

Flat extension and ideal projection

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Abstract

A generalization of the flat extension theorems of Curto and Fialkow and Laurent and Mourrain is obtained by seeing the problem as one of ideal projection.

Key words: flat extension, ideal, projection, multivariate, moment problem, moment matrix, Hankel operator.

1. Introduction

The Hamburger moment problem is to determine the existence and uniqueness of a positive measure whose moments

$$\int_{\mathbb{R}^k} x^\alpha d\mu, \quad \alpha \in \mathbb{Z}_+^k.$$

take a prescribed multi-sequence of values $(y_\alpha)_{\alpha \in \mathbb{Z}_+^k}$. Here, x^α is the α th monomial in Π , the vector space of all polynomials on \mathbb{R}^k . For such a measure to exist, it is necessary and sufficient that the linear functional

$$L : \Pi \rightarrow \mathbb{R} : p = \sum_{\alpha} \hat{p}(\alpha) x^\alpha \mapsto \sum_{\alpha} \hat{p}(\alpha) y_\alpha$$

be nonnegative, that is, $Lp \geq 0$ if $p \geq 0$, and for this to occur, it is necessary that the **moment matrix**

$$[y_{\alpha+\beta}]_{\alpha, \beta \in \mathbb{Z}_+^k} \tag{1}$$

be positive semidefinite, that is, $L(p^2) \geq 0$ for all p . This condition is sufficient only under special circumstances, for instance, when every positive polynomial can be written as a sum of squares, as occurs when $k = 1$ (Pólya and Szego, 1976, §6.6) and again when

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(1) has finite rank, since in that case the functional L is finitely atomic, i.e., a linear combination of finitely many shifts of δ and its derivatives. (Laurent and Rostalski, 2012, Theorem 7).

Finitely atomic measures and their moment matrices are the subject of the **truncated moment problem** of Curto and Fialkow (1996), who address when a such a measure is determined by finitely many of its moments. To be precise, Curto and Fialkow work in \mathbb{C}^k and study the relationship between the complex moment matrix

$$\left[\gamma_{\alpha,\beta} := \int_{\mathbb{C}^k} \bar{x}^\alpha x^\beta d\mu \right]_{\alpha,\beta \in \mathbb{Z}_+^k}, \quad (2)$$

its submatrices of the form

$$[\gamma_{\alpha,\beta}]_{|\alpha|,|\beta| \leq n}, \quad (3)$$

and the size of the support of μ . One of their main results is that if the positive semidefinite (3) can be extended **flatly**, meaning without increasing its rank, to a positive semidefinite

$$[\gamma_{\alpha,\beta}]_{|\alpha|,|\beta| \leq n+1},$$

then the later has a unique, positive semidefinite, flat extension (2) and a finitely-atomic representing measure μ , and the rank of the matrix equals the cardinality of the measure's support.

Curto and Fialkow's flat extension theorem in the special case when \mathbb{C} is replaced by \mathbb{R} is generalized by Laurent and Mourrain (2009). Working with moment matrices of the form (1), they replace \mathbb{C} by any field \mathbb{F} and do not require moment matrices to be positive semidefinite. They then prove that if the finite set $C \subset \mathbb{Z}_+^k$ is **connected to one**, meaning

$$C \neq \emptyset, \text{ and } \alpha \in C \setminus \{0\} \implies \alpha - e_j \in C \text{ for some } j, \quad (4)$$

and if

$$[y_{\alpha+\beta}]_{\alpha,\beta \in C} \quad (5)$$

can be flatly extended to

$$[y_{\alpha+\beta}]_{\alpha,\beta \in C^+} \quad (6)$$

where

$$C^+ := C + \{\alpha + e_j : \alpha \in C, j = 1 : k\}$$

and $e_j \in \mathbb{F}^k$ is the vector given by

$$(e_j)_i = \delta_{i,j}, \quad i, j = 1 : k, \quad (7)$$

then (6) has a unique flat extension (1). In case \mathbb{F} is \mathbb{R} , if (6) is positive semidefinite, then so is (1).

(To be consistent with Laurent and Mourrain, we'll say that the monomial space $\Pi_C := \text{span}\{x^\alpha : \alpha \in C\}$ is connected to one if the set $C \subset \mathbb{Z}_+^k$ is connected to one.)

