0.1: The Binomial Theorem and Pascal's Triangle.

The formulas

$$(x+y)^2 = x^2 + 2xy + y^2, \text{ and}$$
$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3.$$

are special instances of the **Binomial Theorem**, which tells us how to expand binomials of the form $(x+y)^n$ using **Pascal's Triangle**:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \end{array}$$

Details and exercises can be found in http://kunklet.people.cofc.edu/MATH111/pascal.pdf

0.1.e1. Generate the next three rows of Pascal's Triangle.

0.1.e2. Expand the following.

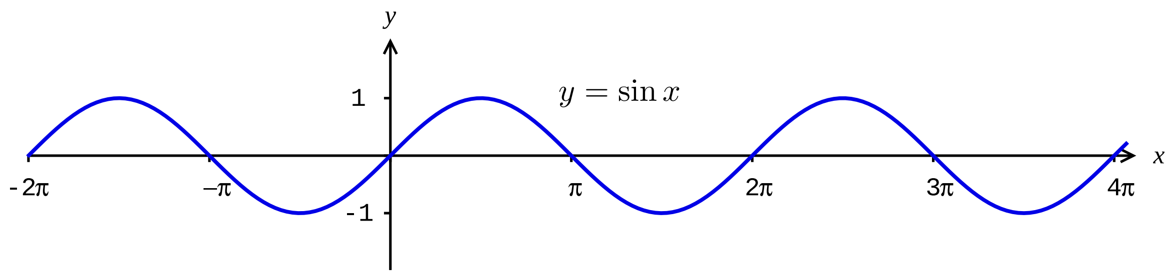
a. $(x+y)^4$ b. $(x+y)^5$ c. $(x-y)^6$

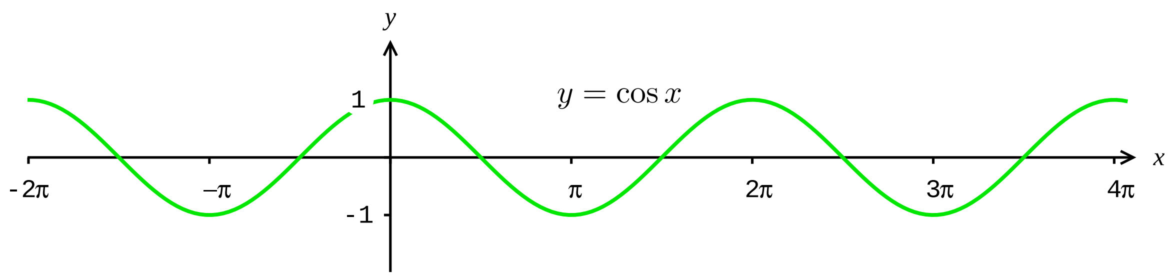
0.1.e3. Expand the following.

a. $(x^2+1)^6$ b. $(x-1)^5(x+1)^5$ c. $\left(x - \dfrac{1}{x}\right)^4$

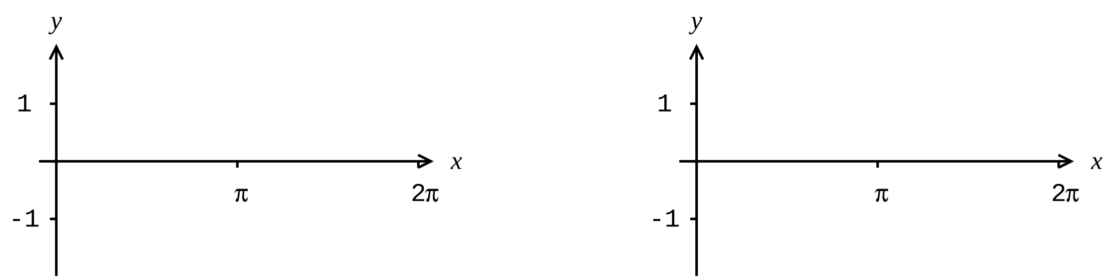
Appendix.D: Trigonometry

The two basic functions in trigonometry are the sine and cosine, graphed here:





