

## Complex Arithmetic

*Math 111, College of Charleston*

A **complex number** is an expression of the form

$$x + iy$$

where  $x$  and  $y$  are real numbers and  $i$  is the “imaginary” square root of  $-1$ . For example,  $2 + 3i$  is a complex number. Just as we use the symbol  $\mathbb{R}$  to stand for the set of real numbers, we use  $\mathbb{C}$  to denote the set of all complex numbers. Any real number  $x$  is also a complex number,  $x + 0i$ ; in set notation,  $\mathbb{R} \subset \mathbb{C}$ .

Assume for this paragraph that

$$z = x + iy.$$

Then  $x$  is called the **real part** of  $z$  and  $y$  is called the **imaginary part** of  $z$ . This is written

$$x = \operatorname{Re}(z) \quad \text{and} \quad y = \operatorname{Im}(z).$$

Two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal. The **conjugate** of  $z$  is the complex number

$$\bar{z} = x - iy$$

and the **absolute value** of  $z$  is

$$|z| = \sqrt{x^2 + y^2}.$$

Note that when  $y = 0$ , this is the same as the absolute value formula for real numbers  $x$ . Note also that, since  $(x + iy)(x - iy) = x^2 - i^2y^2 = x^2 + y^2$ ,

$$z\bar{z} = |z|^2.$$

You can add, subtract, multiply, and divide complex numbers using the usual rules of algebra, keeping in mind that  $i^2 = -1$ .

*Example 1:* Write the sum in  $x + iy$  form:

$$(2 + 3i) + (4 - i) = 6 + 2i$$

*end Example 1*

*Example 2:* Write the sum in  $x + iy$  form:

$$(1 + 5i) + \overline{(2 - 3i)} = 1 + 5i + 2 + 3i = 3 + 8i$$

*end Example 2*

*Example 3:* Write the number in  $x + iy$  form:

$$2(1 - 7i) - \frac{1}{2}(5 + 6i) = 2 - 14i - \frac{5}{2} - 3i = \frac{-1}{2} - 17i.$$

*end Example 3*

*Example 4:* Find the product. Write your answer in  $x + iy$  form:

$$\begin{aligned} (4 - 7i)(2 + 3i) &= 8 - 14i + 12i - 21i^2 \\ &= 8 - (-1)21 - 2i \\ &= 29 - 2i \end{aligned}$$

*end Example 4*

Writing a quotient in  $x + iy$  form requires the use of the conjugate, as the next example demonstrates.

*Example 5: Find the quotient. Write your answer in  $x + iy$  form:*

$$\frac{4 - 7i}{2 - 3i} = \left( \frac{4 - 7i}{2 - 3i} \right) \left( \frac{2 + 3i}{2 + 3i} \right) = \frac{29 - 2i}{4 + 9} = \frac{29}{13} - \frac{2}{13}i$$

*end Example 5*

We sometimes have to use complex numbers in polynomials, as in the next example.

*Example 6: Expand the polynomial:*

$$\begin{aligned} (x - 2 + i)(x - 2 - i) &= x^2 - 2x - ix \\ &\quad - 2x + 4 + 2i \\ &\quad + ix - 2i - i^2 \\ &= x^2 - 4x + 4 - i^2 = x^2 - 4x + 4 + 1 \\ &= x^2 - 4x + 5 \end{aligned}$$

*end Example 6*

The easier way to solve such problems is to use the difference of squares:

*Example 7: Expand the polynomial:*

$$\begin{aligned} (x - 3 + 4i)(x - 3 - 4i) &= ((x - 3) + 4i)((x - 3) - 4i) \\ &= (x - 3)^2 - (4i)^2 \\ &= x^2 - 6x + 9 - 16i^2 = x^2 - 6x + 9 + 16 \\ &= x^2 - 6x + 25 \end{aligned}$$

*end Example 7*

### Exercises

1. Write in  $x + iy$  form:

- |  |   |   |
|--|---|---|
| a. $3 + 2i + 2(1 - i)$                         | b. $3(4 - 5i) - \overline{(2 + 4i)}$                      | c. $-2(2 - i) + \frac{1}{3}(1 + 4i)$                |
| d. $\frac{2}{3}(1 + 8i) + \frac{3}{2}(2 - 7i)$ | e. $\frac{1}{5}(7 - 4i) - \frac{2}{3}\overline{(6 - 5i)}$ | f. $(i + 1)(i - 1)$                                 |
| g. $(2 - 3i)(2 + 3i)$                          | h. $(4 - i)\overline{(5 + 2i)}$                           | i. $(3 + \frac{1}{2}i)(\frac{1}{2} - \frac{1}{3}i)$ |
| j. $(4 + i) \div (1 - 8i)$                     | k. $\overline{(3 - 2i)} \div 2(1 - i)$                    | l. $(1 + 2i) \div (1 - 2i)$                         |
| m. $(3 + 4i) \div (5 + 6i)$                    |   |   |

2. Expand the polynomial:

- |   |   |   |
|---|---|---|
| a. $(x + 2i)(x - 2i)$                           | b. $(x - 3i)(x + 3i)$                         | c. $(x + i\sqrt{5})(x - i\sqrt{5})$         |
| d. $(x - 2 + 4i)(x - 2 - 4i)$                   | e. $(x - 3 + i)(x - 3 - i)$                   | f. $(x + 1 - 2i)(x + 1 + 2i)$               |
| g. $(x + \frac{1}{2} - i)(x + \frac{1}{2} + i)$ | h. $(2x + 1 - i\sqrt{3})(2x + 1 + i\sqrt{3})$ | i. $(x + 3 + i\sqrt{5})(x + 3 - i\sqrt{5})$ |

### Answers

- 1a. 5   1b.  $10 - 11i$    1c.  $-\frac{11}{3} + \frac{10}{3}i$    1d.  $\frac{11}{3} - \frac{31}{6}i$    1e.  $-\frac{13}{5} - \frac{62}{15}i$    1f.  $-2$    1g. 13   1h.  $18 - 13i$    1i.  $\frac{5}{3} - \frac{3}{4}i$    1j.  $-\frac{4+33i}{65}$    1k.  $\frac{1}{4} + \frac{5}{4}i$   
 1l.  $-\frac{3}{5} + \frac{4}{5}i$    1m.  $\frac{39+2i}{61}$    2a.  $x^2 + 4$    2b.  $x^2 + 9$    2c.  $x^2 + 5$    2d.  $x^2 - 4x + 20$    2e.  $x^2 - 6x + 10$    2f.  $x^2 + 2x + 5$    2g.  $x^2 + x + \frac{5}{4}$   
 2h.  $4x^2 + 4x + 4$    2i.  $x^2 + 6x + 14$