_r)$$
where \( l \leq n - 1 \) and \( r \geq q_i = l \) (as \((q_a, q_{a+1}) = (l, n + 1)\) implies that \( a + 1 = i + 1 \) and hence \( q_i = l \)). Note the difference in the coefficient in front of \( \Phi(b_n) \) in the two summands, so that not all the terms with the \( \Phi(b_r) \) in them cancel when \( s = n \). Thus for \( s = n \), we are left with

\[
+ \sum_{r \leq n} \sum_{r_q \geq 1, 3 \leq i \leq n} a_{q_j q_{i+1}}^* \prod_{j=1, j \neq k}^i a_{q_k q_{i+1}}^* \Phi b_r
\]

\[
- \sum_{l=1}^{n-1} \sum_{1 \leq r < n+1} \sum_{r_{q_i} \geq 1, (q_a, q_{a+1}) = (l, n + 1)} a_{q_k q_{i+1}}^* \left( \prod_{k=1, k \neq a}^i a_{q_k q_{i+1}}^* \right) a_{l\delta}^* \Phi b_r
\]

\[
= - \sum_{l=1}^{n-1} \sum_{1 \leq r < n} \sum_{r \geq q_{i+1}} \left( i(q) - 1 \right) \prod_{k=1}^i a_{q_k q_{i+1}}^* a_{l\delta}^* \Phi b_r.
\]

If \( s = n \), then the fourth summation and the tenth are both zero due to the factor \((1 - \delta s, n)\) in both.

Moreover the second summation in (6.3) has products of the form

\[
(6.4) \quad a_{r,s+1} \prod_{j=1, j \neq k}^{i-1} a_{q_j q_{i+1}}^* = a_{r,s+1} a_{q_1 q_i}^* \cdots a_{q_{i-1} q_i}^* a_{r,s+1, q_{i+1}}^* \cdots a_{r,n+1}^*
\]

with \( s \leq r < j = q_i, s + 1 \leq q_i = j \), whereas the seventh summation, after setting \( r = l \), has products of the form

\[
(6.5) \quad a_{r,s+1} \prod_{i=1}^{i-1} a_{q_i q_{i+1}}^* a_{r,n+1}^* = a_{r,s+1} a_{q_1 q_i}^* \cdots a_{q_{i-1} q_i}^* a_{r,n+1}^*,
\]

with \( r < s = q_i \) and \( q_{i-1} \leq r < j \). The eighth summation

\[
a_{r,j} \left( \prod_{k=1, k \neq a}^i a_{q_k q_{i+1}}^* \right) a_{l\delta}^* a_{r,n+1}^* = a_{r,j} a_{q_1 q_i}^* \cdots a_{q_{i-1} q_i}^* a_{r,s+1, q_{i+1}}^* a_{q_{i-1} q_i}^* a_{l\delta}^* a_{r,n+1}^*
\]

where \( r \) has the restriction that \( q_{i-1} \leq r < j = q_i, l \leq s - 1 \) and \( s + 1 \leq q_i = j \). If we consider the eighth summation when \( a = i - 1 \) so \((q_{i-1}, j) = (q_{i-1}, q_i) = (l, s + 1)\), then this part of the eighth summation has products of the form

\[
(6.6) \quad a_{r,s+1} a_{q_1 q_i}^* \cdots a_{q_{i-1} q_i}^* a_{l\delta}^* a_{r,n+1}^*.
\]

These summands cancel when \( r < s \) with summands in the seventh summation in (6.5). When \( r = s \) (and \( a = i - 1 \) so \((q_{i-1}, j) = (q_{i-1}, q_i) = (l, s + 1)\)), the summands (6.6) in the eighth summation cancel with the summands in the second summation in (6.4) where \((r, j) = (s, s + 1)\). If \( a < i - 1 \), then in the eighth summation \( a + 1 \leq i - 1 \), so that \( s + 1 = q_{a+1} \leq q_{i-1} \leq r \) and these terms in the eighth summation cancel with the remaining summands in the second summation of the form (6.4).
Hence for \( s > 1 \), we get from (6.3)

\[
\rho(F_\lambda) = \delta_{s,n} a_{n+1} \sum_{q_i \geq n+1} \prod_{k=1}^{i-1} a_{q_k+1}^* - \delta_{s,n} \mu_{q_i=q} \sum_{l=1}^{n-1} a_{l+1} \sum_{q_j \geq j \geq q, \mu_{q}} \prod_{m=1}^{i-1} a_{q_m+1}^* a_{l_i}^*
\]

This proves the Serre relation for \( s \neq 0, 1, n \). If \( s = n \) we have

\[
\rho(F_\lambda) = a_{n+1} \sum_{q_i=q} \prod_{k=1}^{i-1} a_{q_k+1}^* - \mu_{q_i=q} \sum_{l=1}^{n-1} a_{l+1} \sum_{q_j \geq j \geq q, \mu_{q}} \prod_{m=1}^{i-1} a_{q_m+1}^* a_{l_i}^*
\]

We want to show \([\rho(F_\lambda) \theta(F_n)]_{\mu} a_{n+1} = 0\). To prove this first recall \( \rho(F_n) = a_{n+1} \sum_{p=1}^{n-1} a_{p,n+1}^* a_{p,n} \).

