MATH 120–11 (Kunkle), Exam 2 Name: ______________________

100 pts, 75 minutes Feb 22, 2024

No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points.

You are expected to know the values of all trig functions at multiples of $\pi/4$ and of $\pi/6$.

You will not receive credit for using l'Hospital's Rule (a technique learned later in a calculus course) on any problem on this exam.

1(7 pts). Find y' and y'' if $y = \sin(\pi x^2)$. You are not required to simplify.

2(16 pts). Find $\dfrac{dy}{dx}$ along the curve $\ln|xy| = \dfrac{y}{x^2+1}$. You are not required to simplify.

3a(8 pts). Find the equation of the line tangent to $y = e^x - 3x$ at the point $(x, y) = (2, e^2 - 6)$.

3b(4 pts). Find all points x at which the line tangent to $y = e^x - 3x$ is parallel to the line $y - 2x = 1$.

4. Suppose that, t seconds after a ball is shot vertically upwards from a gun at the top of a tower, its height above ground is $s = 100 - 16(t-2)^2$ feet.

a(8 pts). From what height and with what velocity was the ball fired? Use the correct units in your answers.

b(4 pts). When does the ball reach its maximum height?

c(4 pts). When does the ball hit the ground?

5(4 pts). The figure shows the graphs of the function ℓ and its first two derivatives ℓ' and ℓ''. Identify the graphs of ℓ and ℓ':

graph of ℓ : (circle one) a b c

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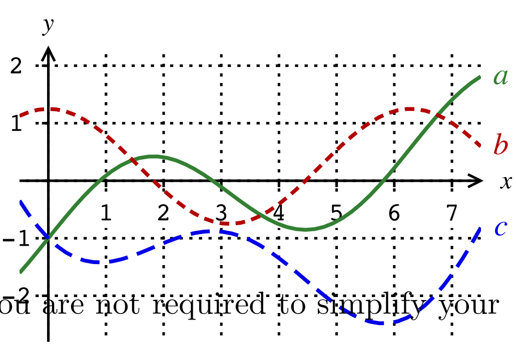

6(45 pts). Find the derivative of the given function. You are not required to simplify your answers.

a. $x\sqrt{x} - \dfrac{2}{x^3} + \sin(\frac{\pi}{4})$

b. $(\tan x + \cot x)^2$

c. $(\ln x)(\sin x)$

d. $\ln(xe^{4x})$

e. $\dfrac{2+e^x}{1-2\tan x}$

f. $\tan^{-1}(\csc^2 x)$

g. $x^3 \cos x \sin^{-1} x$

h. $(\sec x)^x$

i. $x^2 e^{(\frac{1}{3}x^3)}$

1(7 pts).(Source: 3.4.47-50) By the chain rule, $y' = \cos(\pi x^2)(\pi x^2)' = 2\pi x\cos(\pi x^2)$. Then by the product and chain rules,

$$\begin{aligned} y'' &= (2\pi x)'(\cos(\pi x^2)) + (2\pi x)(\cos(\pi x^2))' = 2\pi\cos(\pi x^2) - 2\pi x\sin(\pi x^2)(2\pi x) \\ &= 2\pi\cos(\pi x^2) - (4\pi^2 x^2)\sin(\pi x^2). \end{aligned}$$

2(16 pts).(Source: 3.5.17-20) Think of y as an unknown function of x, differentiate both sides of $\ln|xy| = \frac{y}{x^2+1}$ with respect to x, and solve for $\frac{dy}{dx}$:

$$\begin{aligned} \frac{1}{xy}\left(y + x\frac{dy}{dx}\right) &= \frac{\frac{dy}{dx}(x^2+1) - y(2x)}{(x^2+1)^2} \\ \frac{1}{x} + \frac{1}{y}\frac{dy}{dx} &= \frac{1}{x^2+1}\frac{dy}{dx} - \frac{2xy}{(x^2+1)^2} \end{aligned}$$

$$\begin{aligned} \frac{1}{y}\frac{dy}{dx} - \frac{1}{x^2+1}\frac{dy}{dx} &= -\frac{1}{x} - \frac{2xy}{(x^2+1)^2} \\ \left(\frac{1}{y} - \frac{1}{x^2+1}\right)\frac{dy}{dx} &= -\frac{1}{x} - \frac{2xy}{(x^2+1)^2} \\ \frac{dy}{dx} &= \frac{-\frac{1}{x} - \frac{2xy}{(x^2+1)^2}}{\frac{1}{y} - \frac{1}{x^2+1}} \end{aligned}$$

3a(8 pts).(Source: 3.1.34) Along this curve, $\frac{dy}{dx} = e^x - 3$, which at $x = 2$, equals $e^2 - 3$. The point-slope equation of the line is $y - (e^2 - 6) = (e^2 - 3)(x - 2)$.

3b(4 pts).(Source: 3.1.56,58-60) The line $y = 2x + 1$ has slope 2. Set $\frac{dy}{dx} = 2$ and solve for x:

$$e^x - 3 = 2 \implies e^x = 5 \implies x = \ln 5.$$

4a(8 pts).(Source: 3.7,7,8) Since s equals height, $\frac{ds}{dt} = 0 - 16 \cdot 2(t-2)^1(t-2)' = -32(t-2)$ is the (vertical) velocity of the ball. When the ball was launched at time $t = 0$, its height was $s(0) = 100 - 16(-2)^2 = 36$ ft and its velocity was $s'(0) = -32(-2) = 64$ ft/sec.

