MATH 120–02 (Kunkle), Final Exam Name: ______________________

160 pts, 2 hours Dec 6, 2023

No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

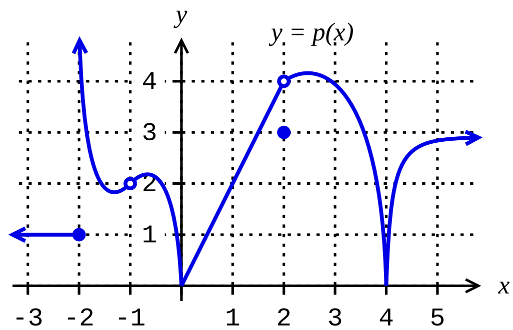
Unless otherwise indicated, supporting work will be required on every problem worth more than 3 points.

You are expected to know the values of all trig functions at multiples of $\pi/4$ and of $\pi/6$.

1(20 pts). Evaluate the limit.

a. $\displaystyle\lim_{x\to 3} \frac{x-4}{(x-3)^2}$ b. $\displaystyle\lim_{x\to\infty} \frac{5x^2}{3x+1}$ c. $\displaystyle\lim_{x\to-\infty} \frac{5x^2}{3x+1}$ d. $\displaystyle\lim_{x\to 0} \frac{\sin x}{\ln(1-x)}$

2(12 pts). Use the graph of $p(x)$ to find the following. Supporting work is not required. Your answer to a-h should be ∞, $-\infty$, a number, or "dne."



a. $\displaystyle\lim_{x\to-2} p(x)$ b. $\displaystyle\lim_{x\to-2^-} p(x)$

c. $\displaystyle\lim_{x\to-2^+} p(x)$ d. $\displaystyle\lim_{x\to-1} p(x)$

e. $\displaystyle\lim_{x\to 4} p(x)$ f. $\displaystyle\lim_{x\to-\infty} p(x)$

g. $p(-1)$ h. $p(2)$

2i. At what x-values, if any, is $p(x)$ discontinuous? Supporting work is not required.

3(12 pts). On the axes provided, sketch the graph of a function $p(x)$ satisfying all of the following:

$p'(x) > 0$ on $(2,\infty)$ $p''(x) > 0$ on $(0,\infty)$ $\displaystyle\lim_{x\to\infty} p(x) = \infty$

$p'(x) < 0$ on $(-\infty,2)$ $p''(x) < 0$ on $(-\infty,0)$ $\displaystyle\lim_{x\to-\infty} p(x) = 0$

4(17 pts). A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 1 m higher than the bow of the boat. If the rope is pulled in at the rate of 1 m/sec, how fast is the boat approaching the dock when it is 4 m from the dock?

(Assume, as suggested by the illustration, that the rope forms a straight line.)

5(20 pts). Differentiate the given function. You are not required to simplify your answers.

a. $2x^5 - \dfrac{2x}{x+4} + \cot(x)$ b. $\ln(2xe^x)$ c. $\sec^3(e^x)$

6(18 pts). Find $\frac{dy}{dx}$ along the curve $\sin^{-1}(xy) = e^{x+y} + 2$.

7(10 pts). Evaluate the definite integral: $\int_0^{\pi} (\cos x + e^x - 2)\, dx$

8(20 pts). Evaluate the indefinite integral.

a. $\int x(2x+3)\, dx$ b. $\int \frac{(\ln x)^3}{x}\, dx$ c. $\int \sin(\pi x)\, dx$

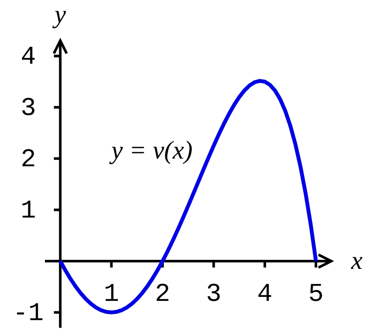
9(8 pts). Find the equation of the line tangent to $y = \tan x$ at the point $(\frac{\pi}{4}, 1)$.

10a(6 pts). Where on the interval $[0, 2\pi]$ is the function $s(x) = 2x + \cos x$ increasing?

10b(6 pts). Where on $[0, 2\pi]$ is the graph of $s(x)$ concave up?

11. Let $v(x)$ be the function graphed here and for x in $[0, 5]$ define the function

$$g(x) = \int_0^x v(t)\, dt.$$



a(3 pts). Find $g'(1)$.

b(4 pts). On what interval(s) is $g(x)$ increasing?

c(4 pts). At what x-value(s) does $g(x)$ have a local min?