Now

\[
\frac{[\rho(F_\lambda) \theta(F_n)]_{\mu} a_{n,n+1}}{= -a_{n+1} \sum_{q_i=q} \prod_{k=1}^{i-2} a_{q_k+1}^*},
\]
whereas

\[- \sum_{p=1}^{n-1} \left[ [\rho(F_0) \lambda \rho(F_n)]_{\mu} a_{p,n+1} a_{p,n}^* \right] =
\]

\[- \sum_{p=1}^{n-1} \left[ (a_{n,n+1} \sum_{q_{i-1}+1}^{q} \prod_{k=1}^{i-1} a_{q_k q_{k+1}}^* - \sum_{1 \leq l < j \leq n+1} a_{lj} \sum_{q_{i-1} \geq q_{i-1}+q_{i} \geq 1}^{m=1} \prod_{l=1}^{i-1} a_{q_{m} q_{m+1}}^* a_{l,l,n}^* \right]
\]

\[= a_{n,n+1} \sum_{q_{i-1}=p}^{n-1} \prod_{k=1}^{i-2} a_{q_k q_{k+1}}^* a_{p,n}^* + \sum_{l=1}^{n-1} a_{l,l,n+1} \sum_{q_{i-1} \geq q_{i-1}+q_{i} \geq p}^{m=1} \prod_{l=1}^{i-2} a_{q_{m} q_{m+1}}^* a_{l,l,n}^* \]

Hence

\[(6.8) \quad [\rho(F_0) \lambda \rho(F_n)]_{\mu} \rho(F_n) = 0.\]

For \( s = 1 \) we get \( \rho(F_1) = a_{1,2} \) and hence

\[\rho(F_0) \lambda \rho(F_1) \]

\[= \sum_{1 \leq r < j \leq n+1} a_{rj} \sum_{q_{i-1} \geq r \geq q_{i-1}} a_{q_{i} q_{i+1}}^* a_{r_{n+1}}^* \]

\[+ \sum_{r \leq n+1} \sum_{r = q_{i-1} \geq q_{i-1} \geq 2} \prod_{j=2}^{i} a_{q_{j} q_{j+1}}^* \Phi b_r + \sum_{1 \leq r \leq n} \sum_{q_{i-1} = r}^{m=1} \prod_{j=2}^{i-1} a_{q_{j} q_{j+1}}^* a_{r,n}^* \]

Thus

\[(6.9) \quad [\rho(F_0) \lambda \rho(F_1)]_{\mu} \rho(F_1) = 0.\]

Next up is the calculation for \( [\rho(F_0) \lambda \rho(F_0) \mu \rho(F_1)] \): For a partition \( q \) = \((1 = q_1, q_2, \ldots, q_i, n+1)\) recall we set \( l(q) = i \). We now write

\[\rho(F_0) \lambda \rho(F_1) \]

\[= A_{01} + B_{01} + C_{01} \quad \text{where}
\]

\[A_{01} = \sum_{1 \leq r \leq n+1} \sum_{q_{i-1} = r}^{l(q)-1} \prod_{j=2}^{l(q)-1} a_{q_{j} q_{j+1}}^* a_{r_{n+1}}^* \]

\[B_{01} = \sum_{r \leq n} \sum_{r = q_{i-1} \geq q_{i-1} \geq 2} \prod_{j=2}^{i} a_{q_{j} q_{j+1}}^* \Phi b_r \]

\[C_{01} = \sum_{1 \leq r \leq n} \sum_{q_{i-1} \geq q_{i-1} \geq r}^{l(q)-1} \prod_{j=2}^{l(q)-1} a_{q_{j} q_{j+1}}^* a_{r,n+1}^* \]
and $F_0 = A + B + C$ where

$$A = \sum_{1 \leq r < j \leq n+1} -a_{rj} \sum_{q,j = q((q_j)^r) \geq q((p_n)^j) - 1}^{l(q)-1} \prod_{l=1}^{l(q)-1} a_{q_{l+1}q_{l+1}}^* a_{r,n+1}^*$$

$$B = - \sum_{1 \leq r < n+1} \sum_{r \geq q((p_n))}^{l(q)} \prod_{j=1}^{l(q)-1} a_{r,j+1}^* \Phi(b_r)$$

$$C = - \sum_{1 \leq r < n+1} \sum_{r = q((p_n))}^{l(q)-1} \prod_{j=1}^{l(q)-1} a_{r,j+1}^* a_{*s,n+1}^*.$$

Then

$$[F_0A_0F_1] = [A_0A_0] + [B_0B_0] + [A_0B_0] + [B_0A_0] + [A_0C_0] + [C_0A_0].$$

Now we calculate each summand above

$$[A_0A_0] = - \sum_{1 \leq r < j \leq n+1} \sum_{1 \leq s < k \leq n+1}^{l(q)-1} \prod_{l=1}^{l(q)-1} a_{q_{l+1}q_{l+1}}^* a_{r,n+1}^* a_{s,n+1}^*$$

$$= - \sum_{1 \leq r < j \leq n+1} \sum_{1 \leq s < k \leq n+1}^{l(q)-1} \prod_{l=1}^{l(q)-1} a_{q_{l+1}q_{l+1}}^* a_{r,n+1}^* a_{s,n+1}^*$$

