No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points.

You are expected to know the values of all trig functions at multiples of $\pi/4$ and of $\pi/6$.

1a(9 pts). Find all critical numbers of the function $g(x) = \dfrac{x}{x^2+9}$.

1b(8 pts). Find the absolute maximum and minimum of $g(x)$ on the interval $[-1, 9]$.

2a(5 pts). Fill in the blank to complete this statement of the Mean Value Theorem:

If $f(x)$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a number c in (a, b) at which ____________________ .

2b(6 pts). Suppose $p(x)$ is continuous and differentiable on $(-\infty, \infty)$, and that $p(-1) = 2$ and $p(1) = 8$. Is it possible that $p'(x) > 3$ for all x? Why or why not?

3a(7 pts). Find the linearization $L(x)$ of the function $f(x) = \frac{1}{x}$ at $a = 2$.

3b(3 pts). Use linear approximation to estimate the number $\frac{1}{2.008}$.

4(20 pts). A Ferris wheel with radius 13 m is rotating at a constant rate. At one moment when a rider is 5 m higher than the center of the wheel, she is rising at the rate of $\frac{1}{2}$ m/sec. At what rate (in radians/sec) is the wheel turning?

5(20 pts). Let $h(x) = 3x^2 + x - \ln x$, and find the following.

a. The domain of $h(x)$.

b. The interval(s) on which h is increasing.

c. The interval(s) on which h is concave up.

d. The x-value(s) at which h has a local maximum, and those at which h has a local minimum. Label these so I can tell which is which.

6(22 pts). Evaluate the following limits.

a. $\displaystyle\lim_{x\to 0} \frac{\sin 3x}{\tan 4x}$ b. $\displaystyle\lim_{x\to 0^+} \frac{\ln x}{x}$ c. $\displaystyle\lim_{x\to\infty} x^2 e^{-x}$

1a(1 pts).(Source: 4.1.36) $g'(x) = \dfrac{x'(x^2+9) - x(x^2+9)'}{(x^2+9)^2} = \dfrac{9-x^2}{(x^2+9)^2}$. Since x^2+9 is nonzero for all real x, both $g(x)$ and $g'(x)$ are defined for all real numbers. The only critical points of g are where $g' = 0$:

$$g'(x) = 0 \implies 9 - x^2 = 0 \implies x = \pm 3.$$

1b(1 pts).(Source: 4.1.54) The absolute extrema of $g(x)$ can occur only at the endpoints of the interval or critical points inside the interval, and so the absolute maximum and minimum must occur in this list of g's values:

x	-1	3	9
$\frac{x}{x^2+9}$	$-\frac{1}{10}$	$\frac{3}{18} = \frac{1}{6}$	$\frac{9}{81+9} = \frac{1}{10}$

The maximum is the largest of these, $\frac{1}{6}$, and the minimum is the smallest, $-\frac{1}{10}$.
Note that a max or a min is a value of $g(x)$, not of x.

2a(2 pts).(Source: 4.2.8,14) The conclusion to the MVT is that there exists c in (a,b) at which

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

2b(1 pts).(Source: 4.2.27) Not possible. Applying the Mean Value Theorem to the function $p(x)$ on the interval $[-1,1]$, we learn that there's a number c in $(-1,1)$ at which

$$p'(c) = \frac{p(1) - p(-1)}{1 - (-1)} = \frac{6}{2} = 3.$$

3a(6 pts).(Source: 3.10.1-4) Differentiate $f(x) = x^{-1}$ to obtain $f'(x) = -x^{-2}$ and evaluate $f(2) = \frac{1}{2}$ and $f'(2) = -2^{-2} = -\frac{1}{4}$. Then

$$L(x) = f(a) + f'(a)(x-a) = f(2) + f'(2)(x-2) = \frac{1}{2} - \frac{1}{4}(x-2).$$

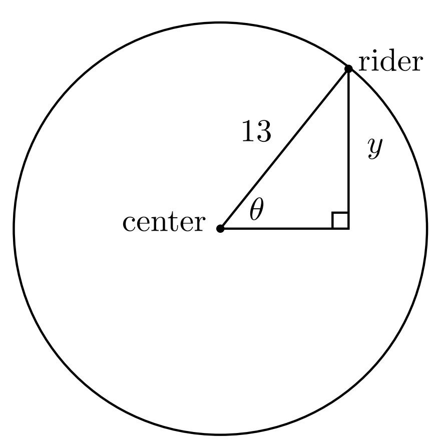
3b(3 pts).(Source: 3.10.24) Since $x = 2.008$ is near $a = 2$, linear approximation tells us that

$$f(2.008) \approx L(2.008) = \frac{1}{2} - \frac{1}{4}(0.008) = 0.5 - 0.002 = 0.498$$

(This compares favorably with the calculator approximation $1/2.008 \approx 0.49800796812$.)