Flat extension is the rank-preserving extension of a quadratic form

$$(p, q) \mapsto L(pq)$$

from one defined for polynomials in some finite-dimensional space B to one defined on all of Π . And, as in the case when B is a monomial space, the dependence relations among the columns of the representing matrix of the quadratic form impose on the moments a set of homogeneous difference equations and force L to vanish on a polynomial ideal of finite codimension. Consequently (see (14)), the quadratic form on Π represented by the

moment matrix (1) has finite rank exactly when $L = LP$ for some **ideal projector** P onto B , that is, a finite-rank projector $\Pi \rightarrow B$ whose kernel is an ideal, and questions of flat extension naturally draw on our understanding of such projections.

As seen in Boor (2005), any ideal projector onto a space B is completely determined by its restriction to

$$B^* := \Pi_0 + \Pi_1 \cdot B \tag{8}$$

(where Π_n is the space of polynomials of total degree n or less). Consequently, every flat extension of (5) to (1) is uniquely determined by the implied extension of (5) to (6). On the other hand, not every projection from B^* to B can be extended to an ideal projection onto B . The flat extension theorems of Curto and Fialkow and Laurent and Mourrain give sufficient conditions for a flat extension of (5) to (6) to have a flat extension to (1), in Curto and Fialkow’s case, that B be a total degree space Π_n , and in Laurent and Mourrain’s case, that B be a monomial space connected to one.

The main result of this paper, Theorem 3.5, is to provide *necessary and sufficient* conditions for the flat extension from B to B^* of a quadratic form to have a flat extension to Π . Although B needn’t have a monomial basis—many examples of bases for ideal projection in approximation theory are nonmonomial—our proofs rely heavily on the results of Laurent and Mourrain. In Examples 3.6 and 3.7, we demonstrate that connection to one is not necessary for the flat extension of (6) and that the necessary and sufficient conditions of Theorem 3.5 can be easy to check.

Shohat and Tamarkin (1943) is the standard reference for the classical moment problem. See Berg (1980) for a concise introduction to the multivariate moment problem from the point of view of harmonic analysis.

When $k = 1$, a matrix of the form (1) is a Hankel matrix. See Iokhvidov (1982) for the general algebraic theory of flat extensions (a.k.a. singular extensions) of Hankel and Toeplitz matrices.

Writing Π as the direct sum of an ideal I and a finite dimensional vector space B is central to the subjects of ideal interpolation and the construction of ideal bases that facilitate the decomposition of a polynomial into its I - and B -components, its so-called **normal form**. For example, see Boor (2005), Boor (2006), Boor and Ron (1991), Boor and Ron (1992), Gasca and Sauer (2000), Kehrein and Kreuzer (2005), Kehrein et al. (2005), Mourrain and Trebuchet (2005), Mourrain (1999), Möller and Sauer (2000a), Möller and Sauer (2000b), Sauer (1998), Sauer (2001), Shekhtman (2009).

Moment matrices for finitely atomic measures and the associated ideal bases have been applied to the optimization of polynomials and the solution of systems of polynomial equations. See Laurent and Rostalski (2012), Lasserre et al. (2009), Laurent (2005), Laurent (2007), for instance.

Analogous to the Hausdorff moment problem, the truncated K -moment [and flat extension] problem is to determine whether a truncated moment matrix has a representing [finitely atomic] measure supported on some compact semi-algebraic set K in \mathbb{C}^k (e.g., Curto and Fialkow, 2000) or in \mathbb{R}^k (e.g., Nie, 2014).

Moments of discrete positive measures and the interpolation of finitely many moments with discrete distributions are of interest both in probability and statistics (Adan et al., 1995), (Norton and Arnold, 1985) and in approximation theory, where the latter is known as cubature (Cools and Rabinowitz, 1993), (Cools, 1999).

2. Definitions and notation.

Here are most of the definitions, including some that are introduced elsewhere.

\mathbb{F} := a field.

k := a positive integer.

$x := (x_1, x_2, \dots, x_k)$: a generic point in \mathbb{F}^k .

$i : j$:=, the integers between i and j , inclusive.

id_S := the identity operator on the set S ; “id” if the domain is clear from context.