Sketching the sine and cosine



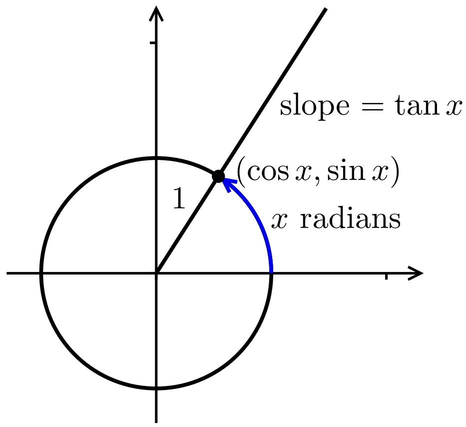
The other four trig functions are defined using sine and cosine:

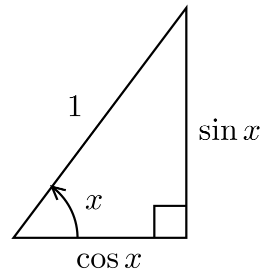
$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

$$\sec x = \frac{1}{\cos x} \qquad \csc x = \frac{1}{\sin x}$$

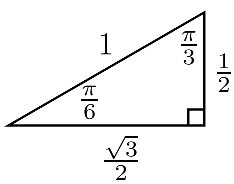
$\sin x$ and $\cos x$ are defined for all real numbers x, but $\tan x$ and $\sec x$ are undefined whenever $\cos x = 0$, and $\cot x$ and $\csc x$ are undefined whenever $\sin x = 0$.

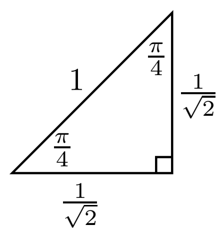
By definition, $\cos x$ and $\sin x$ are the coordinates of the point on the **unit circle** (i.e., the circle of radius one centered at the origin) x radians counterclockwise from the positive horizontal axis.





Consequently, the ray through the origin x radians from the positive horizontal axis has slope $\tan x$, and, when x is an acute angle, $\cos x$ and $\sin x$ are the legs of this right triangle with hypotenuse 1 and interior angle x
These two triangles are often used in examples.





Appendix.D.e1. Find all angles whose sine is $-\frac{1}{2}$.

See also Example 3 and the table which follows on p. A27 of our text.

Trigonometric identities

PYTHAGOREAN IDENTITES

$$\sin^2 x + \cos^2 x = 1$$

$$\tan^2 x + 1 = \sec^2 x$$

$$1 + \cot^2 x = \csc^2 x$$

EVEN & ODD IDENTITIES

$$\cos(-x) = \cos x$$

$$\sin(-x) = -\sin x$$

SUM FORMULAS

$$\sin(x + y) = \sin x \cos y + \sin y \cos x$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

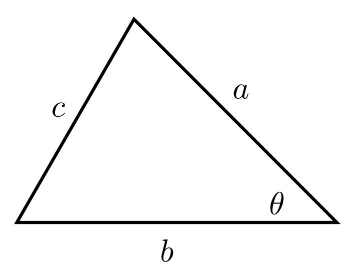
DOUBLE ANGLE FORMULAS

$$\sin(2x) = 2 \sin x \cos x$$

$$\cos(2x) = \cos^2 x - \sin^2 x$$

LAW OF COSINES

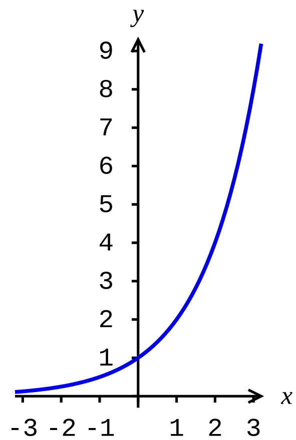
$$c^2 = a^2 + b^2 - 2ab \cos \theta$$



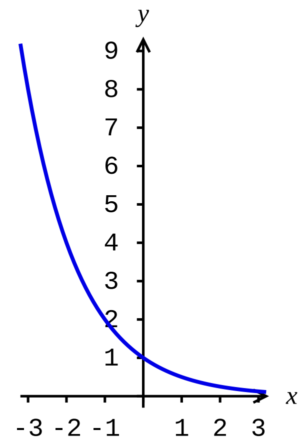
1.4: Exponential Functions

An **exponential function** is one of the form $f(x) = a^x$ for some positive number a.

