MATH 120–02 (Kunkle), Exam 4 — Name: ______________________

100 pts, 75 minutes — Nov 21, 2023 — Page 1 of 1

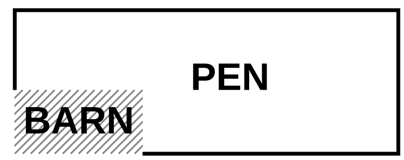
No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Unless otherwise indicated, supporting work will be required on every problem worth more than 3 points.

You are expected to know the values of all trig functions at multiples of $\pi/4$ and of $\pi/6$.

1(20 pts). A farmer plans to enclose an L-shaped pen using fence and the sides of an existing barn. No fencing is needed along the sides of the barn. All angles in the figure are right angles. If the floor of the barn is 4 meters by 8 meters and she has only enough material to make a fence 40 meters long, what is the greatest possible area of the pen?



2(14 pts). Find $q(x)$ if $q''(x) = 2\cos x + 3\sin x$, $q'(0) = 1$, and $q(0) = -1$.

3(6 pts). Approximate $\int_{-1}^{9} 2^x\,dx$ with a Riemann sum using 5 subintervals and their left endpoints.

4(18 pts). Evaluate the definite integral.

a. $\displaystyle\int_1^4 \frac{2x-3}{x}\,dx$ b. $\displaystyle\int_{-2}^{2} \sqrt{4-x^2}\,dx$ c. $\displaystyle\int_{-1}^{2} (3x+2e^x)\,dx$

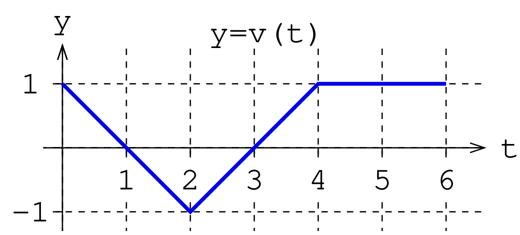
5(8 pts). Find the derivative:

a. $\displaystyle\frac{d}{dx}\int_1^x \sin(t^2)\,dt$ b. $\displaystyle\frac{d}{dx}\int_{x^3}^0 \sin(t^2)\,dt$

6(8 pts). Evalutate the definite integral $\int_1^3 (2f(x)+g(x)+5)\,dx$, given that

$$\int_0^1 f(x)\,dx = 3 \qquad \int_0^3 f(x)\,dx = 2 \qquad \int_3^1 g(x)\,dx = 1$$

7(12 pts). The graph at the right shows the velocity function $v(t)$ for an object moving along a horizontal axis. (v is measured in m/sec and time t is measured in sec.)



a. Over what intervals of time, if any, is the object moving in the positive direction?

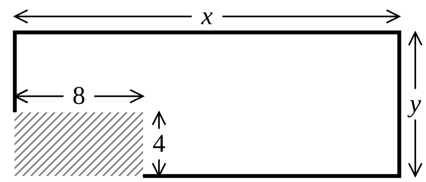
b. Find the net distance traveled by the object between times $t = 0$ and $t = 6$.

c. If the object is at position 14 meters at $t = 0$, what was its position at time $t = 6$?

d. Find the total distance traveled by the object between times $t = 0$ and $t = 6$.

8(14 pts). Evaluate the indefinite integral.

a. $\displaystyle\int (x^3-1)^2\,dx$ b. $\displaystyle\int \frac{1}{x^2+1}\,dx$ c. $\displaystyle\int (\sec x\tan x - e^x + 3\csc^2 x)\,dx$



1(20 pts).(Source: 4.7.18,19) Call the two sides of the pen x and y as in this figure. We wish to maximize the area of the pen:

$$A = xy - 32$$

(that is, xy minus the area of the barn) subject to the constraint that the length of fence is 40:

$$\begin{aligned} 40 &= x + (x-8) + y + (y-4) = 2x + 2y - 12 \\ 26 &= x + y \end{aligned} \tag{1}$$

Solve for y in this constraint and put the result into area:

$$y = 26 - x \quad \Longrightarrow \quad A = x(26-x) - 32 = 26x - x^2 - 32.$$

This construction assumes $x \geq 8$ and $y \geq 4$, and since $y = 26 - x$, this implies $x \leq 22$. Therefore, our problem requires us to find the absolute maximum of A over the interval $8 \leq x \leq 22$.
$A' = 26 - 2x$ exists for all x, and so the only critical points of A are where A' equals zero:

$$A' = 26 - 2x = 0 \quad \Longrightarrow \quad x = 13$$

The absolute maximum of A must occur at 13 or at one of the endpoints, 8 and 22. Evaluate A at these:

x	8	13	22
A	112	137	56

The maximum area of the pen is 137 m^2. **(done)**

You could instead use the first derivative test to conclude the absolute max of A occurs at 13 by observing that 13 is the only critical point of A on $[8, 22]$ (or on $(-\infty, \infty)$, for that matter) and making a sign chart of A', shown at right.

$$\begin{array}{r|l} A' = 2(13-x): & + + + 0 - - - \\ \hline x: & 8 \quad\quad 13 \quad\quad 22 \end{array}$$