as the double contractions are zero in this normal ordering.
\[- \sum_{1 \leq s < k \leq n+1} \sum_{q_{n+1}=q(k)\geq q_l(q)-1}^{l(q)-1} a_{sk} \prod_{p,k:p(p)\geq p(p)-1,P_{1}\geq 2} a^*_p q_{p,q_1+1} a^*_{s,n+1} \]
\[+ \sum_{1 \leq s < k \leq n+1} \sum_{q_{n+1}=q(k)\geq q_l(q)-1}^{l(q)-1} a_{sk} \prod_{l=1}^{l(p)-1} a^*_{q_{l,q_1+1}} a^*_{s,n+1} + a_{sk} \prod_{l=1}^{l(p)-1} a^*_{q_{l,q_1+1}} a^*_{s,n+1} \]
\[- \sum_{1 \leq r < s \leq n+1} \sum_{q_{n+1}=q(r)\geq q_l(q)-1}^{l(q)-1} a_{sk} \prod_{l=1}^{l(p)-1} a^*_{p_{l,p_{1}+1}} a^*_{s,n+1} \]
\[- \sum_{1 \leq r < s \leq n+1} \sum_{q_{n+1}=q(r)\geq q_l(q)-1}^{l(q)-1} a_{sk} \prod_{l=1}^{l(p)-1} a^*_{q_{l,q_1+1}} a^*_{s,n+1} \]
\[- \sum_{1 \leq r < s \leq n+1} \sum_{q_{n+1}=q(r)\geq q_l(q)-1}^{l(q)-1} a_{sk} \prod_{l=1}^{l(p)-1} a^*_{q_{l,q_1+1}} a^*_{s,n+1} \]
\[+ \sum_{1 \leq r < s \leq n+1} \sum_{q_{n+1}=q(r)\geq q_l(q)-1}^{l(q)-1} a_{sk} \prod_{l=1}^{l(p)-1} a^*_{q_{l,q_1+1}} a^*_{s,n+1} \]
\[- \sum_{1 \leq r < s \leq n+1} \sum_{q_{n+1}=q(r)\geq q_l(q)-1}^{l(q)-1} a_{sk} \prod_{l=1}^{l(p)-1} a^*_{q_{l,q_1+1}} a^*_{s,n+1} \]
\[+ \sum_{1 \leq r < s \leq n+1} \sum_{q_{n+1}=q(r)\geq q_l(q)-1}^{l(q)-1} a_{sk} \prod_{l=1}^{l(p)-1} a^*_{q_{l,q_1+1}} a^*_{s,n+1} \]
To show that the above summation is zero reduces to showing the following are zero:

\[ I_1 := - \sum_{q,n+1 \leq q(l(q))} \prod_{p,k = p(l(p))^{p_2}} a_{p_1 p_{\xi + 1}}^{l(p) - 1} \sum_{l=1}^{l(q) - 1} a_{q_1 q_{l+1}}^{l(q) - 1} \]

\[ + \sum_{1 \leq r < j \leq n+1} \sum_{q,k = q(l(q))} \prod_{p,l = p(l(p))^{p_2}} a_{p_1 p_{\xi + 1}}^{l(p) - 1} \sum_{l=1}^{l(q) - 1} a_{q_1 q_{l+1}}^{l(q) - 1} \prod_{l=1}^{l(p) - 1} a_{q_1 q_{l+1}}^{l(p) - 1} a_{r_1 r_{n+1}}^{l(p) - 1} \]

Note that the first summation is over all partitions \( p \) and \( q \) and has summands of the form

\[ a_{2p_3}^{l(p)} \cdots a_{p_1 p_{l+1}^{l(p)}}^{l(p) - 1} a_{1.q_2}^{l(q) - 1} \cdots a_{q_1 q_{l+1}}^{l(q) - 1} n+1 \]

with \( p_{l(p)} - 1 \leq s \) and \( q_{l(q)} - 1 \leq s \). The second summation is over all partitions \( p \) and \( q \) and has summands of the form

\[ a_{1.q_2}^{l(q)} \cdots a_{q_1 q_{l+2}}^{l(q) - 1} \cdots a_{q_1 q_{l+1}}^{l(q) - 1} a_{2p_3}^{l(p)} \cdots a_{p_1 p_{l+1}^{l(p)}}^{l(p) - 1} a_{r_1 r_{n+1}}^{l(p) - 1} \]

\[ = \left( a_{2p_3}^{l(p)} \cdots a_{p_1 p_{l+1}^{l(p)}}^{l(p) - 1} a_{1.q_2}^{l(q) - 1} \cdots a_{q_1 q_{l+1}}^{l(q) - 1} \right) a_{2p_3}^{l(p)} \cdots a_{p_1 p_{l+1}^{l(p)}}^{l(p) - 1} a_{r_1 r_{n+1}}^{l(p) - 1} \]

where \( s \geq q_{l(q)} - 1, s \geq r \geq p_{l(p)} - 1, r < j \leq k \). But these two sets of partitions are the same so \( I_1 = 0 \).

