No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Unless otherwise indicated, supporting work will be required on every problem worth more than 3 points.

You are expected to know the values of all trig functions at multiples of $\pi/4$ and of $\pi/6$.

1a(13 pts). Find $\frac{dy}{dx}$ along the curve $3 + e^{(xy)} = x^2 - \sin y$

1b(5 pts). Find an equation of the line tangent to the curve in 1a at the point $(2, 0)$.

2(6 pts). Evaluate the limit: $\lim_{x\to 0} \frac{\sin(4x)}{5x}$. You will not receive credit for using l'Hospital's Rule (a technique learned later in this class) on this problem.

3(11 pts). Find the x-values in $[0, 2\pi]$ where the line tangent to $y = x + 2\cos x$ is ...

a. horizontal.

b. parallel to $4x + y = 5$.

4(38 pts). Find the derivative of the following functions. Simplification is not required.

a. $\dfrac{7}{x^8} - 3\sqrt[5]{x} + e^2$

b. $\sec^{-1}(e^x)$

c. $(1 + x)^{\tan x}$

d. $(\cot x)(\tan^{-1} x)$

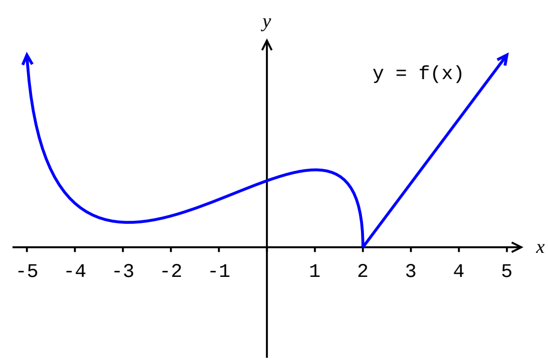
e. $\dfrac{\sec x}{1 + \csc x}$

f. $\ln\big(\tan(\cot^{-1} x)\big)$

g. $\sin^{-1} x + \cos^{-1} x$

h. $\ln\left(\dfrac{x^2 + 1}{(x + 2)^2}\right)$

5(10 pts). The graph of the function $f(x)$ appears in the figure at right. Sketch the graph of $f'(x)$ on the axes provided.



6(17 pts). Here are the values of $f(x)$ and $g(x)$ and their derivatives at $x = 2$, 3, and 4.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
2	4	−2	3	5
3	2	6	4	−3
4	3	−4	2	7

Evaluate the **derivatives** of the following functions at $x = 3$. You can leave unfinished arithmetic (for example "$5(4 - (-7))/(1 + 2 \cdot 3)$") in your final answers.

a. $g(f(x))$

b. $f(x)g(x)$

c. $\dfrac{f(x) - g(x)}{g(x)}$

1a(13 pts). (Source: 3.5.17,19) Think of y as an unspecified function of x, differentiate both sides of $3 + e^{(xy)} = x^2 - \sin y$ with respect to x, and solve for $\frac{dy}{dx}$:

$$e^{(xy)}\left(y + x\frac{dy}{dx}\right) = 2x - \cos y\frac{dy}{dx}$$

$$ye^{(xy)} + xe^{(xy)}\frac{dy}{dx}$$

$$= 2x - \cos y\frac{dy}{dx}$$

$$\cos y\frac{dy}{dx} + xe^{(xy)}\frac{dy}{dx} = 2x - ye^{(xy)}$$

$$(\cos y + xe^{(xy)})\frac{dy}{dx} = 2x - ye^{(xy)}$$

$$\frac{dy}{dx} = \frac{2x - ye^{(xy)}}{\cos y + xe^{(xy)}}$$

1b(5 pts). (Source: 3.5.25-32) At $(2, 0)$, we compute $\frac{dy}{dx} = \frac{4-0}{\cos 0+2} = \frac{4}{3}$, so the point-slope equation of the line is $y = \frac{4}{3}(x - 2)$.