1 := the constant polynomial 1.

$$\delta_{x,y} := \begin{cases} 1 & \text{if } x = y \text{ and} \\ 0 & \text{otherwise} \end{cases}$$

\mathbb{Z}_+^k := the set of all k -tuples of nonnegative integers.

multi-sequence := a function $\mathbb{Z}_+^k \rightarrow \mathbb{F}$.

$|\alpha| := \sum_{j=1:k} \alpha_j$ for $\alpha \in \mathbb{Z}_+^k$.

e_j := the vector in \mathbb{F}^k given by $(e_j)_i := \delta_{i,j}$ for $i, j = 1 : k$.

$\mathbb{F}[x_1, x_2, \dots, x_k]$:= the ring of polynomials in k variables over the field \mathbb{F} .

Π := a convenient abbreviation for $\mathbb{F}[x_1, x_2, \dots, x_k]$ if \mathbb{F} and k are clear from context.

S_j := the map $\Pi \rightarrow \Pi : p \mapsto x_j p$, for $j = 1 : k$

monomial space := the span of a set of monomials.

$x^\alpha := \prod_{j=1:k} x_j^{\alpha_j}$, the α th monomial for $\alpha \in \mathbb{Z}_+^k$.

For the polynomial $p \in \Pi$,

$\hat{p}(\alpha) :=$ the coefficient of x^α in the monomial expansion of $p(x)$.

The **total degree** of $p :=$ the nonnegative integer $\max\{|\alpha| : \hat{p}(\alpha) \neq 0\}$.

$\Pi_n :=$ the set of polynomials in Π having total degree less or equal n .

$A \subseteq B :=$ A is a subspace of B , possibly equal to B .

$\text{ideal}(S) :=$ the ideal generated by $S :=$ either a set of polynomials or a variety.

$B + C :=$ the vector space spanned by all sums $b + c$ of elements $b \in B$ and $c \in C$.

$B \cdot C :=$ the vector space spanned by all products bc of elements $b \in B$ and $c \in C$.

$B^* := \Pi_0 + \Pi_1 \cdot B$

B' := depending on context, either the algebraic dual of the vector space B (i.e., the space of all linear maps from B into \mathbb{F}), or the dual of the map $B : C \rightarrow D$ given by

$$B' : D' \rightarrow C' : \lambda \mapsto \lambda B.$$

For $L \in \Pi'$:

moments of $L := \{Lx^\alpha : \alpha \in \mathbb{Z}_+^k\}$.

moment matrix of L . See (1).

$H_\Pi^L :=$ the **Hankel operator** associated to L given by $H_\Pi^L : \Pi \rightarrow \Pi' : b \mapsto L(b \cdot)$. See also (9).

$I_L := \ker H_\Pi^L = \{p \in \Pi : L(p\Pi) = 0\}$.

ideal of **finite codimension** := one whose cosets form a finite dimensional vector space.

Also known as a **zero-dimensional** ideal.

ideal projector := a projector on Π whose kernel is an ideal.

\xrightarrow{b} , **flat extension**. See (10).

$X, b(X)$. See (15), (16).

connected to one. See (4) and the paragraph which follows it.

3. A generalized flat extension theorem

In this section, we generalize an ideal projection theorem of de Boor and the flat extension theorems of Curto and Fialkow and Laurent and Mourrain.

In any field \mathbb{F} , if $C \subseteq \Pi$, and if $L \in (C \cdot C)'$, then the **Hankel operator** associated to L is the map

$$H_C^L : C \rightarrow C' : p \mapsto L(p \cdot) \quad (9)$$

(Laurent and Mourrain, 2009). If B is a finite dimensional subspace of C , and if $\text{rank } H_B^L = \text{rank } H_C^L$, then we say H_C^L is a **flat extension** of H_B^L and write

$$H_B^L \xrightarrow{b} H_C^L. \quad (10)$$

If we consider a basis \mathcal{B} for B to be a row vector of polynomials indexed by the elements of \mathcal{B} themselves, then $\mathcal{B}^T \mathcal{B}$ is a matrix of polynomials whose rows and columns are indexed by \mathcal{B} , and the representer of H_B^L relative to \mathcal{B} is the matrix