1.4.e1.

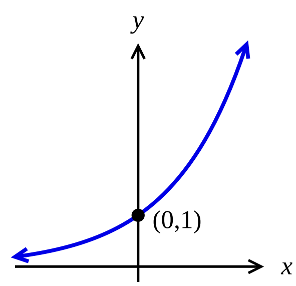


$y = 2^x$

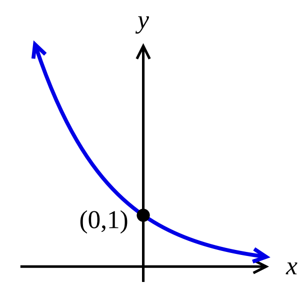


$y = \left(\frac{1}{2}\right)^x$

All exponential functions have the same graph, in the sense that the graph of any one can be obtained from that of any other by rescaling in the x-direction and, possibly, reflecting about the y-axis:



$y = a^x$
$a > 1$



$y = a^x$
$0 < a < 1$

The symbol e represents a mathematical constant, roughly equal to 2.7. For reasons we'll go into in section 3.1, e^x is the most commonly occuring exponential function in calculus.

Exponential growth and decay

A quantity y that depends on time t is said to **grow** or **decay exponentially** if y can be expressed

$$y = Ca^t$$

for some $a > 1$ (in the case of exponential growth) or $0 < a < 1$ (in the case of exponential decay). This can also be expressed with base e:

$$y = Ce^{kt}$$

for some constant $k > 0$ (in the case of exponential growth) or $k < 0$ (exponential decay).

A quantity y grows/decays exponentially if and only if it grows/decays is a constant percent per unit of time. Radioactive isotopes exhibit exponential decay. For example, a sample of Fe-59 (iron 59) will lose about 1.55% of its mass per day.

When a population of living organisms exhibits a constant reproduction rate and constant death rate (as measured as a percent of the population per unit of time), the population grows exponentially when

$$\text{reproduction rate} - \text{death rate} > 0$$

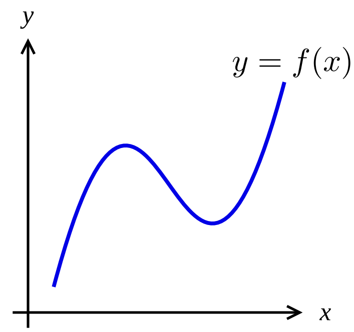
and decays exponentially when

$$\text{reproduction rate} - \text{death rate} < 0.$$

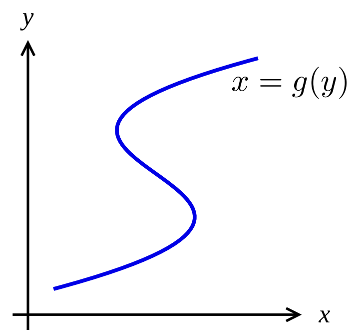
1.4.e2. A population P equals 4 at time $t = 1$ and 12 at time $t = 3$. Assuming exponential growth, express P as a function of t.

1.5: Inverse Functions, including logartihms and inverse trig functions

A curve that passes the **vertical line test** has an equation the form $y = f(x)$ for some function f.

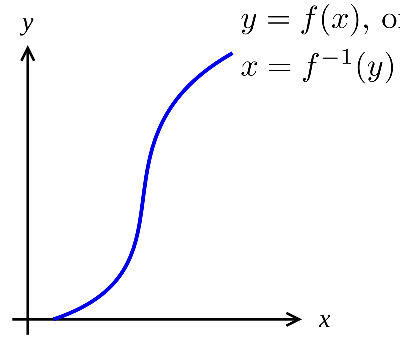


A curve that passes the **horizontal line test** has an equation the form $x = g(y)$ for some function g.



