On ternary designs with a specified number of blocks with repeated elements

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Abstract

The blocks of a balanced ternary design, BTD$(V, B; \rho_1, \rho_2, R; K, \Lambda)$, can be partitioned into two sets: the b_1 blocks that each contain no repeated elements, and the $b_2 = B - b_1$ blocks containing repeated elements. In this note, we address, and answer in some particular cases, the following question. For which partitions of the integer B as $b_1 + b_2$ does there exist a BTD$(V, B; \rho_1, \rho_2, R; K, \Lambda)$?

1. Introduction

A balanced n-ary design (BnD) with parameters (V, B, R, K, Λ), said by some to be proper and equireplicate, is an arrangement of V elements into B multisets, called blocks, under the following stipulations. Letting m_{vb} denote the multiplicity of the vth element in the bth block, we insist that each block have cardinality K:

$$\forall b \leq B, \quad \sum_{v=1}^{V} m_{vb} = K;$$

that any element appears at most $n-1$ times in a block: $m_{vb} < n$; that each element appears R times in the design:

$$\forall v \leq V, \quad \sum_{b=1}^{B} m_{vb} = R;$$

and that every pair of distinct elements appears Λ times:

$$\forall v, w \leq V, v \neq w, \quad \sum_{b=1}^{B} m_{vb} m_{wb} = \Lambda.$$

When $n = 3$, such a design is called a balanced ternary design, or BTD, and in this case one can show that all elements appear with multiplicity one in a constant number of blocks, say ρ_1, and with multiplicity two in a constant number of blocks, say ρ_2. Customarily, one lists the parameters of a BTD as

$$(V, B; \rho_1, \rho_2, R; K, \Lambda). \tag{1.1}$$

The existing work on BTDs is extensive, beginning with the introduction of n-ary designs by Tocher [**24**]. Saha [**17**] proved that, in a BTD, $\sum_b m_{vb}^2$ is independent of v. For a general introduction to n-ary and ternary designs, see Billington's [**1**], [**2**], and Billington and Robinson's [**7**], which lists parameters of BTDs whose existence or nonexistence is known, or whose existence could not be ruled out at that time. Since its publication, new designs have been constructed; see, for instance, Dillon and Wertheimer [**8**] as well as Sinha [**21**]. An upcoming paper of Billington will include an improved list. Billington [**3**] has completely settled the question of existence of ternary designs with block size three, and Donavan has obtained several families of BTDs with block size four. By using Bhaskar Rao designs, Sarvate and Seberry [**20**] have constructed certain designs not isomorphic to those listed in [**7**]. Many authors, for example Billington, Dey, Francel, Hoffman, Khodkar, Mahmoodian, Mathur, Misra, Nigam, Patwardhan, Saha, Sarvate, Sharma, Sinha (see bibliography), have studied partially balanced and balanced ternary designs. See Vartak and Diwanji [**25**] for constructions of column-regular BTDs.

Motivated by a comment of Professor D.A. Preece at Billington's talk [**4**], we consider designs with a prescribed number of blocks each having at least one repeated element. Specifically, define a balanced part n-ary design with parameters

$$(V; b_1, b_2, \dots, b_{n-1}, B; R; K, \Lambda) \tag{1.2}$$

to be a BnD$(V, B; R; K, \Lambda)$ such that exactly b_k blocks satisfy

$$\min m_{vb} = k.$$

Obviously, $b_1 + \cdots + b_{n-1} = B$.

We focus in this paper on the case $n = 3$, referring to the design as a balanced part ternary design and listing its parameters as

$$(V; b_1, b_2, B; \rho_1, \rho_2, R; K, \Lambda) \tag{1.3}$$

From the definition, every BTD is a BPTD for some choice of b_1 and b_2, and conversely. However, the parameters (1.1) need not determine b_1 and b_2. For example,

1	1	1	2	2	3	1	1
1	1	1	2	2	3	2	2
2	3	4	3	4	4	3	3
2	3	4	3	4	4	4	4

and

1	1	1	2	2	3	1	1
1	1	1	2	2	3	2	2
2	3	4	3	4	4	3	3
2	4	4	4	4	4	3	3

are both BTDs with parameters (4, 8; 2, 3, 8; 4, 6), but the first has $b_1 = 2$, $b_2 = 6$ and the second has $b_1 = 0$, $b_2 = 8$. This raises the question, "Given a BTD with parameters (1.1), for which b_1 and b_2 in (1.3) does a BPTD exist?". We provide partial answers to this general question and analyze completely some particular cases.