4(13 pts).(Source: 3.9.46) The units radians/sec indicate that we're to find the derivative of an angle, $\frac{d\theta}{dt}$ in the figure, given that $\frac{dy}{dt} = \frac{1}{2}$ when $y = 5$. To find an equation relating these two rates, begin with the equation relating θ and y

$$\frac{y}{13} = \sin\theta \quad \text{or} \quad y = 13\sin\theta$$



and differentiate both sides implicitly with respect to time:

$$\frac{dy}{dt} = 13\cos\theta\frac{d\theta}{dt} \tag{1}$$

When $y = 5$, find the other leg of the triangle x by the Pythagorean theorem:

$$x^2 + 5^2 = 13^2 \quad\Longrightarrow\quad x = 12$$

Then $\cos\theta = \frac{12}{13}$, so $13\cos\theta = 12$. Plug this into (1), along with $\frac{dy}{dt} = \frac{1}{2}$, and solve for $\frac{d\theta}{dt}$:

$$\frac{1}{2} = 12\frac{d\theta}{dt} \quad\Longrightarrow\quad \frac{d\theta}{dt} = \frac{1}{24}.$$

5.(Source: 4.3.17) a(1 pts). Since $3x^2 + x$ is defined for all x, the domain of $h(x)$ is the same as the domain of $\ln x$, that is, $(0, \infty)$.

b(9 pts). To find where $h'(x) = 6x + 1 - x^{-1} \geq 0$, factor $h'(x) = x^{-1}(6x^2 + x - 1) = x^{-1}(2x - 1)(3x + 1)$. On the domain of $h(x)$, both x^{-1} and $(3x + 1)$ are always positive, so $h'(x) \geq 0$ if and only if $2x - 1 \geq 0$ if and only if $x \geq \frac{1}{2}$.

c(6 pts). $h''(x) = 6 + x^{-2} = 6 + \frac{1}{x^2}$ is always positive, so the graph of h is concave up on its entire domain, $(0, \infty)$.

d(3 pts). The only critical point of h is $x = \frac{1}{2}$. Since its graph is concave up, h must have a local minimum at $x = \frac{1}{2}$, by the Second Derivative Test.
You could make the same conclusion by the First Derivative Test, since h is decreasing on $(0, \frac{1}{2}]$ and increasing on $[\frac{1}{2}, \infty)$.

6a(5 pts).(Source: 4.4.14) The limit looks like $\frac{0}{0}$, so try l'Hospital's Rule.

$$\lim_{x\to 0}\frac{f'(x)}{g'(x)} = \lim_{x\to 0}\frac{3\cos 3x}{4\sec^2 4x} = \frac{3\cos 0}{4\sec^2 0} = \frac{3}{4}.$$

Therefore, $\lim_{x\to 0}\frac{\sin 3x}{\tan 4x}$ must also equal $\frac{3}{4}$.

6b(4 pts).(Source: 4.4.21) $\ln x \to -\infty$ as $x \to 0^+$, so the limit looks like $\frac{\infty}{0}$, implying that the limit is infinite. Sign analysis: $\ln x \to -\infty$, so $\ln x < 0$, and $x \to 0^+$, so $x > 0$. Therefore, $\frac{\ln x}{x}$ is negative, and the limit must be $-\infty$.
Note that it is illegitimate to apply l'Hospital's Rule to 6b, since the limit is not of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Using l'Hospital's leads to the wrong answer, since $\frac{1/x}{1} \to +\infty$ as $x \to 0^+$.

6c(8 pts).(Source: 4.4.47, Example 2) Rewrite the product as a quotient and apply l'Hospital's twice:

$$\lim_{x\to\infty} x^2 e^{-x} = \lim_{x\to\infty}\frac{x^2}{e^x} = \text{“}\frac{\infty}{\infty}\text{”} \overset{HR}{\longrightarrow} \lim_{x\to\infty}\frac{2x}{e^x} = \text{“}\frac{\infty}{\infty}\text{”} \overset{HR}{\longrightarrow} \lim_{x\to 0\infty}\frac{2}{e^x} = \text{“}\frac{1}{\infty}\text{”} = 0.$$

Therefore, the original limit must also equal zero.