And a curve that passes both has both types of equations. In this case, we say that f is **invertible**, and we name the other function f^{-1} (read "f-inverse") due to the relation

$$\begin{aligned} x &= f^{-1}\big(f(x)\big) \quad \text{if} \quad x \in \operatorname{domain} f = \operatorname{range} f^{-1} \\ y &= f\big(f^{-1}(y)\big) \quad \text{if} \quad y \in \operatorname{domain} f^{-1} = \operatorname{range} f \end{aligned}$$



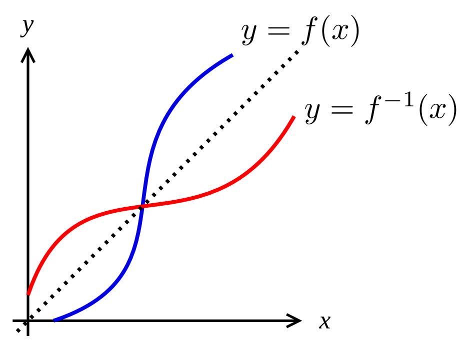
Since the equation

$$y = f^{-1}(x)$$

is obtained from

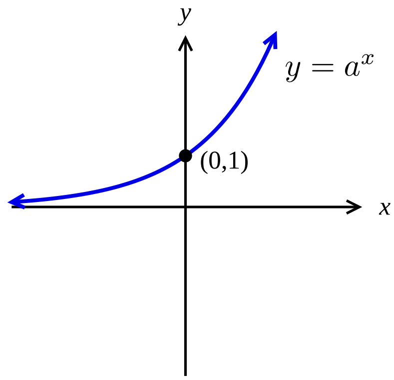
$$x = f^{-1}(y)$$

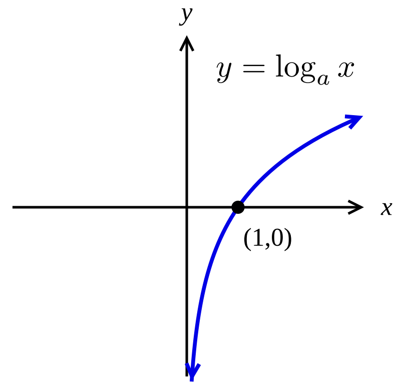
by interchanging x and y, the graph of f^{-1} is obtained by reflecting the graph of f across the line $y = x$.



Logarithms

The inverse of the exponential function a^x is called $\log_a x$ (read "log base a of x").





Some important facts about logs:

1. $\operatorname{domain} \log_a x = (0, \infty) = \operatorname{range} a^x$
2. $\operatorname{range} \log_a x = (-\infty, \infty) = \operatorname{domain} a^x$
3. $x = a^{\log_a x}$ for all $x > 0$.
4. $x = \log_a(a^x)$ for all real numbers x.
5. $\log_a(AB) = \log_a A + \log_a B$ for all A and $B > 0$.
6. $\log_a\left(\frac{A}{B}\right) = \log_a A - \log_a B$ for all A and $B > 0$.
7. $\log_a(A^t) = t \log_a A$ for all $A > 0$ and for any real t.

By convention, "ln" stands for $\log_e$, the **natural logarithm**. "log" stands for $\log_{10}$, the **common logarithm**. In calculus, we work with the natural log almost exclusively, thanks in part to the **change of base** formula:

8. $\log_b x = \dfrac{\ln x}{\ln b}$ for all b and $x > 0$.

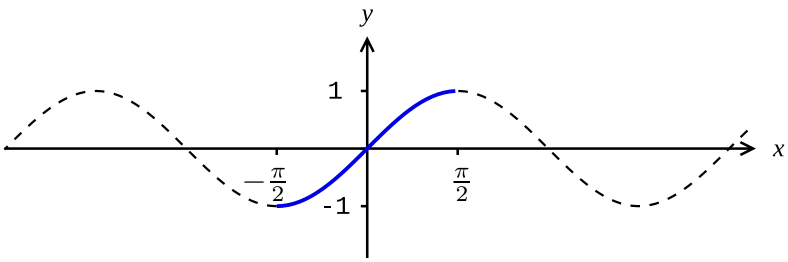
1.5.e1. Simplify the expression $\ln\left(e^2(x^2+1)^3\right)$
Logs are useful in solving equations in which the variable appears in an exponent.