One can easily generalize this question for n-ary designs: for which partition of B into $b_1 + b_2 + \cdots + b_{n-1}$ does a BPnD with parameters (1.2) exist?

Many existing constructions of BTDs are equally useful as BPTD constructions, in the sense that they provide ternary designs with specific values of the parameters b_1 and b_2. As examples, we restate four results from [**7**] (their Theorem 1.1 and Results 1.2, 1.3, 1.4) from a BPTD point of view.

Theorem 1.4: *If there exists a BPTD with parameters* $(V; b_1, b_2, V; V-4, 1, V-2; V-2, V-4)$, *then a BPTD with parameters* $(V+4; b_1, b_2+4, V+4; V, 1, V+2; V+2, V)$ *exists.*

Result 1.5: *From a BIBD with parameters* (v, b, r, k, λ), *where* $k = \frac{1}{2}v$, *a BPTD with parameters* $V = v$, $b_1 = L$, $b_2 = b$, $B = b+L$, $\rho_1 = L$, $\rho_2 = r$, $R = L + 2r$, $K = 2k = v$, *and* $\Lambda = 4\lambda + L$ *can be constructed for every nonnegative integer* L.

Result 1.6: *From a BIBD with parameters* (v, b, r, k, λ), *where* $k = \frac{1}{2}(v-1)$, *a BPTD with parameters* $V = v$, $b_1 = v$, $b_2 = b$, $B = b + v$, $\rho_1 = v - 1$, $\rho_2 = r$, $R = v - 1 + 2r$, $K = 2k = v - 1$, *and* $\Lambda = 4\lambda + v - 2$ *can be constructed.*

Result 1.7: *From a BPTD with parameters* $(V; b_1, b_2, B; \rho_1, \rho_2, R; K, \Lambda)$ *and a BIBD with parameters* $(V, b, r, \frac{1}{2}K, \lambda)$, *one can construct a BPTD with parameters* $(V; b_1, b_2 + b, B + b; \rho_1, \rho_2 + r, R + 2r; K, \Lambda + 4\lambda)$.

We give a simple generalization of 1.5 and 1.6 in section 2. In section 3, we derive some necessary conditions on (1.3). Section 4 covers the case $\min(K, \Lambda) \leq 3$. In sections 5 and 6, we find all possible values of b_1 and b_2 for certain BPTDs satisfying $K = \Lambda$.

2. Constructions of BPTDs.

The constructions in Results 1.5 and 1.6 produce BPTDs by doubling the block size of a given BIBD by writing the elements of each block twice and appending a suitable design. In 1.5, the second design consists of L complete blocks, and, in (1.6), it consists of all blocks generated by the initial block $\{1, 2, \ldots, v-1\}$ modulo v. In general, it is not necessary to require more of the second design than that its parameters be related correctly to the first's. One can also use s copies of the first design and t copies of the second, where s and t are nonnegative integers, to produce the final BPTD. The resulting BPTD, like those constructed in 1.5 and 1.6, has the property that any block containing repeated elements contains no single elements. We summarize this construction in the following result.

Theorem 2.1: *If there exists a BIBD with parameters* $(V, b_1, \rho_1, K, \lambda_1)$ *and a BIBD with parameters* $(V, b_2, \rho_2, \frac{K}{2}, \lambda_2)$*, then one can construct a BPTD with parameters* $(V; sb_1, tb_2, B; s\rho_1, t\rho_2, R; K, s\lambda_1 + 4t\lambda_2)$ *for any nonnegative integers* s *and* t*. In the resulting BPTD, no block with repeated elements contains single elements.*

For example, there exist BIBDs with parameters (16, 30, 15, 8, 7) and (16, 20, 5, 4, 1) ([**14**]), from which we can construct a BPTD with parameters (16; 30, 20, 50; 15, 5, 25; 8, 11). Observe that the second BIBD is of a form different from that in results 1.5 or 1.6.