Similarly if we look at the partitions for the summands of

\[ I_2 := - \sum_{q,k = q(l(q))} \prod_{p,n+1 = p(l(p))^{p_2}} a_{p_1 p_{\xi + 1}}^{l(p) - 1} \sum_{l=1}^{l(q) - 1} a_{q_1 q_{l+1}}^{l(q) - 1} \prod_{l=1}^{l(p) - 1} a_{q_1 q_{l+1}}^{l(p) - 1} a_{r_1 r_{n+1}}^{l(p) - 1} \]

\[ - \sum_{2 \leq r < j \leq n+1} \sum_{q,k = q(l(q))} \prod_{p,l = p(l(p))^{p_2}} a_{p_1 p_{\xi + 1}}^{l(p) - 1} a_{q_1 q_{l+1}}^{l(q) - 1} \prod_{l=1}^{l(p) - 1} a_{q_1 q_{l+1}}^{l(p) - 1} a_{r_1 r_{n+1}}^{l(p) - 1} \]

we can check that \( I_2 = 0 \). Hence \([A_{\lambda}A_{\mu}] = 0\).
\[ [B_\lambda B_{01}] = - \sum_{1 \leq r < n+1} \sum_{1 \leq \ell < n+1} \sum_{\lambda \geq \ell(q)} \sum_{r,k \geq \ell(p)} \frac{l(q)}{n+1} a_{q,j} q_{j+1} \Phi(b_r) \prod_{l=2}^{l(p)} a_{\ell(p) + 1} \Phi(b_k) \]

\[ = \sum_{1 \leq r < n+1} \sum_{1 \leq k < n+1} A_{r,k} \sum_{\lambda \geq \ell(q)} \sum_{r,k \geq \ell(p)} \kappa \cdot (\lambda + D) \left( \prod_{j=1}^{l(q)} a_{q,j} q_{j+1} \right) \prod_{l=2}^{l(p)} a_{\ell(p) + 1} \]

\[ = \sum_{1 \leq r < n+1} \sum_{1 \leq k < n+1} (-\delta_{r,k-1} + 2\delta_{r,k} - \delta_{r,k+1}) \sum_{\lambda \geq \ell(q)} \sum_{r,k \geq \ell(p)} \kappa \cdot (\lambda + D) \left( \prod_{j=1}^{l(q)} a_{q,j} q_{j+1} \right) \prod_{l=2}^{l(p)} a_{\ell(p) + 1} \]

\[ = - \sum_{1 \leq r < n+1} \sum_{q,r \geq \ell(q)} \sum_{r\ell + 1 \geq \ell(p)} \sum_{p \geq \ell(p)} \kappa \cdot (\lambda + D) \left( \prod_{j=1}^{l(q)} a_{q,j} q_{j+1} \right) \prod_{l=2}^{l(p)} a_{\ell(p) + 1} + 2 \sum_{1 \leq r < n+1} \sum_{q,r \geq \ell(q)} \sum_{r\ell + 1 \geq \ell(p)} \sum_{p \geq \ell(p)} \kappa \cdot (\lambda + D) \left( \prod_{j=1}^{l(q)} a_{q,j} q_{j+1} \right) \prod_{l=2}^{l(p)} a_{\ell(p) + 1} \]

\[ - \sum_{2 \leq r < n+1} \sum_{q,r \geq \ell(q)} \sum_{r\ell - 1 \geq \ell(p)} \sum_{p \geq \ell(p)} \kappa \cdot (\lambda + D) \left( \prod_{j=1}^{l(q)} a_{q,j} q_{j+1} \right) \prod_{l=2}^{l(p)} a_{\ell(p) + 1} \]

\[ = - \sum_{1 \leq r < n+1} \sum_{q,r \geq \ell(q)} \sum_{r\ell + 1 \geq \ell(p)} \sum_{p \geq \ell(p)} \kappa \cdot (\lambda + D) \left( \prod_{j=1}^{l(q)} a_{q,j} q_{j+1} \right) \prod_{l=2}^{l(p)} a_{\ell(p) + 1} + 2 \sum_{1 \leq r < n+1} \sum_{q,r \geq \ell(q)} \sum_{r\ell + 1 \geq \ell(p)} \sum_{p \geq \ell(p)} \kappa \cdot (\lambda + D) \left( \prod_{j=1}^{l(q)} a_{q,j} q_{j+1} \right) \prod_{l=2}^{l(p)} a_{\ell(p) + 1} \]

\[ - \sum_{1 \leq r < n+1} \sum_{q,r \geq \ell(q)} \sum_{r\ell - 1 \geq \ell(p)} \sum_{p \geq \ell(p)} \kappa \cdot (\lambda + D) \left( \prod_{j=1}^{l(q)} a_{q,j} q_{j+1} \right) \prod_{l=2}^{l(p)} a_{\ell(p) + 1} \]

\[ = - \sum_{1 \leq r < n+1} \sum_{q,r \geq \ell(q)} \sum_{r\ell + 1 \geq \ell(p)} \sum_{p \geq \ell(p)} \kappa \cdot (\lambda + D) \left( \prod_{j=1}^{l(q)} a_{q,j} q_{j+1} \right) \prod_{l=2}^{l(p)} a_{\ell(p) + 1} \]

\[ - \sum_{1 \leq r < n+1} \sum_{q,r \geq \ell(q)} \sum_{r\ell - 1 \geq \ell(p)} \sum_{p \geq \ell(p)} \kappa \cdot (\lambda + D) \left( \prod_{j=1}^{l(q)} a_{q,j} q_{j+1} \right) \prod_{l=2}^{l(p)} a_{\ell(p) + 1} \]