2(14 pts).(Source: 4.9.41) Integrate $q''(x)$ to find $q'(x)$, and then use $q'(0) = 1$ to find the constant of integration:

$$\begin{aligned} q'(x) &= 2\sin x - 3\cos x + C & 1 &= -3 + C \quad \Longrightarrow \quad 4 = C \\ 1 &= 2\sin 0 - 3\cos 0 + C & q'(x) &= 2\sin x - 3\cos x + 4 \end{aligned}$$

Repeat to find $q(x)$:

$$\begin{aligned} q(x) &= -2\cos x - 3\sin x + 4x + D & -1 &= -2 + D \quad \Longrightarrow \quad 1 = D \\ -1 &= -2\cos 0 - 3\sin 0 + 4 \cdot 0 + D & q(x) &= -2\cos x - 3\sin x + 4x + 1 \end{aligned}$$

3(6 pts).(Source: 5.2.1) Divide the length of the interval by 5 to find $\Delta x = 10/5 = 2$. Endpoints of the subintervals are $-1, 1, 3, 5, 7, 9$. The left endpoints are $-1, 1, 3, 5, 7$, and so the Riemann sum is

$$\Delta x\big(f(-1) + f(1) + f(3) + f(5) + f(7)\big) = 2(2^{-1} + 2^1 + 2^3 + 2^5 + 2^7)$$

(or 341).

4a(8 pts).(Source: 5.4.29,5.3.35, 4.9.21) $\int_1^4 \left(\frac{2x}{x} - \frac{3}{x}\right)\,dx = \int_1^4 \left(2 - \frac{3}{x}\right)\,dx = (2x - 3\ln|x|)\,\Big|_1^4 =$ $(8 - 3\ln 4) - (2 - 3\ln 1) = 6 - 3\ln 4$. ($\ln 1 = 0$.)

4b(6 pts).(Source: 5.2.37-38) The graph of $y = \sqrt{4 - x^2}$ is the upper half of the circle of radius 2 centered at the origin. To see this, square both sides to obtain $y^2 = 4 - x^2$, or $x^2 + y^2 = 4$. $y = -\sqrt{4 - x^2}$ is the lower half.
The area beneath the semicircle is $\frac{1}{2}\pi 2^2 = 2\pi$.

4c(4 pts).(Source: 5.3.19,5.3.37,4.9.27) $\left(\frac{3}{2}x^2 + 2e^x\right)\Big|_{-1}^{2} = \left(\frac{3}{2}\cdot 4 + 2e^2\right) - \left(\frac{3}{2} + 2e^{-1}\right) = \frac{9}{2} + 2e^2 - 2e^{-1}$.

5a(2 pts).(Source: 5.3.7-10) By the Fundamental Theorem of Calculus, part 1 (FTC1), $\frac{d}{dx}\int_1^x \sin(t^2)\,dt = \sin(x^2)$.

5b(6 pts).(Source: 5.3.17-18) This requires the chain rule. To see how, let $u = x^3$ and $I = \int_{x^3}^0 \sin(t^2)\,dt = -\int_0^u \sin(t^2)\,dt$. By the FTC1,

$$\frac{dI}{du} = -\sin(u^2) = -\sin(x^6),$$

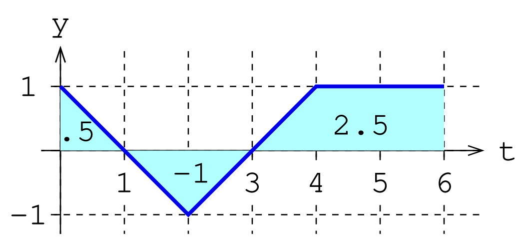
and by the chain rule,

$$\frac{dI}{dx} = \frac{dI}{du}\frac{du}{dx} = -3x^2\sin(x^6).$$

6(8 pts).(Source: 5.2.47-50) $\int_1^3 (2f(x) + g(x) + 5)\,dx =$

$$\int_1^3 2f(x)\,dx + \int_1^3 g(x)\,dx + \int_1^3 5\,dx = 2\int_1^3 f(x)\,dx - \int_3^1 g(x)\,dx + 5(3-1)$$

$$= 2\left(\int_0^3 f(x)\,dx - \int_0^1 f(x)\,dx\right) - 1 + 5\cdot 2 = 2(2-3) - 1 + 10 = 7.$$



7a(2 pts).(Source: 5.4.59-60,5.3.2-3,5.2.33) Object is moving forward when $v(t) \geq 0$, on the intervals $[0, 1]$ and $[3, 6]$.

7b(4 pts). Net distance traveled is net signed area between the velocity curve and the horizontal axis:

$$\int_0^6 v(t)\,dt = 0.5 - 1 + 2.5 = 2$$

7c(2 pts). Stopping position = starting position plus displacement = $14 + 2 = 16$.

7d(4 pts). Counting all areas as positive, total distance is $0.5 + 1 + 2.5 = 4$ ($= \int_0^6 |v(t)|\,dt$).

8a(6 pts).(Source: 5.3.33) Expand the integrand. $\int (x^6 - 2x^3 + 1)\,dx = \frac{1}{7}x^7 - \frac{1}{2}x^4 + x + C$

8b(3 pts). (Source: 5.4.12,41,4.9.33,5.3.42) $\tan^{-1} x + C$

8c(5 pts).(Source: 5.4.27,5.3.32) $\sec x - e^x - 3\cot x + C$

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