Finally, the BPTD$(8; 16, 4, 20; 8, 1, 10; 4, 4)$

1	3	5	7	1	1	1	1	1	1	1	1	2	2	2	2	2	2	2	2
1	3	5	7	3	3	3	3	4	4	4	4	3	3	3	3	4	4	4	4
2	4	6	8	5	5	6	6	5	5	6	6	5	5	6	6	5	5	6	6
2	4	6	8	7	8	7	8	7	8	7	8	7	8	7	8	7	8	7	8

demonstrates that there exist BPTDs not arising from (2.1) but still having the property that no block with repeated elements contains single elements.

3. Necessary Conditions

In a BPTD$(V; b_1, b_2, B; \rho_1, \rho_2, R; K, \Lambda)$, there occurs at least one repeated element in each of the b_2 blocks. Thus $b_2 \leq V\rho_2$, the total number of such occurrences. At the same time, b_2 cannot exceed B.

Let M be the maximum number of repeated elements in any single block of a BPTD. Then $V\rho_2 \leq Mb_2$. Combining this with the upper bound yields

$$\frac{V\rho_2}{M} \leq b_2 \leq \min(V\rho_2, B). \tag{3.1}$$

Since $2M \leq K$, we also have

$$\frac{V\rho_2}{\left\lfloor \dfrac{K}{2} \right\rfloor} \leq b_2 \leq \min(V\rho_2, B). \tag{3.2}$$

Let c_l be the number of blocks with l repeated elements (and, consequently, $K-2l$ single elements). For instance, $c_0 = b_1$ and

$$\sum_{l=1}^{M} c_l = b_2. \tag{3.3}$$

Furthermore, the total number of repeated elements is

$$\sum_{l=1}^{M} c_l l = V\rho_2. \tag{3.4}$$

As usual, we say that the pair (x, y) occurs singly within a block that contains both x and y with multiplicity one. Let a_t be the number of pairs that appear singly t times in the design. Since each element appears in ρ_1 blocks with multiplicity one, and no pair appears singly more than Λ times, $a_t = 0$ for $t > \min(\rho_1, \Lambda)$. Thus

$$\sum_{t=0}^{\min(\Lambda,\rho_1)} a_t = \binom{V}{2} \tag{3.5}$$

In a block with l repeated elements, $\binom{K-2l}{2}$ pairs occur singly. Thus the total number of single occurrences of pairs is

$$\sum_{t=1}^{\min(\Lambda,\rho_1)} a_t t = \sum_{l=0}^{M} c_l \binom{K-2l}{2}. \tag{3.6}$$

In a ternary design, every multiple occurrence of a pair contributes an even number of its total occurrences. Thus, each pair must occur singly an odd number of times if Λ is odd, and an even number of times if Λ is even. That is, $a_t = 0$ if t and Λ have different parity.

Lemma 3.7: *If K equals 4 or 5, and if Λ is odd, then a necessary condition for a BPTD to exist is that*

$$4b_2 \le B\binom{K}{2} - \binom{V}{2} - V\rho_2\binom{K-2}{2}. \tag{3.7}$$

Proof: There can be at most 2 repeated elements per block, so (3.3) and (3.4) become a 2×2 linear system in c_1 and c_2 from which we find that

$c_1 = 2b_2 - V\rho_2$. Because Λ is odd, we need sum only over the odd integers in (3.5) and (3.6). Consequently,

$$\begin{aligned}\binom{V}{2} &= a_1 + a_3 + a_5 + \cdots \\ &\le a_1 + 3a_3 + 5a_5 + \cdots \\ &= c_0\binom{K}{2} + c_1\binom{K-2}{2} + c_2\binom{K-4}{2} \\ &= (B - b_2)\binom{K}{2} + (2b_2 - V\rho_2)\binom{K-2}{2}.\end{aligned}$$

Because $\binom{K}{2} - 2\binom{K-2}{2} = 4$, the above simplifies to (3.7). ■

This condition is not sufficient, as we will see in a later section.

In a private communication, Professor Billington has told the authors that (3.7) is part of a result appearing in her upcoming paper.

As a further application of (3.3), (3.4), (3.5), and (3.6), we have the following result (which can be stated without reference to the concept of a BPTD).