\[ + 2 \sum_{1 \leq r < n+1} \sum_{q,r \geq \ell(q)} \sum_{r\ell + 1 \geq \ell(p)} \sum_{p \geq \ell(p)} \kappa \cdot (\lambda + D) \left( \prod_{j=1}^{l(q)} a_{q,j} q_{j+1} \right) \prod_{l=2}^{l(p)} a_{\ell(p) + 1} \]

\[ + 2 \sum_{1 \leq r < n+1} \sum_{q,r \geq \ell(q)} \sum_{r\ell - 1 \geq \ell(p)} \sum_{p \geq \ell(p)} \kappa \cdot (\lambda + D) \left( \prod_{j=1}^{l(q)} a_{q,j} q_{j+1} \right) \prod_{l=2}^{l(p)} a_{\ell(p) + 1} \]
\[
\begin{align*}
&= - \sum_{1 \leq r < n} \sum_{q: r \geq q_{(q)}} \sum_{p: r+1 = p_{(p)} + p_2 = 2} \kappa \cdot (\lambda + D) \left( \prod_{j=1}^{l(q)} a_{q,j}^* \right) \left( \prod_{l=2}^{l(p)} a_{p_l,p_{l+1}}^* \right) \\
&+ 2 \sum_{q: n \geq q_{(q)}} \sum_{p: n \geq p_{(p)} + p_2 = 2} \kappa \cdot (\lambda + D) \left( \prod_{j=1}^{l(q)} a_{q,j}^* \right) \left( \prod_{l=2}^{l(p)} a_{p_l,p_{l+1}}^* \right) \\
&- \sum_{1 \leq r < n} \sum_{r+1 = q_{(q)}} \sum_{p: r+1 = p_{(p)} + p_2 = 2} \kappa \cdot (\lambda + D) \left( \prod_{j=1}^{l(q)} a_{q,j}^* \right) \left( \prod_{l=2}^{l(p)} a_{p_l,p_{l+1}}^* \right).
\end{align*}
\]

Now

\[
[A_1 B_{01}] = - \sum_{1 \leq r < n + 1} \sum_{q: j = q_{(q)} + r \geq q_{(q)} - 1} \sum_{p: r + 1 = p_{(p)} + p_2 = 2} \frac{l(q)-1}{l(p)} \prod_{l=1}^{l(q)} a_{q,l,j}^* \lambda \prod_{j=1}^{l(p)} a_{p_j,p_{j+1}}^* \Phi \left( b_s \right)
\]

\[
[B_1 A_{01}] = - \sum_{1 \leq r < n + 1} \sum_{q: j = q_{(q)} + r \geq q_{(q)} - 1} \sum_{p: r + 1 = p_{(p)} + p_2 = 2} \frac{l(q)-1}{l(p)} \prod_{l=1}^{l(q)} a_{q,l,j}^* \left( \prod_{j=1}^{l(p)} a_{p_j,p_{j+1}}^* \lambda \right) \Phi \left( b_s \right)
\]

Hence \([A_1 B_{01}] + [B_{01} A_1] = 0\) follows from the calculation below each fixed \(1 \leq s \leq n\), and \(r \geq 2\)

\[
\begin{align*}
&- \sum_{r < j \leq n + 1} \sum_{q: j = q_{(q)} + r \geq q_{(q)} - 1} \sum_{p: r + 1 = p_{(p)} + p_2 = 2} \frac{l(q)-1}{l(p)} \prod_{l=1}^{l(q)} a_{q,l,j}^* \lambda \prod_{j=1}^{l(p)} a_{p_j,p_{j+1}}^* \\
&- \sum_{r < j \leq n + 1} \sum_{q: j = q_{(q)} + r \geq q_{(q)} - 1} \sum_{p: r + 1 = p_{(p)} + p_2 = 2} \frac{l(q)-1}{l(p)} \prod_{l=1}^{l(q)} a_{q,l,j}^* \left( \prod_{j=1}^{l(p)} a_{p_j,p_{j+1}}^* \lambda \right) \\
&= - \sum_{r < j \leq n + 1} \sum_{q: j = q_{(q)} + r \geq q_{(q)} - 1} \sum_{p: r + 1 = p_{(p)} + p_2 = 2} \frac{l(q)-1}{l(p)} \prod_{l=1}^{l(q)} a_{q,l,j}^* \left( \prod_{j=1}^{l(p)} a_{p_j,p_{j+1}}^* \right) \\
&+ \sum_{r < j \leq n + 1} \sum_{q: j = q_{(q)} + r \geq q_{(q)} - 1} \sum_{p: r + 1 = p_{(p)} + p_2 = 2} \frac{l(q)-1}{l(p)} \prod_{l=1}^{l(q)} a_{q,l,j}^* \left( \prod_{j=1}^{l(p)} a_{p_j,p_{j+1}}^* \right)
\end{align*}
\]
The factors appearing in the first summation is over all partitions \( \lambda \) and \( q \) and looks like
\[
a_1^{\lambda} \cdots a_n^{\lambda} \prod_{q_i(q_i'-2i)} a_{q_i(q_i'-2i)} \cdots a_{q_i(q_i'-2i)} a_{q_i(q_i'-2i)} \cdots a_{q_i(q_i'-2i)} a_{q_i(q_i'-2i)} \cdots a_{q_i(q_i'-2i)}
\]
with \( l(q) - 1 \leq r \) and \( p_{t-1} \leq r \) whereas the second summation is over all partitions \( \lambda \) and \( q \) resulting in
\[
a_1^{\lambda} \cdots a_n^{\lambda} \prod_{q_i(q_i'-2i)} a_{q_i(q_i'-2i)} \cdots a_{q_i(q_i'-2i)} a_{q_i(q_i'-2i)} \cdots a_{q_i(q_i'-2i)} a_{q_i(q_i'-2i)} \cdots a_{q_i(q_i'-2i)}
\]
with \( l(q) - 1 \leq r \) and \( p_{t-1} \leq r \). Renaming \((1, q_1, \ldots, q_{l(q)-2})\) to \((1, p_1', \ldots, p_{t-1}')\), and \((p_3, \ldots, p_{t-1})\) to \((q_{l(q)-1}', q_{l(q)-1})\) we see that these two summations cancel.