1.5.e2. At what time was the population in example 1.4.e2 equal 8?

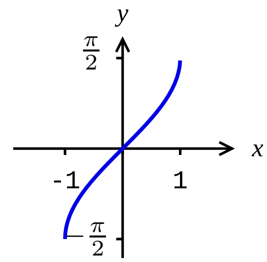
The inverse trig functions

To define inverses of the trig functions, we restrict each to a domain on which its graphs passes the horizontal line test. The three we'll see most often in calculus are the inverses of the sine, tangent, and secant.

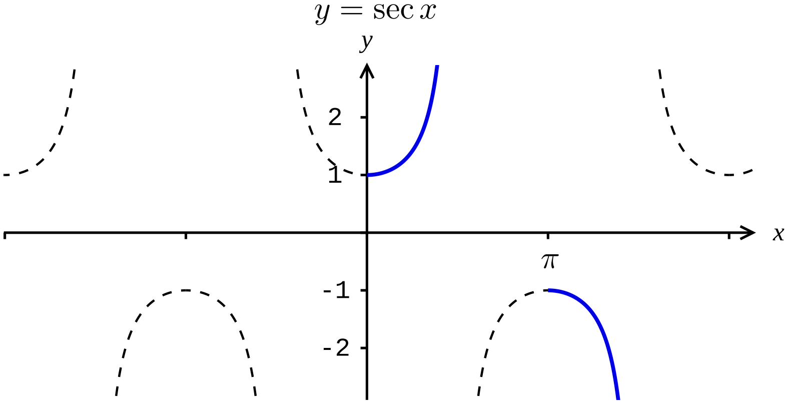
$y = \sin x$



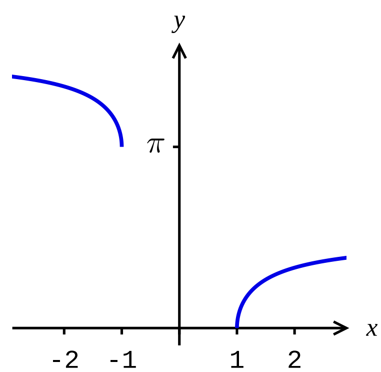
$y = \sin^{-1} x$



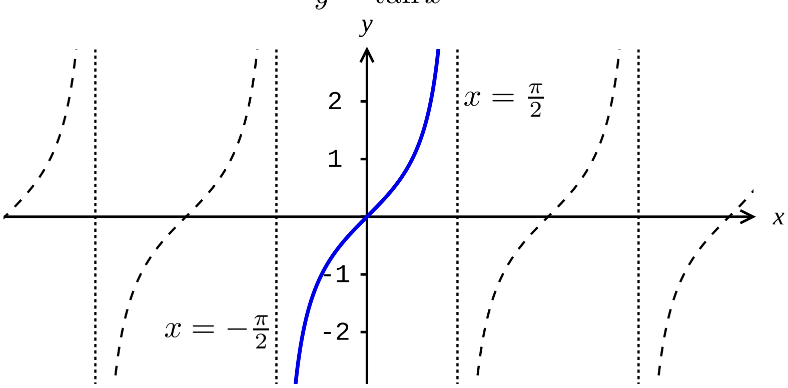
$y = \sec x$



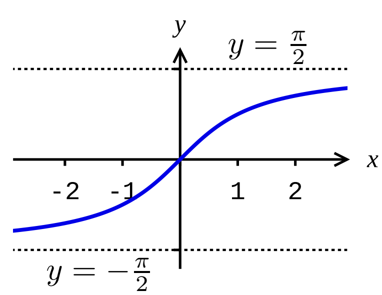
$y = \sec^{-1} x$



$y = \tan x$



$y = \tan^{-1} x$



Traditionally , each of the inverse trig functions has two names. The inverse function of $\sin x$ is called $\sin^{-1} x$ or $\arcsin x$, the inverse function of $\cos x$ is called $\cos^{-1} x$ or $\arccos x$, etc..