Lemma 3.8: *Let K equal 4 or 5. The following conditions are necessary for the existence of a BPTD, or, for that matter, a BTD.*

(3.8.a) *If Λ is even, then $V\rho_2$ is even.*

(3.8.b) *If Λ is odd, then $V\rho_2\binom{K-2}{2} + \binom{V}{2}$ is even.*

Proof: As in the proof of (3.7), $c_1 = 2b_2 - V\rho_2$, regardless of the parity of Λ.

If Λ is even, we need sum only over the even integers in (3.6):

$$\begin{aligned}2a_2 + 4a_4 + 6a_6 + \cdots &= (B - b_2)\binom{K}{2} + (2b_2 - V\rho_2)\binom{K-2}{2} \\ &= B\binom{K}{2} - V\rho_2\binom{K-2}{2} - 4b_2.\end{aligned}$$

Because the left side of the above is even, because $B\binom{K}{2}$ and $4b_2$ are even, and because $\binom{K-2}{2}$ is odd, we conclude that $V\rho_2$ must be even.

If Λ is odd, we sum over the odd integers in (3.5) and (3.6):

$$\begin{aligned}a_1 + a_3 + a_5 + \cdots &= \binom{V}{2} \\ a_1 + 3a_3 + 5a_5 + \cdots &= B\binom{K}{2} - V\rho_2\binom{K-2}{2} - 4b_2.\end{aligned}$$

By subtracting equations and repeating the arguments used when Λ is even, one obtains the desired result. ■

The following lemma is given here, as it shortens the proofs of some later results.

Lemma 3.9: *If Λ equals 6 or 7, and if, within two blocks of a BPTD, there appear $\{x, x, y, y\}$ and $\{x, x, y\}$, then it is necessary that*

$$2(\rho_1 + \rho_2) - \Lambda + 4 \leq B. \tag{3.9}$$

Proof: Since $\Lambda < 8$, the multiplicity with which y appears within the second block must be one. Among the other $B - 2$ blocks there must be ρ_1 single occurrences of x and $\rho_1 - 1$ single occurrences of y. Furthermore, the pair $\{x, y\}$ must occur singly in exactly $\Lambda - 6$ of these blocks. There must also appear among these $B - 2$ blocks $\rho_2 - 2$ double occurrences of x and $\rho_2 - 1$ double occurrences of y. Hence

$$2\rho_1 - 1 - (\Lambda - 6) + 2\rho_2 - 3 \leq B - 2,$$

which, after simplifying, yields the desired result. ■

The following well-known fact ([**7**], equation 1.6) is so useful for our purposes that we state it here as a theorem.

Theorem 3.10: *There exists a BPTD with parameters*

$$(V; b_1, b_2, B; \rho_1, \rho_2, R; V, \Lambda) \tag{3.10.a}$$

if and only if there exists one with parameters

$$(V; 0, b_2, b_2; \rho_1 - b_1, \rho_2, R - b_1; V, \Lambda - b_1). \tag{3.10.b}$$

Proof: Since the block size is V, any block without repeated elements must be a complete block. Given a BPTD with parameters (3.10.a) or (3.10.b), one can construct a BPTD with the other parameters by adding or deleting b_1 complete blocks. ■

4. BPTDs with $\min(K, \Lambda) \leq 3$

As an immediate application of (3.2), we prove the following theorem.

Theorem 4.1: *Let $\min(K, \Lambda) \leq 3$. Then there exists a BPTD with parameters*

$$(V; b_1, b_2, B; \rho_1, \rho_2, R; K, \Lambda)$$

iff $b_2 = V\rho_2$ and a BTD exists with parameters

$$(V, B; \rho_1, \rho_2, R; K, \Lambda).$$

Proof: Assume first that the BPTD exists.

For any Λ, when $K = 3$, (3.2) implies that $b_2 = V\rho_2$.

For any K, when $\Lambda \leq 3$, each of the b_2 blocks must have exactly one repeated element, so again $b_2 = V\rho_2$.

The converse is immediate. ∎

Since theorem (4.1) completely covers the possible values of b_1 and b_2 when neither K nor Λ exceeds 3, we next investigate designs for which at least one of these is greater or equal to 4.

5. Some BPTDs with $K = V =$ 4 or 5 and $\Lambda \geq 4$

In this section, we deal with BPTDs for which $K = V$. In view of theorem 3.10, we first limit ourselves to the case $b_1 = 0$.