Next we calculate
\[
[A_\lambda C_{01}] = - \sum_{1 \leq r < j \leq n+1} \sum_{q_{i,j}=q_{i,(q_{i}-1)} \geq q_{i,(q_{i}-1)}} \prod_{l=1}^{l(q)-1} a_{q_i(q_i'-1)} a_{r,n+1} a_{r,n+1} \lambda \sum_{1 \leq s < n+1} \sum_{p:s=p(p), p_2=2} \prod_{k=2}^{l(p)-1} a_{p_k p_{k+1}} \cdot D_{a_{s,n+1}}
\]

While on the other hand we have
\[
[C_\lambda A_{01}] = - \sum_{1 \leq r < j \leq n+1} \sum_{q_{i,j}=q_{i,(q_{i}-1)} \geq q_{i,(q_{i}-1)}} \prod_{l=1}^{l(p)-1} a_{r,n+1} \lambda \sum_{1 \leq r < j \leq n+1} \sum_{q_{i,j}=q_{i,(q_{i}-1)} \geq q_{i,(q_{i}-1)}} \prod_{l=2}^{l(q)-1} a_{q_i(q_i'-1)} a_{s,n+1} \cdot (\lambda + D) \prod_{k=1}^{l(p)-1} a_{p_k p_{k+1}}
\]

Thus
\[
[A_\lambda C_{01}] + [C_\lambda A_{01}] = - \sum_{1 \leq r < j \leq n+1} \sum_{q_{i,j}=q_{i,(q_{i}-1)} \geq q_{i,(q_{i}-1)}} \prod_{l=1}^{l(q)-1} a_{q_i(q_i'-1)} a_{r,n+1} \lambda \sum_{1 \leq s < n+1} \sum_{p:s=p(p), p_2=2} \prod_{k=2}^{l(p)-1} a_{p_k p_{k+1}} \cdot D_{a_{s,n+1}}
\]

\[
- \sum_{1 \leq s < n+1} \sum_{q_{i,n+1}=q_{i,(q_{i}-1)} \geq q_{i,(q_{i}-1)}} \sum_{p:s=p(p), p_2=2} \prod_{k=2}^{l(p)-1} a_{p_k p_{k+1}} \cdot D_{a_{s,n+1}}
\]

\[
+ \sum_{1 \leq r < j \leq n+1} \sum_{q_{i,j}=q_{i,(q_{i}-1)} \geq q_{i,(q_{i}-1)}} \prod_{l=2}^{l(q)-1} a_{q_i(q_i'-1)} a_{s,n+1} \lambda \sum_{1 \leq k \leq l} \prod_{k=1}^{l(p)-1} a_{p_k p_{k+1}} \cdot D_{a_{s,n+1}}
\]

\[
- \sum_{1 \leq s < n+1} \sum_{q_{i,n+1}=q_{i,(q_{i}-1)} \geq q_{i,(q_{i}-1)}} \sum_{p:s=p(p)} \prod_{l=2}^{l(q)-1} a_{q_i(q_i'-1)} a_{s,n+1} \lambda \sum_{1 \leq k \leq l} \prod_{k=1}^{l(p)-1} a_{p_k p_{k+1}}
\]
Then
\[
[B_{\lambda}B_{01}] + [A_{\lambda}C_{01}] + [C_{\lambda}A_{01}]
\]

\[
= - \sum_{1 \leq r < n} \left( \sum_{q: r \geq q_1(q)} \kappa \cdot (\lambda + D) \left( \prod_{j=1}^{l(q)} a_{j_0, q_{j+1}} \right) \right) \left( \sum_{p: r + 1 = p_1(p), p_2 = 2} \prod_{s=2}^{l(p)} a_{s, p_{s+1}} \right) + 2 \left( \sum_{q: n \geq q_1(q)} \kappa \cdot (\lambda + D) \left( \prod_{j=1}^{l(q)} a_{j_0, q_{j+1}} \right) \right) \left( \sum_{p: n \geq p_1(p), p_2 = 2} \prod_{s=2}^{l(p)} a_{s, p_{s+1}} \right) - \sum_{1 \leq r < n} \left( \sum_{q: r+1 = q_1(q)} \kappa \cdot (\lambda + D) \left( \prod_{j=1}^{l(q)} a_{j_0, q_{j+1}} \right) \right) \left( \sum_{p: r \geq p_1(p), p_2 = 2} \prod_{s=2}^{l(p)} a_{s, p_{s+1}} \right)
\]