4b(4 pts). The ball reaches its max height when $s'(t) = -32(t-2) = 0$, at time $t = 2$ sec.

4c(4 pts). $s(t) = 100 - 16(t-2)^2 = 0 \implies 16(t-2)^2 = 100$

$$\implies (t-2)^2 = \frac{100}{16} \implies t - 2 = \pm\sqrt{\frac{100}{16}} = \pm\frac{10}{4} \implies t = 2 \pm \frac{5}{2}.$$

Since the ball must strike the ground *after* it reaches its maximum height at time 2, disregard $t = 2 - \frac{5}{2}$ and conclude that it hits the ground at time $t = 2 + \frac{5}{2}$, or $\frac{9}{2}$ sec.

5(4 pts).(Source: 2.8.49,50) When the graph of b is horizontal near $x = 0$, neither a nor c is zero; since neither a nor c is b', b must be ℓ''. Near $x = 4.5$, where b is zero, a is horizontal and c is not. b must be the derivative of a, and so $a = \ell'$ and therefore $c = \ell$.
You could also start by observing that c is negative throughout, but neither a nor b always has negative slope. Therefore c must be ℓ, since it is not the derivative of either a or b.
You can see the three functions at https://www.desmos.com/calculator/gd0pmp8mkt.

$f(x)$	$f'(x)$
x^n	nx^{n-1}
$\ln\lvert x\rvert$	x^{-1}
e^x	e^x
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\csc^2 x$
$\sec x$	$\sec x \tan x$
$\csc x$	$-\csc x \cot x$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\cot^{-1} x$	$\frac{-1}{1+x^2}$
$\sec^{-1} x$	$\frac{1}{x\sqrt{x^2-1}}$
$\csc^{-1} x$	$\frac{-1}{x\sqrt{x^2-1}}$

6a(5 pts).(Source: 3.1.4,16,19) Rewrite, then differentiate:
$\left(x^{3/2} - 2x^{-3} + \text{constant}\right)' = \frac{3}{2}x^{1/2} + 6x^{-4}$.

6b(4 pts).(Source: 3.3.8,9. 3.4.7) Chain rule:
$((\tan x + \cot x)^2)' = 2(\tan x + \cot x)^1(\tan x + \cot x)'$
$= 2(\tan x + \cot x)(\sec^2 x - \csc^2 x)$.

6c(4 pts).(Source: 3.3.3) Product rule:
$\frac{dy}{dx} = (\ln x)'(\sin x) + (\ln x)(\sin x)' = \frac{1}{x}\sin x + (\ln x)\cos x$

6d(5 pts). (Source: 3.6.9,19) Simplify before differentiating:

$$\ln(xe^{4x}) = \ln(e^{4x}) + \ln x = 4x + \ln x$$

and so the derivative is $4 + \frac{1}{x}$.

6e(6 pts).(Source: 3.2.23,3.3.9) Quotient rule:

$$\frac{(2+e^x)'(1-2\tan x) - (2+e^x)(1-2\tan x)'}{(1-2\tan x)^2} =$$
$$\frac{e^x(1-2\tan x) - (2+e^x)(-2\sec^2 x)}{(1-2\tan x)^2}$$

6f(3 pts).(Source: 3.5.54) Use the chain rule twice to differentiate this composition of three functions:
$\frac{dy}{dx} = \frac{1}{1+(\csc^2 x)^2}(\csc^2 x)' = \frac{1}{1+(\csc^2 x)^2}2\csc x(\csc x)'$
$= \frac{1}{1+\csc^4 x}2\csc x(-\csc x\cot x)$, or $-\frac{2\csc^2 x\cot x}{1+\csc^4 x}$.

6g(6 pts).(Source: 3.3.15,3.3.more.1) Use the product rule for three functions:

$$(x^3\cos x\sin^{-1} x)' = (x^3)'\cos x\sin^{-1} x + x^3(\cos x)'\sin^{-1} x + x^3\cos x(\sin^{-1} x)'$$
$$= 3x^2\cos x\sin^{-1} x - x^3\sin x\sin^{-1} x + x^3\cos x\frac{1}{\sqrt{1-x^2}}$$

6h(8 pts).(Source: 3.6,45,47) Rewrite $(\sec x)^x$ as $e^{\ln((\sec x)^x)} = e^{x\ln(\sec x)}$. Now differentiate with the chain rule, the product rule, and then the chain rule again:

$$e^{x\ln(\sec x)}(x\ln\sec x)' = e^{x\ln(\sec x)}(\ln(\sec x) + x\frac{1}{\sec x}\sec x\tan x)$$

6i(5 pts).(Source: 3.4.36) Product rule, and then chain rule:

$$(x^2)'(e^{(\frac{1}{3}x^3)}) + (x^2)(e^{(\frac{1}{3}x^3)})' = 2xe^{(\frac{1}{3}x^3)} + x^2e^{(\frac{1}{3}x^3)}(\frac{1}{3}x^3)'$$
$$= 2xe^{(\frac{1}{3}x^3)} + x^2e^{(\frac{1}{3}x^3)}x^2, \text{ or } 2xe^{(\frac{1}{3}x^3)} + x^4e^{(\frac{1}{3}x^3)}$$