Lemma 5.1: *There do not exist BPTDs with the following parameters:*

(5.1.a)	$(4; 0, 7, 7; 1, 3, 7; 4, 5)$
(5.1.b)	$(5; 0, 6, 6; 2, 2, 6; 5, 5)$
(5.1.c)	$(5; 0, 7, 7; 3, 2, 7; 5, 6)$
(5.1.d)	$(5; 0, 8, 8; 4, 2, 8; 5, 7)$

Proof of (5.1.a) *and* **(5.1.b):** Both (5.1.a) and (5.1.b) fail to satisfy lemma 3.7, so there cannot exist BPTDs with these parameters. ∎

Proof of (5.1.c): Assume that there exists a BPTD with parameters (5.1.c). Then we must have three blocks each with two repeated elements, and among these three blocks, there must be two having a repeated element in common. Without loss of generality (WLOG), we let two blocks contain $\{1, 1, 2, 2\}$ and $\{1, 1, 3, 3\}$, and the third contains either $\{2, 2, 3, 3\}$ (case 1), $\{2, 2, 4, 4\}$ (case 2), or $\{4, 4, 5, 5\}$ (case 3).

Assume case 1. The four remaining blocks must contain the double occurrences of 4 and 5, and, to satisfy $\Lambda = 6$, they must also contain $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$, and $\{1, 2, 3\}$. WLOG, we have the following partial structure:

1	1	2	4	4	5	5
1	1	2	4	4	5	5
2	3	3	1	1	2	1
2	3	3	2	3	3	2
						3

There is only one way to complete the fourth, fifth, and sixth blocks:

1	1	2	4	4	5	5
1	1	2	4	4	5	5
2	3	3	1	1	2	1
2	3	3	2	3	3	2
			5	5	4	3

But now the pair $\{2, 5\}$ cannot appear 6 times. Case 1 is therefore impossible.

Assume case 2. The blocks containing $\{3, 3\}$ and $\{4, 4\}$ are determined by lemma 3.9:

$$\begin{array}{ccccccc}
1 & 1 & 2 & 3 & 4 & 5 & 5 \\
1 & 1 & 2 & 3 & 4 & 5 & 5 \\
2 & 3 & 4 & 2 & 1 & & \\
2 & 3 & 4 & 4 & 3 & & \\
 & & & 5 & 5 & &
\end{array}$$

But then the pair $\{2, 5\}$ cannot appear 6 times. Case 2 is therefore impossible.

Assume case 3. Then the remaining blocks are likewise determined by lemma 3.9:

$$\begin{array}{ccccccc}
1 & 1 & 4 & 2 & 3 & 4 & 5 \\
1 & 1 & 4 & 2 & 3 & 4 & 5 \\
2 & 3 & 5 & 3 & 2 & 1 & 1 \\
2 & 3 & 5 & 4 & 4 & 2 & 2 \\
 & & & 5 & 5 & 5 & 4
\end{array}$$

Again, the pair $\{2, 5\}$ cannot appear 6 times. Case 3 is therefore impossible.
■

Proof of (5.1.d): Assume that a BPTD exists with parameters (5.1.d). Then there must be exactly two blocks with two repeated elements. WLOG, one of these contains $\{1, 1, 2, 2\}$, and the other contains either $\{1, 1, 3, 3\}$ (case 1) or $\{3, 3, 4, 4\}$ (case 2).

Assume case 1. The blocks containing $\{2, 2\}$ and $\{3, 3\}$ are determined by lemma 3.9:

$$\begin{array}{cccccccc}
1 & 1 & 2 & 3 & 4 & 4 & 5 & 5 \\
1 & 1 & 2 & 3 & 4 & 4 & 5 & 5 \\
2 & 3 & 3 & 2 & & & & \\
2 & 3 & 4 & 4 & & & & \\
 & & 5 & 5 & & & &
\end{array}$$

But now the pair $\{4, 5\}$ cannot appear 7 times, so case 1 is impossible.

Assume case 2. Then, by lemma 3.9, the remaining blocks must be of the form

$$\begin{array}{cccccccc}
1 & 3 & 1 & 2 & 3 & 4 & 5 & 5 \\
1 & 3 & 1 & 2 & 3 & 4 & 5 & 5 \\
2 & 4 & 3 & 3 & 1 & 1 & & \\
2 & 4 & 4 & 4 & 2 & 2 & & \\
 & & 5 & 5 & 5 & 5 & &
\end{array}$$

This violates $\rho_1 = 3$, so case 2 is impossible. ■

Note that (5.1.c) and (5.1.d) both satisfy (3.7), so (3.7) is not sufficient for the existence of a BPTD.