$$L(\mathcal{B}^T \mathcal{B}) := [L(bc)]_{b,c \in \mathcal{B}}$$

(i.e., the square matrix whose row b , column c entry is $L(bc)$). Expand \mathcal{B} to a basis $\mathcal{B} \cup \mathcal{P}$ for C , and the corresponding representer of H_C^L is the block matrix

$$L((\mathcal{B} \cup \mathcal{P})^T (\mathcal{B} \cup \mathcal{P})) = \begin{bmatrix} L(\mathcal{B}^T \mathcal{B}) & L(\mathcal{B}^T \mathcal{P}) \\ L(\mathcal{P}^T \mathcal{B}) & L(\mathcal{P}^T \mathcal{P}) \end{bmatrix}. \quad (11)$$

For this and $L(\mathcal{B}^T \mathcal{B})$ have the same rank, it is necessary and sufficient that there exists a matrix A with rows and columns indexed by \mathcal{B} and \mathcal{P} , respectively, so that

$$\begin{bmatrix} L(\mathcal{B}^T \mathcal{B}) & L(\mathcal{B}^T \mathcal{B})A \\ A^T L(\mathcal{B}^T \mathcal{B}) & A^T L(\mathcal{B}^T \mathcal{B})A \end{bmatrix} = \begin{bmatrix} \text{id}_{\mathcal{B}} \\ A^T \end{bmatrix} L(\mathcal{B}^T \mathcal{B}) \begin{bmatrix} \text{id}_{\mathcal{B}} & A \end{bmatrix}. \quad (12)$$

That is, $H_B^L \xrightarrow{b} H_C^L$ if and only if

$$H_C^L = P' H_B^L P \quad (13)$$

with P the projector from C onto B represented by the block matrix

$$\begin{bmatrix} \text{id} & A \end{bmatrix}$$

and P' its dual. In case $\mathbb{F} = \mathbb{R}$, (13) shows that if H_B^L is positive semidefinite, then so is any flat extension of H_B^L .

For this paragraph, assume that H_B^L is invertible. Then, when $H_B^L \xrightarrow{b} H_C^L$, the projector P is uniquely determined by (13). In the special case that $C = \Pi$ then

$$H_B^L \xrightarrow{b} H_{\Pi}^L$$

if and only if

$$H_{\Pi}^L = P' H_B^L P \quad (14)$$

where P is an **ideal projector**, that is, a finite-rank projector whose kernel is an ideal, specifically,

$$\ker P = \ker H_{\Pi}^L = \{p \in \Pi : L(p\Pi) = 0\} =: I_L.$$

Given a projector N from B^* (8) onto $B \subseteq \Pi$, define

$$X_j : B \rightarrow B : b \mapsto N(x_j b), \quad j = 1 : k, \quad (15)$$

If the operators X_j commute, then we can unambiguously define the operators

$$X^\alpha := \prod_{j=1}^k X_j^{\alpha(j)}$$

for any multiinteger α (taking X^0 to be id_B) and

$$b(X) := \sum_{\alpha} \hat{b}(\alpha) X^\alpha \quad (16)$$

for any polynomial b .

Proposition 3.1. *If B is a finite-dimensional subspace of $\mathbb{F}[x_1, x_2, \dots, x_k]$ and N is a projector from B^* onto B , then N is the restriction to B^* of a ideal projector P onto B if and only if both*

$$X_i X_j = X_j X_i \quad \forall i, j = 1 : k \quad (17)$$

and

$$b = b(X)N1 \quad \forall b \in B. \quad (18)$$

In that case, P is unique and is given by

$$Pp := p(X)N1 \quad (19)$$

Proposition 3.1 is a generalization of the earlier result Mourrain (1999, 3.1) in which B assumed to be connected to one. Condition (18) is a modification of the conclusion of Mourrain (1999, 3.2).

Proof. If $N = P|_{B^*}$ for some ideal projector, then (17) and (18) follow from Boor (2005, pp. 61-62).