1a(6 pts).(Source: 2.2.33) As $x \to 3$, $\frac{x-4}{(x-3)^2} \to$ "$\frac{-7}{0}$," which indicates blow up to $\pm\infty$. When x is near 3, $x - 4$ is near -1 and is therefore negative. The denominator is positive, since it is a square. Therefore $\frac{x-4}{(x-3)^2}$ is negative when x is near 3. The limit must be $-\infty$.
1b(5 pts).(Source: 2.6.26) You can solve this problem by l'Hospital's Rule. Here' s different solution. Factor out highest power of x in the top and bottom:

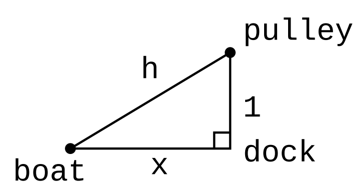
$$\frac{5x^2}{3x+1} = \frac{5x^2}{x(3+\frac{1}{x})} = \frac{5x}{3+\frac{1}{x}} \tag{1}$$

As $x \to \infty$, the top goes to ∞ and the bottom goes to $3 + 0 = 3$, so the limit is ∞.
1c(3 pts).(Source: 2.6.26) Using (1) again, as $x \to -\infty$, $\frac{5x}{3+\frac{1}{x}} \to \frac{-\infty}{3} = -\infty$.
1d(6 pts).(Source: 4.4.15,35) The easiest solution is by l'Hôpital's Rule:

$$\frac{\sin x}{\ln(1-x)} \longrightarrow \text{``}\frac{0}{0}\text{''} \overset{HR}{\hookrightarrow} \frac{\cos x}{-\frac{1}{1-x}} \longrightarrow \frac{1}{-1} = 1$$

Therefore, $\lim_{x\to 0} \frac{\sin x}{\ln(1-x)}$ also equals -1.
2(12 pts).(Source: 2.2.4-9, 2.6.3-4) Parts a-h were worth 1 pt each. Part i was worth 4 pts.
a. DNE. b. 1. c. ∞. d. 2. e. 0. f. 1. g. DNE. h. 3.
Be careful not to confuse function values with limit values. To answer part i, p is discontinuous at those points where function values disagree with limit values, at $x = -2, -1$, and 2. (At $x = 0$ and $x = 4$, p is continuous but not differentiable.)
3(12 pts).(Source: 4.3.29,31) A graph is at https://www.desmos.com/calculator/hyz4r02e2y

4(17 pts).(Source: 3.9.22) Let x denote the distance from the boat to the dock and h the length of rope from the boat to the pulley. The question gives use that $\frac{dh}{dt} = -1$ and asks for $\frac{dx}{dt}$ when $x = 4$. The lengths x and h are related by the Pythagorean theorem:



$$(*) \qquad x^2 + 1 = h^2$$

Differentiate both sides with respect to time t (and cancel the two on both sides):

$$(**) \qquad \not{2}\, x\frac{dx}{dt} = \not{2}\, h\frac{dh}{dt}.$$

When $x = 4$, use equation (*) to find $h = \sqrt{4^2+1} = \sqrt{17}$. Plug these values and $\frac{dh}{dt} = -1$ into (**) and solve:

$$4\frac{dx}{dt} = -\sqrt{17} \quad \Longrightarrow \quad \frac{dx}{dt} = -\frac{\sqrt{17}}{4}$$

That is, at the moment in question, the distance between boat and dock is decreasing $\frac{\sqrt{17}}{4}$ m/sec.

5a(7 pts).(Source: 3.1.5-8,3.2.8,3.3.4) By the quotient rule, $\left(\frac{2x}{x+4}\right)' = \frac{2(x+4)-2x\cdot 1}{(x+4)^2}$, and so

$$\left(2x^5 - \frac{2x}{x+4} + \cot(x)\right)' = 10x^4 + \frac{8}{(x+4)^2} - \csc^2 x$$

5b(6 pts).(Source: 3.6.9) It pays to simplify the function before differentiating:

$$\ln(2xe^x) = \ln 2 + \ln x + \ln(e^x) = \ln 2 + \ln x + x$$

$\ln 2$ is a constant, so the derivative is $\frac{1}{x} + 1$.
5c(7 pts).(Source: 3.4.37-42) $\sec^3(e^x) = f(g(h(x)))$ where f is the cubing function, g is the secant, and h is the exponential function. Differentiate using the chain rule twice:

$$\begin{aligned}(\sec^3(e^x))' &= \overbrace{3\sec^2(e^x)}^{f'(g(h(x)))} \cdot \overbrace{\sec(e^x)\tan(e^x)}^{g'(h(x))} \cdot \overbrace{e^x}^{h'(x)} \\ &= 3e^x \sec^3(e^x)\tan(e^x)\end{aligned}$$

6(18 pts).(Source: 3.5.13,17) Differentiate both sides implicitly with respect to x and then solve for $\frac{dy}{dx}$.