Theorem 5.2: *There exists a BPTD*$(4; 6+L-b_2, b_2, 6+L; L, 3, 6+L; 4, 4+L)$ *for* L *a nonnegative integer iff* b_2 *is one of* $\{6, 8, 9, 10, 11, 12\}$ *and* $6+L-b_2 \geq 0$.

Theorem 5.3: *There exists a BPTD*$(5; 5+L-b_2, b_2, 5+L; 1+L, 2, 5+L; 5, 4+L)$ *for* L *a nonnegative integer iff* b_2 *is one of* $\{5, 9, 10\}$ *and* $5+L-b_2 \geq 0$.

Proof of (5.2) *and* **(5.3):** For the existence a BPTD with parameters $(4; 6+L-b_2, b_2, 6+L; L, 3, 6+L; 4, 4+L)$ it is necessary by (3.2) that $6 \leq b_2 \leq 12$; for the existence a BPTD with parameters $(5; 5+L-b_2, b_2, 5+L; 1+L, 2, 5+L; 5, 4+L)$ it is necessary by (3.2) that $5 \leq b_2 \leq 10$. The proof is completed by the examples in the appendix and by Lemmas 3.10 and 5.1. ∎

Note that the condition $K = V = 4$ or 5 and $\Lambda \geq 4$ do not determine that B must be $\Lambda + 2$. For example, if $K = V = \Lambda = 4$, then BTDs exist with $B = 4$ (take 4 complete blocks) or 6 (see appendix).

In general, we can derive bounds on B from the conditions $K = V = 4$ or 5 and $\Lambda \geq 4$. By combining $K = V$ with the standard BTD identities

$$VR = BK$$
$$\rho_1 + 2\rho_2 = R$$
$$\rho_1(K-1) + 2\rho_2(K-2) = V(\Lambda - 1),$$

we obtain

$$\rho_2 = (B - \Lambda)(K-1)/2$$
$$\rho_1 = (K-1)\Lambda - (K-2)B.$$

In the case $V = K = 4$, setting both ρ_1 and $\rho_2 \geq 0$ implies

$$\Lambda \leq B \leq \frac{3}{2}\Lambda; \quad (5.4)$$

and, in the case $V = K = 5$, setting $\rho_2 \geq 0$ and $V\rho_1 \geq B$ (ρ_1 cannot equal zero) yields

$$\Lambda \leq B \leq \frac{5}{4}\Lambda. \quad (5.5)$$

In either case, for any Λ, the lower bounds are attained by the BPTD consisting of Λ complete blocks (although it will simply be a complete binary design). The upper bounds are attained by the BPTDs consisting of t copies of either BPTD$(4; 0, 6, 6; 0, 3, 6; 4, 4)$ or BPTD$(5; 0, 5, 5; 1, 2, 5; 5, 4)$, depending on the case. See the appendix for designs with these parameters.

In a later paper, we will deal BPTDs with $K = V$ and $\Lambda \geq 4$ in a more general setting.

6. BTD$(2t+1, 2t+2; 2, t, 2t+2; 2t+1, 2t+1)$

For some BTDs, there is only one choice of b_1 and b_2. For instance, we can prove the following result.

Theorem 6.1: *There exists a BTD with parameters*

$$(2t+1, 2t+2; 2, t, 2t+2; 2t+1, 2t+1) \tag{6.1.a}$$

if and only if there exists a BPTD with parameters

$$(2t+1; 1, 2t+1, 2t+2; 2, t, 2t+2; 2t+1, 2t+1). \tag{6.1.b}$$

Proof: By (3.2), the only possible values of b_2 in a BTD with parameters (6.1.a) are $2t+1$ and $2t+2$. We will show that $b_2 = 2t+2$ is impossible.