As indicated in the graphs above,

	Domain	Range
$\sin^{-1}$	$[-1, 1]$	$\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$
$\sec^{-1}$	$(-\infty, -1] \cup [1, \infty)$	$\left[0, \dfrac{\pi}{2}\right) \cup \left(\pi, \dfrac{3\pi}{2}\right]$
$\tan^{-1}$	$(-\infty, \infty)$	$\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

although, to tell the truth, there are few times in calculus when you'll need to know this. For the sake of completeness, here are the table entries for the other inverse trig functions.

$\cos^{-1}$	$[-1, 1]$	$[0, \pi]$
$\cot^{-1}$	$(-\infty, \infty)$	$(0, \pi)$
$\csc^{-1}$	$(-\infty, -1] \cup [1, \infty)$	$\left(0, \dfrac{\pi}{2}\right] \cup \left[\pi, \dfrac{3\pi}{2}\right)$

It is helpful to remember the definitions of the inverse trig functions in words:

If $-1 \le x \le 1$, then $\sin^{-1} x$ is the angle in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is x. That is,

$$\sin(\sin^{-1} x) = x \quad \text{for any } x \in [-1, 1].$$

If $-1 \le x \le 1$, then $\cos^{-1} x$ is the angle in $[0, \pi]$ whose cosine is x. That is,

$$\cos(\cos^{-1} x) = x \quad \text{for any } x \in [-1, 1].$$

If $-\infty < x < \infty$, then $\tan^{-1} x$ is the angle in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ whose tangent is x. That is,

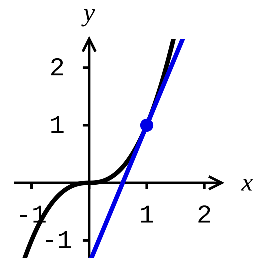
$$\tan(\tan^{-1} x) = x \quad \text{for any real number } x.$$

1.5.e3. Evaluate: $\sin^{-1}\left(-\frac{1}{2}\right)$

2.1: The Tangent Line and Velocity Problems

Calculus can be divided into two parts, the study of derivatives, or *differential calculus*, and the study of integrals, or *integral calculus*. Today we'll see two problems that gave rise to differential calculus.

The **Tangent Line Problem** is this: how can we find the slope of the line tangent to the graph of a function at a given point?



2.1.e1. Approximate the slope of the line tangent to $y = x^3$ at $x = 1$ with slopes of secant lines. What appears to be the true slope of the tangent line?

Here's an interactive graph: https://www.desmos.com/calculator/y4nvy6epy8

h	0.1	0.01	0.001	0.0001
secant slope	3.31	3.0301	3.003001	

h	-0.1	-0.01	-0.001	-0.0001
secant slope	2.71	2.9701	2.997001	

Definition 2.1.1. *The expression*

$$\frac{f(a+h) - f(a)}{h}$$

is called a **difference quotient** *for the function f at the point a. It can be interpreted as the slope of the secant line passing through the graph of f at $x = a$ and $x = a + h$.*

The precalculus formula $\frac{\Delta\text{distance}}{\Delta\text{time}}$ can be used to calculate an object's average velocity over an interval of time. What formula is there for an object's velocity at a single moment, its **instantaneous velocity**?

The **Velocity Problem** is this: what is instantaneous velocity, and how do we find it?

From now on, the word **velocity** will mean instantaneous velocity by default. When we mean **average velocity**, we'll use those words exactly.

2.1.e2. The table below gives the position s at several times of an object moving along an axis (e.g., the real number line). Estimate the object's velocity at time $t = 5$ seconds.

t (sec)	4.0	4.5	5.0	5.5	6.0
s (m)	2.2	2.33	2.46	2.60	2.70

To better *estimate* the object's velocity, we'd need to its position at more times, e.g., every 0.1 second. To have any hope of finding the object's velocity *exactly*, we'd have to know its position at all times, i.e., an expression of position as a function of time.