$$\begin{aligned}\frac{1}{\sqrt{1-(xy)^2}}(xy)' &= e^{x+y}(x+y)' \\ \frac{1}{\sqrt{1-x^2y^2}}\left(y + x\frac{dy}{dx}\right) &= e^{x+y}\left(1 + \frac{dy}{dx}\right) \\ y + x\frac{dy}{dx} &= e^{x+y}\sqrt{1-x^2y^2}\left(1 + \frac{dy}{dx}\right) \\ &= e^{x+y}\sqrt{1-x^2y^2} + e^{x+y}\sqrt{1-x^2y^2}\frac{dy}{dx} \\ y - e^{x+y}\sqrt{1-x^2y^2} &= x\frac{dy}{dx} + e^{x+y}\sqrt{1-x^2y^2}\frac{dy}{dx} \\ &= (x + e^{x+y}\sqrt{1-x^2y^2})\frac{dy}{dx} \\ \frac{y - e^{x+y}\sqrt{1-x^2y^2}}{x + e^{x+y}\sqrt{1-x^2y^2}} &= \frac{dy}{dx}\end{aligned}$$

7(10 pts).(Source: 5.3.19,25,37,5.4.21,27)

$$\begin{aligned}\int_0^{\pi} (\cos x + e^x - 2)\,dx &= (\sin x + e^x - 2x)\Big|_0^{\pi} \\ &= (\sin\pi + e^{\pi} - 2\pi) - (\sin 0 + e^0 - 2\cdot 0) \\ &= e^{\pi} - 2\pi - 1\end{aligned}$$

8a(6 pts).(Source: 5.3.27-28,5.4.26) Rewrite the integrand without a product, and then use the power rule:

$$\int (2x^2 + 3x)\,dx = \frac{2}{3}x^3 + \frac{3}{2}x^2 + C$$

8b(8 pts).(Source: 5.5.21) Substitute $u = \ln x$. Then $du = \frac{1}{x}\,dx$, and the integral becomes

$$\int u^3\,du = \frac{1}{4}u^4 + C = \frac{1}{4}(\ln x)^4 + C$$

8c(6 pts).(Source: 5.5.11-12) Substitute $u = \pi x$. Then $du = \pi\,dx$, so that $\frac{1}{\pi}\,du = dx$. The integral becomes

$$\frac{1}{\pi}\int \sin u\,du = -\frac{1}{\pi}\cos(u) + C = -\frac{1}{\pi}\cos(\pi x) + C$$

9(8 pts).(Source: 3.3.21-24) $\frac{dy}{dx} = \sec^2 x$. To evaluate $\frac{dy}{dx}$ at $x = \frac{\pi}{4}$, use either the definition of the secant

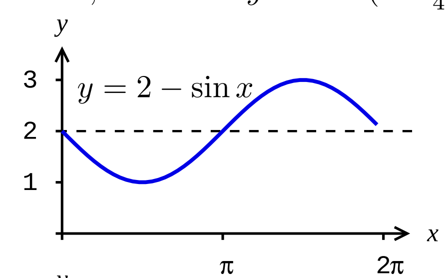
$$\sec(\frac{\pi}{4}) = \frac{1}{\cos(\frac{\pi}{4})} = \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2}$$

or the Pythagorean identity

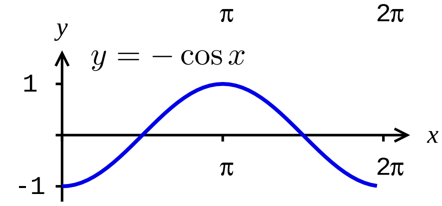
$$\sec^2(\frac{\pi}{4}) = \tan^2(\frac{\pi}{4}) + 1 = 1 + 1 = 2.$$

Slope of the line in question is $\sec^2(\frac{\pi}{4}) = 2$. In point-slope form, the line is $y - 1 = 2(x - \frac{\pi}{4})$.

10a(6 pts).(Source: 4.5.34) $s'(x) = 2 - \sin x$ is positive everywhere, and so s is increasing on $[0, 2\pi]$.



10b(6 pts). The graph of s is concave up where $s''(x) = -\cos x$ is positive. That's $[\frac{\pi}{2}, \frac{3\pi}{2}]$.



11a(3 pts).(Source: 5.3.7-10) By the Fundamental Theorem of Calculus part 1, $g'(x) = f(x)$. Therefore $g'(1) = f(1) = -1$.

11b(4 pts).(Source: 4.3.8,35-36,5.3.3-4) $g(x)$ is increasing where $g'(x) = f(x)$ is positive: $[2, 5]$.

11c(4 pts).(Source: 4.3.8,35-36,5.3.3-4) Since $g(x)$ decreases on $[0, 2]$ and increases on $[2, 5]$, g has a local minimum at $x = 2$.