Conversely, suppose (17) and (18). Then the map P given by (19) is well-defined and satisfies $P|_B = \text{id}_B$. Its kernel is an ideal, since

$$P(pq) = (pq)(X)N1 = q(X)p(X)N1 = q(X)P(p).$$

To see that $P|_{B^*} = N$, observe that

$$P1 = 1(X)N1 = N1$$

and that, for $b \in B$ and $j = 1 : k$,

$$P(x_j b) = (X_j b(X))N1 = X_j(b(X)N1) = X_j b = N(x_j b).$$

The uniqueness of P follows from Boor (2005, 1.7). \square

We next see that when such an N can be extended to an ideal projection P , the ideal $\ker P$ is generated by $\ker N$. This generalizes Boor (2005, 2.2), modifying the original proof to remove the requirement that $1 \in B$.

Proposition 3.2. *If P is an ideal projector with range B and $N := P|_{B^*}$, then*

$$\ker P = \text{ideal}(\ker N).$$

The generating set $\{(\text{id} - N)b\}_b$, where b ranges over a basis for B^* , is called a **border basis** for $\ker P$. (Kehrein and Kreuzer, 2005), (Kehrein et al., 2005).

Proof. Since $\ker N$ lies in the ideal $\ker P$,

$$\text{ideal}(\ker N) \subset \ker P.$$

Let $F := \text{span}(B \cup 1)$. Since F contains 1,

$$\Pi = \bigcup_{m \geq 0} \Pi_m \cdot F,$$

and so, to prove

$$\ker P \subset \text{ideal}(\ker N),$$

it will suffice to show by induction that

$$(\Pi_m \cdot F) \cap \ker P \subset \text{ideal}(\ker N) \tag{20}$$

for all nonnegative integers m .

The case $m = 0$ is straightforward:

$$(\Pi_0 \cdot F) \cap \ker P \subset B^* \cap \ker P = \ker N.$$

Assume (20) is true for some $m \geq 0$. Because F is closed under addition,

$$\Pi_{m+1} \cdot F = \Pi_1 \cdot (\Pi_m \cdot F),$$

and so, if $p \in \Pi_{m+1} \cdot F$, then there exist polynomials $q_i \in \Pi_m \cdot F$ so that

$$p = q_0 + \sum_{i=1:d} x_i q_i,$$

or, taking $x_0 := 1$,

$$p = \sum_{i=0:d} x_i q_i = \sum_{i=0:d} x_i (Pq_i + (\text{id} - P)q_i).$$

Observe that $(\text{id} - P)q_i$ is in both $\ker P$ and $\Pi_m \cdot F + B \subset \Pi_m \cdot F$. The induction hypothesis therefore implies that $(\text{id} - P)q_i$, and therefore $x_i(\text{id} - P)q_i$, belong to $\text{ideal}(\ker N)$. If p is also in $\ker P$, then $\sum_{i=0:d} x_i Pq_i$ is in both $\ker P$ and $\Pi_1 \cdot B \subset B^*$, and hence in $\ker N$. Consequently,

$$p \in \ker N + \text{ideal}(\ker N) \subset \text{ideal}(\ker N),$$

completing the inductive step. \square

Our main result depends heavily on a result of Laurent and Mourrain (2009), which we restate here in a slightly different form.

Proposition 3.3. *Let $L \in (B^* \cdot B^*)'$ for some finite dimensional $B \subseteq \Pi$ and let H_B^L be invertible. Suppose that $H_B^L \xrightarrow{b} H_{B^*}^L$, and let N be the projector from B^* onto B satisfying $H_{B^*}^L = N' H_B^L N$ as in (13). Then the operators $\{X_j : j = 1 : k\}$ (15) commute.*

In the original formulation of Proposition 3.3 in Laurent and Mourrain (2009, Lemma 2.1), B was a monomial space containing 1. We reproduce Laurent and Mourrain's original proof below only to demonstrate that this assumption is unnecessary.

Proof. Define

$$F := \text{id}_{B^*} - N : B^* \rightarrow B^*.$$

and

$$S_j : \Pi \rightarrow \Pi : p \mapsto x_j p, \quad j = 1 : k, \quad (21)$$

so that $X_j = NS_j$ for $j = 1 : k$. Then

$$X_i X_j - X_j X_i = N(S_i NS_j - S_j NS_i) = N(S_j FS_i - S_i FS_j). \quad (22)$$