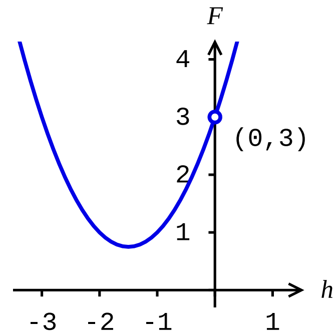
2.1.e3. Suppose an object's position s at time t is given by $s = t^3$. Estimate the object's instantaneous velocity at time $t = 1$.

As this example illustrates, if $f(t)$ is an object's position, then the difference quotient

$$\frac{f(a+h) - f(a)}{h}$$

is *both* the object's average velocity between times a and $a + h$ *and* the slope of secant line passing through the graph of f at times a and $a + h$. Consequently, the two things they approximate must also be equal: the velocity of the object at time a equals the slope of the line tangent to its position curve at the point $\big(a, f(a)\big)$.

2.2: Limits



In Example 2.1.e1, we saw that the slope of the secant line to $y = x^3$ at the points 1 and $1+h$ is

$$F(h) = \frac{(1+h)^3 - 1}{h} = \begin{cases} h^2 + 3h + 3 & \text{if } h \neq 0, \text{ and} \\ \text{undefined} & \text{if } h = 0. \end{cases}$$

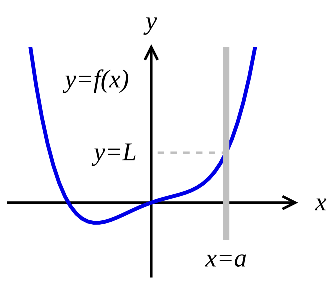
We concluded that the slope of the tangent line is 3, not because $F(0)$ equals 3 (it doesn't), but because 3 seems to be the natural altitude of the graph of F at $h = 0$, based on its altitudes at h's *near* 0. The accepted way to indicate this is to write

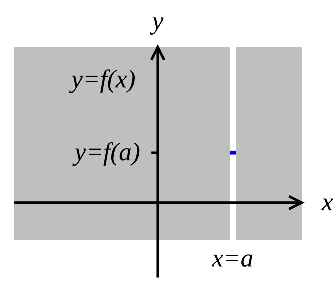
$$\lim_{h \to 0} \frac{(1+h)^3 - 1}{h} = 3,$$

which is read, "the limit as h goes to 0 of $\frac{(1+h)^3-1}{h}$ equals 3."

Limit at a vs. function value at a

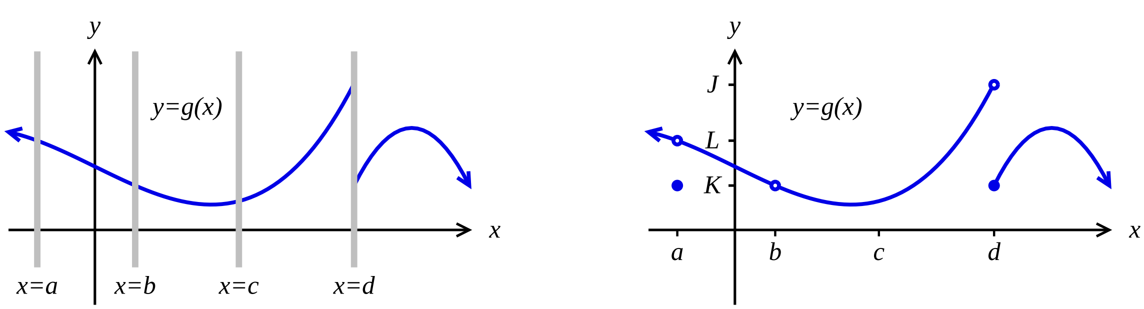
The $\lim_{x \to a} f(x)$ is based on the values of f at x's near a, ignoring what the function does at a itself. The function value $f(a)$ is the opposite. It's based solely on what f does at a, and ignores all other values:





Most of the time, these two are the same thing.

2.2.e1. Consider the limits at a, b, c, and d of the function $g(x)$ graph below.