If we look at the forth summation we have factors with \( q: j = q_1(q) \geq q_1(q) - 1; \ p, s = p_1(p); p_2 = 2; \exists t: (r, j) = (q_t, q_{t+1}) \) and are of the form
\[
- \left( \prod_{l=1}^{\lfloor l(q) \rfloor} a_{q_l, q_{l+1}} \right) a_{r, n+1} \prod_{k=2}^{l(p)} a_{s, k_{s+1}} \kappa \cdot D a_{n, n+1}^{s, n+1}
\]

This is the same as summing over factors with of the form
\[
(6.10)
\]

\[
- a_{q_1, q_1}^s \cdots a_{q_l, q_{l+1}}^s \kappa \cdot D \left( a_{s, n+1}^* \prod_{l=2}^{\lfloor l(p') \rfloor} a_{p, p_{s+1}} \right)
\]

where \( q' := (1, q_2, \ldots, l(q) - 1, r, p_{l+2}, \ldots, l(p) - 1, s, n + 1) \) and \( p' := (1, 2, p_3, \ldots, p_{l-1}, r, n + 1) \). In the above we have \( r < s \) and \( l(q') = s \) and \( l(p') < s \).
If we look at the fifth summation we have factors with \( q; n + 1 = q_l(q); s \geq q_l(q)-1 \), \( p, s = p_l(p), p_2 = 2 \) and are of the form

\[
-\kappa \cdot (\lambda + D) \left( \prod_{l=1}^{l(q)-1} a_{q_l(q_l+1)}^* \right) \prod_{k=2}^{l(p)-1} a_{p_k,p_{k+1}}^* a_{s,n+1}^* - \kappa \cdot (\lambda + D) \left( a_{1q_2}^* \cdots a_{q_{q(q)-1},n+1}^* \right) a_{s,n+1}^* a_{2p_3}^* \cdots a_{l(p)-1,s}^*
\]

\[
= -\kappa \cdot (\lambda + D) \left( a_{1q_2}^* \cdots a_{q_{q(q)-1},n+1}^* \right) a_{2p_3}^* \cdots a_{l(p)-1,s}^* a_{s,n+1}^*
\]

\[
= -\kappa \cdot (\lambda + D) \left( \prod_{l=1}^{l(q')-1} a_{q_l(q_l'+1)}^* \right) \prod_{k=2}^{l(p)} a_{p_k,p_{k+1}}^*
\]

where \( q' := (1, q_2, \cdots, q_{q(q)-1} + 1) \) (so \( q_l(q'+1) = n + 1 \) and \( q_l(q') = q_l(q)-1 \)) and \( l(q') \leq s = l(p) \).

If we look at the sixth summation we have factors with \( q; n + 1 = q_l(q); s \geq q_l(q)-1 \), \( p, s = p_l(p), p_2 = 2 \) and are of the form

\[
(6.11) \quad - \left( \prod_{l=1}^{l(q)-1} a_{q_l(q_l+1)}^* \right) \kappa \cdot D(a_{s,n+1}^*) \prod_{k=2}^{l(p)-1} a_{p_k,p_{k+1}}^* = -a_{1q_2}^* \cdots a_{q_{q(q)-1},n+1}^* \kappa \cdot D(a_{s,n+1}^*) a_{2p_3}^* \cdots a_{l(p)-1,s}^*
\]

\[
= - \left( \prod_{l=1}^{l(q')-1} a_{q_l(q_l'+1)}^* \right) \kappa \cdot D(a_{s,n+1}^*)
\]

where \( q' := (1, q_2, \cdots, q_{q(q)-1} + 1) \) (so \( q_l(q'+1) = n + 1 \) and \( q_l(q') = q_l(q)-1 \)) and \( l(q') \leq s \).

In the second to the last summation we have after we switch the \( q \) with the \( p \) are summed over \( p; j = p_l(p); r \geq p_l(p)-1, p_2 = 2 \), \( q, s = q_l(q) \) \( \exists t : (q_l, t_{l+1}) = (r, j) \) and the factors have the form

\[
(6.12) \quad \left( \prod_{l=2}^{l(p)-1} a_{p_l,p_{l+1}}^* \right) a_{r,n+1}^* \prod_{k=1}^{l(q)-1} a_{q_{q(k+1),q_k+1}}^* \kappa \cdot D(a_{s,n+1}^*)
\]

\[
= a_{1q_2}^* \cdots a_{q_{q(q)-1},n+1}^* \cdots a_{s,n+1}^* \kappa \cdot D(a_{s,n+1}^*) a_{2p_3}^* \cdots a_{l(p)-1,s}^* a_{r,n+1}^*
\]

\[
= (a_{1q_2}^* \cdots a_{q_{q(q)-1},n+1}^*)(a_{2p_3}^* \cdots a_{l(p)-1,r}^*) a_{j_{q_{q(q)-1},s}^*} \kappa \cdot D(a_{s,n+1}^*)
\]

Thus this second to last summation is cancelled by terms of the sixth summation (6.11) leaving us with a summation over \( q; n + 1 = q_l(q); s = q_l(q)-1 \), \( p, s = p_l(p), p_2 = 2 \) in the sixth summation. Thus the remaining summation has factors that can be rewritten in the form

\[
\left( \prod_{l=1}^{l(q')-1} a_{q_l(q_l'+1)}^* \right) a_{s,n+1}^* = \kappa \cdot D(a_{s,n+1}^*) a_{2p_3}^* \cdots a_{l(p)-1,s}^* a_{s,n+1}^* \prod_{l=1}^{l(q')-1} a_{q_l(q_l'+1)}^*
\]