For any $q \in B^*$, because H_B^L is invertible, Nq is the unique element of B for which $H_{B^*}^L q = H_{B^*}^L Nq$. Consequently, (22) equals zero if and only if

$$\begin{aligned} 0 &= H_{B^*}^L (S_j FS_i - S_i FS_j) \\ \Leftrightarrow \forall b \in B^* \quad 0 &= L(b(S_j FS_i - S_i FS_j)) \\ \Leftrightarrow \forall b \in B^* \quad 0 &= L((Nb)(S_j FS_i - S_i FS_j)) \\ \Leftrightarrow \forall b \in B \quad 0 &= L(b(S_j FS_i - S_i FS_j)) \\ &= L((S_j b)FS_i - (S_i b)FS_j) \end{aligned}$$

and this last equals zero since $H_{B^*}^L F = H_{B^*}^L (\text{id} - N) = 0$. \square

Proposition 3.3, when combined with Proposition 3.1 and the preceding remarks, has the following corollary.

Corollary 3.4. *Under the same hypotheses as in Proposition 3.3, L has an extension $K \in \Pi'$ for which*

$$H_{B^*}^L \xrightarrow{b} H_{\Pi}^K$$

if and only if

$$b = b(X)N1 \quad \forall b \in B, \quad (23)$$

and in that case, the extension is unique.

Proof. Suppose first that $H_{B^*}^L \xrightarrow{b} H_{\Pi}^K$. Then, by (13), there exist projectors $Q : \Pi \rightarrow B^*$ and $N : B^* \rightarrow B$ for which

$$H_{\Pi}^K = Q' H_{B^*}^L Q = Q' N' H_B^L N Q. \quad (24)$$

We claim that NQ is the ideal projection onto B with kernel I_K promised by (14). It's easy to see that it's a projection, so all that remains is to observe that, $(NQ)' : B' \rightarrow \Pi'$ is one-to-one, and being invertible, H_B^L is also one-to-one, and so $\ker H_{\Pi}^K = \ker(NQ)$ by (24).

Proposition 3.1 now implies (23) and that the extension of N to an ideal projection NQ is unique. By (14), the flat extension H_{Π}^K of $H_{B^*}^L$ is also unique, completing the first half of the proof.

Conversely, suppose (18) is true. By Propositions 3.1 and 3.3, N can be uniquely extended to an ideal projector $P : \Pi \rightarrow B$. Then $H_B^L \xrightarrow{b} P' H_B^L P =: H_{\Pi}^K$.

Since H_{Π}^K , H_B^L , and $H_{B^*}^L$ all have the same rank, to prove $H_{B^*}^L \xrightarrow{b} H_{\Pi}^K$, we need only show that the map $H_{\Pi}^K : \Pi \rightarrow \Pi'$ is an extension of $H_{B^*}^L : B^* \rightarrow B^{*'}$. For that, note that, for any polynomials $p, q \in B^*$,

$$q' H_{\Pi}^K p = q' P' H_B^L P p = q' N' H_B^L N p = q' H_{B^*}^L p.$$

□

Corollary 3.4 allows us to prove our main result.

Theorem 3.5. *Suppose that $C \subseteq \Pi$ is finite dimensional, that $L \in (C^* \cdot C^*)'$, and that*

$$H_C^L \xrightarrow{b} H_{C^*}^L.$$

Choose a $B \subseteq C$ so that H_B^L is invertible and of the same rank as H_C^L , and, consequently, both $H_{B^}^L$ and $H_{C^*}^L$ are flat extensions of H_B^L . Let N and M be the associated projections from B^* and C^* onto B as in (13), and define X_i for $i = 1 : k$ as in (15).*

Then L can be extended to $K \in \Pi'$ so that $H_{C^}^L \xrightarrow{b} H_{\Pi}^K$ if and only if*

$$M c = c(X) N 1 \quad \forall c \in C, \tag{25}$$

and in that case, the flat extension is unique.

In contrast, Laurent and Mourrain (2009, Theorem 1.4) state that if $H_C^L \xrightarrow{b} H_{C^*}^L$, then for $H_{C^*}^L$ to have a flat extension all of Π , it is sufficient that C be a monomial space connected to one. It follows, of course, that if C is connected to one, then (25) must always be true; indeed, this is the content of Laurent and Mourrain (2009, Lemma 2.2), since one can assume safely that $1 \in B$ (Laurent and Mourrain, 2009, p. 92).