2(6 pts).(Source: 3.3.39-40) Because $4x \to 0$ as $x \to 0$, and $\lim_{h\to 0} \frac{\sin h}{h} = 1$, the limit in question is

$$\lim_{x\to 0} \frac{\sin(4x)}{5x} = \lim_{x\to 0} \frac{\sin(4x)}{4x}\frac{4}{5} = 1 \cdot \frac{4}{5} = \frac{4}{5}.$$

3a(7 pts).(Source: 3.3.33) Set $\frac{dy}{dx}$ equal zero and solve for x. $\frac{dy}{dx} = 1 - 2\sin x = 0$ implies $\sin x = \frac{1}{2}$. The x's in $[0, 2\pi]$ whose sine is $\frac{1}{2}$ are $x = \frac{\pi}{6}$ and $x = \frac{5\pi}{6}$.

3b(4 pts). The line $y = 5-4x$ has slope -4, but $\frac{dy}{dx} = 1-2\sin x = -4$ implies $\sin x = \frac{5}{2} > 1$, which has no solutions.

4. This problem uses the product, quotient, and chain rules, as well as the derivatives of some of the functions in the table below.

$f(x)$	$f'(x)$
x^n	nx^{n-1}
$\ln\|x\|$	x^{-1}
e^x	e^x
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\csc^2 x$
$\sec x$	$\sec x \tan x$
$\csc x$	$-\csc x \cot x$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\cot^{-1} x$	$\frac{-1}{1+x^2}$
$\sec^{-1} x$	$\frac{1}{x\sqrt{x^2-1}}$
$\csc^{-1} x$	$\frac{-1}{x\sqrt{x^2-1}}$

a(4 pts).(Source: 3.1.4,15,19)
$\big(7x^{-8} - 3x^{1/5} + \text{constant}\big)' = -56x^{-9} - \frac{3}{5}x^{-4/5}$.

b(3 pts).(Source: 3.4.23) Chain rule:
$(\sec^{-1}(e^x))' = \frac{1}{e^x\sqrt{(e^x)^2-1}}e^x$, or $\frac{1}{\sqrt{e^{2x}-1}}$.

c(7 pts).(Source: 3.6,43-50) Rewrite before differentiating: $(1+x)^{\tan x} = e^{\ln((1+x)^{\tan x})} = e^{\tan x \ln(1+x)}$. Now differentiate with the chain rule, the product rule, and then the chain rule again:

$$e^{\tan x \ln(1+x)}(\tan x \ln(1+x))' =$$
$$e^{\tan x \ln(1+x)}(\sec^2 x \ln(1+x) + \sec x \cdot \frac{1}{1+x} \cdot 1)$$

d(4 pts).(Source: 3.3.15,16) Remember that $\tan^{-1} x$ is the arctangent of x, not $\frac{1}{\tan x}$. Product rule:

$$(\cot x)'(\tan^{-1} x) + (\cot x)(\tan^{-1} x)'$$
$$= (-\csc^2 x)(\tan^{-1} x) + (\cot x)\left(\frac{1}{x^2+1}\right)$$

e(6 pts).(Source: 3.3.4,11) Quotient rule:

$$\frac{\sec' x(1+\csc x) - \sec x(1+\csc x)'}{(1+\csc x)^2} =$$
$$\frac{\sec x \tan x(1+\csc x) + \sec x \csc x \cot x}{(1+\csc x)^2}$$

f(5 pts).(Source: 3.6.12,3.4.42,3.5.54) $\tan(\cot^{-1} x) = \frac{1}{x}$, because $\cot(\cot^{-1} x) = x$, and so $\ln\big(\tan(\cot^{-1} x)\big) = \ln(\frac{1}{x}) = -\ln x$, the derivative of which is $-\frac{1}{x}$.
If, instead, you differentiated the function as it was originally given, then you'd use the chain rule twice:

$$\frac{1}{\tan(\cot^{-1} x)} \cdot (\tan(\cot^{-1} x))' = \frac{1}{\tan(\cot^{-1} x)} \cdot \sec^2(\cot^{-1} x) \cdot \frac{-1}{x^2+1}.$$

g(2 pts).(Source: 3.5.55,58) $(\sin^{-1} x + \cos^{-1} x)' = \frac{1}{\sqrt{1-x^2}} + \frac{-1}{\sqrt{1-x^2}} = 0.$