In the last summation we have after we switch the \( q \) with the \( p \) are summed over \( p; n + 1 = p_l(p); s \geq p_l(p)-1, p_2 = 2 \), \( q, s = q_l(q) \) and the factors have the form

\[
\left( \prod_{l=2}^{l(p)-1} a_{p_l,p_{l+1}}^* \right) a_{s,n+1}^* = \kappa \cdot (\lambda + D) (a_{1q_2}^* \cdots a_{q_{q(q)-1},n+1}^*) a_{s,n+1}^* \prod_{l=2}^{l(q')-1} a_{p_l,p_{l+1}}^*
\]
where \( \mathbf{p}' := (1, 2, p_3, \ldots, p_{l(p)}-1, n + 1) \) so that \( l(\mathbf{p}') \leq s \). Now the last summation combines with (6.10) and (6.13) to give us a summation over \( \mathbf{p}'; s \geq p_{l(\mathbf{p})}, p_2' = 2, q', s = q'(q') \) with factors the form

\[
- \prod_{l=2}^{l(p)-1} a_{p_l p_{l+1}} a_{s,n+1}^* \kappa \cdot (\lambda + D) \prod_{k=1}^{l(q)-1} a_{q_k q_{k+1}}^* = -\kappa \cdot (\lambda + D) \left( \prod_{l=1}^{l(q')} a_{q_l q_{l+1}}^* \right) \prod_{l=2}^{l(p')} a_{p'_l p'_{l+1}}^*
\]

As a consequence we obtain

\[
[A_\lambda C_{01}] + [C_\lambda A_{01}] = - \sum_{1 \leq s<n+1} \sum_{q:s\geq q_0(q)} \kappa \cdot (\lambda + D) \left( \prod_{l=1}^{l(q)} a_{q_l q_{l+1}}^* \right) \left( \sum_{p,s=p_1(p), p_2=2}^l a_{p_k p_k+1}^* \right)
\geq \sum_{s<n+1} \sum_{q:s=q_0(q)} \kappa \cdot (\lambda + D) \left( \prod_{l=1}^{l(q)} a_{q_l q_{l+1}}^* \right) \left( \sum_{p,s=p_1(p), p_2=2}^l a_{p_k p_k+1}^* \right)
\]

Now if we set

\[
B(t) := \sum_{q:t=q_0(q)} \prod_{q_i=q_{i-1}(q)}^{l(q)} a_{q_i q_{i+1}}^* \quad \text{and} \quad C(t) := \sum_{p,t=p_1(p), p_2=2}^l \prod_{p_k p_{k+1}}^l a_{p_k p_k+1}^*
\]

then by induction one can show that

\[
\sum_{s=1}^n \left( \prod_{q=1}^s B(q) \right) C(s) + \sum_{s=1}^n \left( B(s) \prod_{q=1}^s C(q) \right) = 2 \left( \prod_{q=1}^n B(q) \right) \left( \sum_{q=1}^s C(q) \right) - \sum_{s=1}^n \left( \sum_{q=1}^s B(q) \right) C(s+1) - \sum_{s=1}^n B(s+1) \left( \sum_{q=1}^s C(q) \right).
\]

(The identity above holds for any elements \( B(q) \) and \( C(s) \) in an algebra with coefficients in \( \mathbb{Z} \).) Thus we conclude

\[
[B_\lambda B_{01}] + [A_\lambda C_{01}] + [C_\lambda A_{01}] = 0.
\]

This completes the proof that

\[
(6.14) \quad [\rho(F_0\lambda)[\rho(F_0)_\mu \rho(F_1)]] = 0.
\]

Now the proof that \( [\rho(F_0)_\lambda[\rho(F_0)_\mu \rho(F_0)]] = 0 \) and \( [\rho(F_0)_\lambda \rho(F_0)] = 0 \) are proven in a similar manner, where in the end it boils down to the following formal identities:

\[
\sum_{r=1}^n \left( \sum_{s=1}^{n-1} (-\delta_{r,s-1} + 2\delta_{r,s} - \delta_{r,s+1}) \left( \sum_{l=1}^r \left( \sum_{v=1}^s B(t) C(v) \right) \right) \right) = \sum_{l=1}^{n-1} B(t) C(l) - B(n) \left( \sum_{l=1}^{n-1} C(l) \right)
\]

and

\[
\left( \sum_{r=1}^n \left( \sum_{q=1}^s A(q) \right) \right) A(r+1) - \left( \sum_{q=1}^n A(q) \right)^2 + \sum_{s=1}^n \left( \left( \sum_{q=1}^s A(q) \right) A(s) \right) = 0.
\]

For the Serre type relations for the \( E_r \), the calculations are the same as those in [Cox02, Lemma 3.5] where

\[
-\gamma b_r(z) - \frac{1}{2} \left( b_{r-1}^+(z) + b_{r+1}^+(z) \right)
\]
is replaced by $\Phi(b_r)$ and
\[-\frac{\gamma}{2} \hat{a}_{r,r+1}(z)\]
is replaced by $\kappa \cdot Da_{r,r+1}^*$. We refer the interested reader to that paper for the proof. □

REFERENCES