Proof. Suppose that

$$H_B^L \xrightarrow{b} H_{B^*}^L \xrightarrow{b} H_{C^*}^L \xrightarrow{b} H_{\Pi}^K.$$

Then N is extended to an ideal projection P onto B which satisfies

$$H_{\Pi}^K = P' H_B^L P. \tag{26}$$

Since

$$H_{C^*}^L = M' H_B^L M, \tag{27}$$

and $H_{C^*}^L$ is the restriction of $H_{\Pi}^K : \Pi \rightarrow \Pi'$ to a map $C^* \rightarrow C^{*'}$,

$$H_{C^*}^L = (P|_{C^*})' H_B^L P|_{C^*},$$

and since H_B^L is invertible, $M = P|_{C^*}$. Now (25) follows from (19).

Conversely, suppose that (25) is true. Since H_B^L is invertible, $M|_{B^*} = N$, and therefore for all $c \in C$ and $i = 1 : k$, $M(x_i c) = M(x_i c(X) N 1) = N(x_i c(X) N 1) = X_i c(X) N 1$. That is, equation (25) is true for all $c \in C^*$.

Being a special case of (25), condition (23) is satisfied, and so Corollary 3.4 implies that $H_{B^*}^L$ has a unique flat extension to Π . This by itself does not guarantee that H_{Π}^K (26) is an extension of $H_{C^*}^L$ (27), but we prove this by noting that (19) and (25) (with C replaced by C^*) guarantee that $P|_{C^*} = M$. □

It may be worth mentioning that (25)—or (23) in case H_C^L is invertible and $B = C$ —should be straightforward to verify. Calculating the projectors M in (27) and N in $H_{B^*}^L = N'H_B^L N$ amounts to finding the linear dependence relations in H_C^L and $H_{B^*}^L$. If one encounters roundoff error in doing this, one could then calculate (the coefficient vectors of) both sides of (25) at each basis element of C and see if their difference has norm less than some prescribed tolerance. In the following examples, we are able to check (23) and (25) using exact arithmetic.

Example 3.6. Laurent and Mourrain (2009) give the following univariate example. Let $B := \text{span}\{1, x^3\}$, so that $B^* = \text{span}\{1, x^1, x^3, x^4\}$, and choose real numbers a and b so that $b \neq a^2$. Then the rank-2 matrix

$$\begin{array}{c} 1 \quad x^3 \\ 1 \quad \left(\begin{array}{cc} 1 & a \\ a & b \end{array} \right) \\ x^3 \end{array} \quad (28)$$

can be flatly extended to

$$\begin{array}{c} 1 \quad x^3 \quad x^1 \quad x^4 \\ 1 \quad \left(\begin{array}{cccc} 1 & a & 1 & a \\ a & b & a & b \\ 1 & a & 1 & a \\ a & b & a & b \end{array} \right), \\ x^3 \\ x^1 \\ x^4 \end{array} \quad (29)$$

but the latter's only possible extension to Π_4 has rank at least 3:

$$\begin{array}{c} 1 \quad x^3 \quad x^1 \quad x^4 \quad x^2 \\ 1 \quad \left(\begin{array}{ccccc} 1 & a & 1 & a & 1 \\ a & b & a & b & a \\ 1 & a & 1 & a & a \\ a & b & a & b & b \\ 1 & a & a & b & a \end{array} \right). \\ x^3 \\ x^1 \\ x^4 \\ x^2 \end{array}$$

From the point of view of Corollary 3.4, the reason why (29) has no flat extension is not that B fails to be connected to 1, but rather that the corresponding projection N from B^* onto B violates (23) for some b in B . Indeed, from the dependence relations in the columns of (29),

$$N : x^1 \mapsto 1 \quad \text{and} \quad N : x^4 \mapsto x^3,$$

which means that $X1 = N(x^1) = 1$, and $X^3 N1 = X^3 1 = 1 \neq x^3$.