Suppose that $b_2 = 2t+2$, so that every block in the design has at least one repeated element. Letting c_l be as in §3, we have that $L \leq t$, so (3.3) and (3.4) become

$$\sum_{l=1}^{t} c_l = 2t+2. \tag{6.2}$$

and

$$\sum_{l=1}^{t} c_l l = (2t+1)t. \tag{6.3}$$

Because Λ is odd and $\rho_1 = 2$, the only r for which a_t is not zero is $r = 1$. Consequently, (3.5) implies that $a_1 = \binom{V}{2}$. Equation (3.6) then reads

$$\sum_{l=0}^{t} c_l \binom{2t+1-2l}{2} = \binom{2t+1}{2}.$$

Combining this with (6.2) and (6.3), one finds

$$\sum_{l=1}^{t} c_l l^2 = (2t+1)t^2.$$

Because each c_l is nonnegative,

$$\sum_{l=1}^{t} c_l l^2 = \left(\sum_{l=1}^{t} c_l l \right) t$$

implies that $c_l = 0$ for $l < t$, but this is inconsistent with (6.2) and (6.3). ■

Saha and Dey [**18**] prove the existence of a BTD with parameters

$$(p^\alpha, p^\alpha; 1, \frac{1}{2}(p^\alpha - 1), p^\alpha; p^\alpha, p^\alpha - 1)$$

where p is an odd prime. By adjoining a complete block to this design, one can form a BTD as in (6.1.a), and therefore a BPTD as in (6.1.b).

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Appendix: Some BPTDs with $K = V =$ 4 or 5 and $\Lambda \geq 4$.

$(4; 0, 12, 12; 6, 3, 12; 4, 10)$:

1	1	1	2	2	2	3	3	3	4	4	4
1	1	1	2	2	2	3	3	3	4	4	4
2	2	3	1	1	3	1	1	2	1	1	2
3	4	4	3	4	4	2	4	4	2	3	3

$(4; 0, 9, 9; 3, 3, 9; 4, 7)$:

1	1	1	2	2	3	3	4	4
1	1	1	2	2	3	3	4	4
2	3	4	1	3	1	2	1	2
2	3	4	4	4	2	4	3	3

$(4; 0, 10, 10; 4, 3, 10; 4, 8)$:

1	1	1	2	2	3	3	3	4	4
1	1	1	2	2	3	3	3	4	4
2	3	3	3	3	4	1	1	1	1
2	4	4	4	4	4	2	2	2	2

$(4; 0, 11, 11; 5, 3, 11; 4, 9)$:

1	1	1	2	2	3	3	3	4	4	4
1	1	1	2	2	3	3	3	4	4	4
2	2	3	3	3	1	1	1	1	1	2
2	4	4	4	4	2	2	4	3	2	3

$(4; 0, 8, 8; 2, 3, 8; 4, 6)$:

1	1	1	2	2	3	3	3
1	1	1	2	2	3	3	3
2	3	4	3	4	1	1	4
2	4	4	4	4	2	2	4

$(4; 0, 6, 6; 0, 3, 6; 4, 4)$:

1	1	1	2	2	3
1	1	1	2	2	3
2	3	4	3	4	4
2	3	4	3	4	4

$(5; 0, 5, 5; 1, 2, 5; 5, 4)$:

1	2	3	4	5
2	1	1	2	3
2	1	1	2	3
3	4	5	5	4
3	4	5	5	4

$(5; 0, 9, 9; 5, 2, 9; 5, 8)$:

1	1	2	3	3	4	4	5	5
1	1	2	3	3	4	4	5	5
2	3	3	1	1	1	1	1	2
2	4	4	2	2	2	2	3	3
5	5	5	5	4	3	5	4	4

$(5; 0, 10, 10; 6, 2, 10; 5, 9)$:

1	1	2	2	3	3	4	4	5	5
1	1	2	2	3	3	4	4	5	5
2	2	1	3	1	2	1	1	1	1
3	3	4	4	2	4	3	2	2	3
4	5	5	5	4	5	5	5	3	4

Along with many useful suggestions, the referee has kindly supplied us with the efficiency factor $(V\Lambda/RK)$ of each of the designs above and in the introduction. From left to right, top to bottom, they are: $\frac{5}{6}$, $\frac{7}{9}$, $\frac{4}{5}$, $\frac{9}{11}$, $\frac{3}{4}$, $\frac{2}{3}$, $\frac{4}{5}$, $\frac{8}{9}$, and $\frac{9}{10}$. Both designs on pages 2 and 3 have efficiency factor $\frac{3}{4}$.