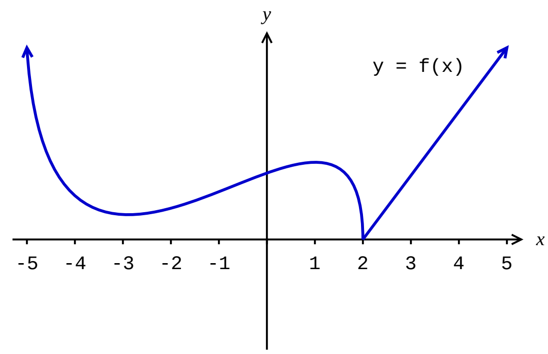
h(7 pts).(Source: 3.6.13) It helps to simplify before differentiating:

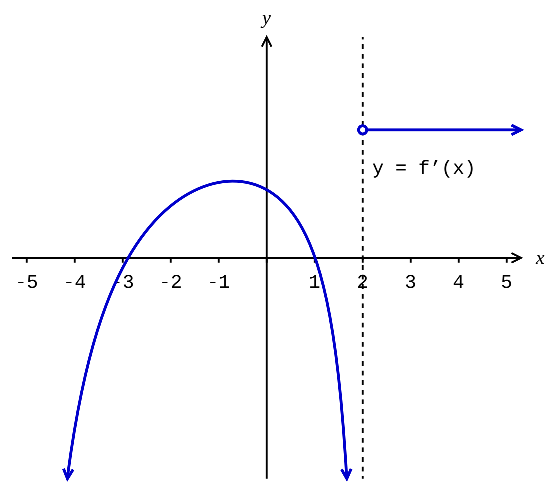
$$\ln\left(\frac{x^2+1}{(x+2)^2}\right) = \ln(x^2+1) - \ln((x+2)^2) = \ln(x^2+1) - 2\ln(x+2).$$

Now the derivative is

$$\frac{1}{x^2+1}\cdot 2x - 2\frac{1}{x+2}\cdot 1.$$

5(10 pts). (Source: 2.8.3,5,7) Here's the graph of f and its derivative.





Note that when $y = f(x)$ has positive [negative] slope, $y = f'(x)$ has positive [negative] altitude. The line tangent to $y = f(x)$ is horizontal near $x = -3$ and $x = 1$, so $y = f'(x)$ has zeros near -3 and 1. $f' \to -\infty$ as $x \to 2^-$ and as $x \to -\infty$, because the tangent line is becoming vertical and negatively sloped. To the right of 2, f' is a positive constant, since f has positive constant slope. $f'(2)$ does not exist because at $x = 2$, f has a corner (or cusp, or vertical tangent line, depending on your point of view).

6.(Source: 3.2.43, 3.4.63) In each part below, we must differentiate the given function, and then evaluate the result at $x = 3$. Most errors on this problem were due to misunderstanding function notation, so read these solutions carefully.

6a(4 pts). Chain Rule: At $x = 3$, $g'\big(f(x)\big)f'(x)$ is $g'\big(f(3)\big)f'(3) = g'(2)f'(3) = 5\cdot 6 = 30$

6b(5 pts). Product Rule: At $x = 3$,
$f'(x)g(x) + f(x)g'(x)$ becomes
$f'(3)g(3) + f(3)g'(3) = 6\cdot 4 + 2\cdot(-3) = 18$

6c(8 pts). Quotient Rule: The derivative is

$$\frac{\big(f'(x) - g'(x)\big)g(x) - \big(f(x) - g(x)\big)g'(x)}{\big(g(x)\big)^2}$$

At $x = 3$, this is

$$\frac{\big(f'(3) - g'(3)\big)g(3) - \big(f(3) - g(3)\big)g'(3)}{\big(g(3)\big)^2} = \frac{(6-(-3))\cdot 4 - (2-4)\cdot(-3)}{4^2} = -\frac{30}{16}.$$

Note that $\dfrac{f(x) - g(x)}{g(x)} = \dfrac{f(x)}{g(x)} - \dfrac{g(x)}{g(x)} = \dfrac{f(x)}{g(x)} - 1$, so its derivative is the same as the derivative of $\dfrac{f(x)}{g(x)}$.