The matrix (28) has flat extensions to $\Pi = \mathbb{C}[x]$. For instance, when $b = -1$,

$$\begin{array}{c} 1 \quad x^3 \quad x^1 \quad x^4 \\ 1 \quad \left(\begin{array}{cccc} 1 & a & -a & 1 \\ a & -1 & 1 & a \\ -a & 1 & -1 & -a \\ 1 & a & -a & 1 \end{array} \right), \\ x^3 \\ x^1 \\ x^4 \end{array} \quad (30)$$

is a flat extension of (28) for which the corresponding projection

$$N : x^1 \mapsto -x^3 \quad \text{and} \quad N : x^4 \mapsto 1.$$

satisfies $X^3N1 = x^3$, as required. Therefore, by Theorem 3.4, (30) has a unique flat extension to Π . In fact, the flat extension is given by the finitely atomic functional

$$L : p \mapsto \left(\frac{1+ia}{2}\right)p(i) + \left(\frac{1-ia}{2}\right)p(-i).$$

In general, (28) can be extended flatly to Π . To do so, it suffices to find real numbers u_0, u_1, m_0, m_1 satisfying

$$1 = m_0 + m_1, \quad a = m_0u_0^3 + m_1u_1^3, \quad b = m_0u_0^6 + m_1u_1^6, \quad u_0 \neq u_1, \quad (31)$$

since then the moment matrix

$$[Lx^{i+j}]_{i,j \geq 0}$$

of the functional

$$L : p \mapsto m_0p(u_0) + m_1p(u_1).$$

is an extension of (28) and has rank 2 by Laurent and Rostalski (2012, Theorem 7) One solution to (31) is to choose some

$$x_0 > \max \left\{ 0, \sqrt[3]{a + \sqrt{|a^2 - b|}} \right\},$$

and then define

$$u_1 := \sqrt[3]{\frac{a^2 - b}{u_0^3 - a}} + a \quad m_0 := \frac{a - u_1^3}{u_0^3 - u_1^3} \quad m_1 := \frac{u_0^3 - a}{u_0^3 - u_1^3}$$

Example 3.7. Let

$$\begin{aligned} B &= \text{span}\{1\} & B^* &= \text{span}\{1, x\} \\ C &= \text{span}\{1, x^3\} & C^* &= \text{span}\{1, x^3, x, x^4\} \end{aligned}$$

and for some real number $a \neq 1$,

$$H_{C^*}^L = \begin{matrix} & 1 & x^3 & x^1 & x^4 \\ \begin{matrix} 1 \\ x^3 \\ x^1 \\ x^4 \end{matrix} & \begin{pmatrix} 1 & 1 & a & a \\ 1 & 1 & a & a \\ a & a & a^2 & a^2 \\ a & a & a^2 & a^2 \end{pmatrix} \end{matrix}.$$

Then, as in the hypotheses of Theorem 3.5,

$$H_B^L \xrightarrow{b} H_C^L \xrightarrow{b} H_{C^*}^L,$$

and also

$$H_B^L \xrightarrow{b} H_{B^*}^L = \frac{1}{x} \begin{pmatrix} 1 & x \\ a & a^2 \end{pmatrix}.$$

The projections N and M can be seen by factoring $H_{B^*}^L$ and $H_{C^*}^L$, respectively:

$$H_{B^*}^L = N' H_B^L N = \begin{pmatrix} 1 \\ a \end{pmatrix} (1)(1 \ a) \quad \text{and} \quad H_{C^*}^L = M' H_B^L M = \begin{pmatrix} 1 \\ 1 \\ a \\ a \end{pmatrix} (1)(1 \ 1 \ a \ a).$$

That is,

$$N : B^* \rightarrow B : \begin{Bmatrix} 1 \\ x \end{Bmatrix} \mapsto \begin{Bmatrix} 1 \\ a \end{Bmatrix} \quad \text{and} \quad M : C^* \rightarrow B : \begin{Bmatrix} 1 \\ x^3 \\ x \\ x^4 \end{Bmatrix} \mapsto \begin{Bmatrix} 1 \\ 1 \\ a \\ a \end{Bmatrix}.$$

N satisfies (23), since $N1 = 1$, but M violates (25), because $Mx^3 = 1$ and $X^3N1 = X^31 = N(xN(xN(x \cdot 1))) = a^3$. Therefore, $H_{B^*}^L$ has a unique flat extension to Π , but $H_{C^*}^L$ has none. Indeed, the flat extension of $H_{B^*}^L$ is obtained by extending L to the functional $p \mapsto p(a)$, but the only possible extension of $H_{C^*}^L$ to Π_4 has rank greater than one.

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