Example 2.2.e1 illustrates a defining property of limits:

Rule 2.2.1. *The (two-sided) limit*

$$\lim_{x \to a} f(x)$$

exists if and only if both one-sided limits

$$\lim_{x \to a^-} f(x) \text{ and } \lim_{x \to a^+} f(x)$$

exist and are equal. In that case, all three limits are the same.

2.2.e2. Evaluate the limit, or explain why it does not exist: $\lim_{x\to 0} \frac{x}{|x|}$.

Infinite limits

2.2.e3. Evaluate the limit, or explain why it does not exist:

a. $\lim_{x \to 0^+} \ln x$

b. $\lim_{x \to \frac{\pi}{2}^-} \tan x$

Definition 2.2.2. *The line $x = a$ is a* **vertical asymptote** *of the graph of $f(x)$ if either*

$$\lim_{x \to a^-} f(x) \text{ or } \lim_{x \to a^+} f(x)$$

is infinite.

2.2.e4. Sketch the graph of a function h(x) satisfying all of the following:

a. $\lim_{x \to -1} h(x) = 0$

b. $h(-1) = 1$

c. $\lim_{x \to 0} h(x) = 2$

d. $h(0) = 2$

e. $\lim_{x \to 1^-} h(x) = -2$

f. $\lim_{x \to 1} h(x)$ dne

g. $\lim_{x \to 2^-} h(x) = \infty$

h. $\lim_{x \to 2^+} h(x) - \infty$

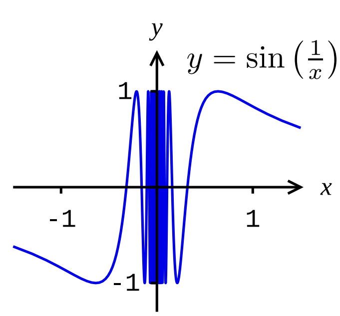
Tip: $\frac{\text{nonzero}}{0}$ indicates a quantity that's blowing up to either $\pm\infty$. To decide between these two, determine the sign of the quotient.

2.2.e5. Evaluate the limit, or explain why it does not exist:

a. $\lim_{x \to 0} \frac{1}{x^2}$ b. $\lim_{x \to 3} \frac{x-4}{(x-3)^4}$ c. $\lim_{x \to 0} \frac{1}{x^3}$ d. $\lim_{x \to -1^+} \frac{x}{x^2 - x - 2}$

Another nonexistent limit

2.2.e6. Explain the graph of $\sin\left(\frac{1}{x}\right)$ and why $\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ does not exist.



2.3: Limit Laws

The Limit Laws are rules that allow us to calculate limit more easily. They come in two types:

I. Examples of some elementary continuous functions.

1. $\lim_{x\to a} x = a$ 2. $\lim_{x\to a} c = c$ 3. $\lim_{x\to a} e^x = e^a$

II. Combination laws. If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist and are finite, then:

4. $\lim_{x\to a} \big(f(x) + g(x)\big) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

5. $\lim_{x\to a} \big(f(x) - g(x)\big) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$

6. $\lim_{x\to a} \big(f(x) \times g(x)\big) = \lim_{x\to a} f(x) \times \lim_{x\to a} g(x)$

7. $\lim_{x\to a} \big(f(x) \div g(x)\big) = \lim_{x\to a} f(x) \div \lim_{x\to a} g(x)$ (except in the case of division by zero).

If c is a constant, then

8. $\lim_{x\to a} cf(x) = c \lim_{x\to a} f(x)$

If r is a rational number, then, except in case of division by zero or roots that do not exist,

9. $\lim_{x\to a} \big(f(x)\big)^r = \big(\lim_{x\to a} f(x)\big)^r$

As a consequence of rules 1. and 9.,

10. $\lim_{x\to a} x^r = a^r$, (except in the case of division by zero or roots that do not exist)

2.3.e1. Explain how the limit laws above tell us that $\lim_{x\to 2} 4x^3 - 2x^2 + 7 = 31$.

By the same logic as in Example 2.3.e1, we obtain these two very useful limit laws:

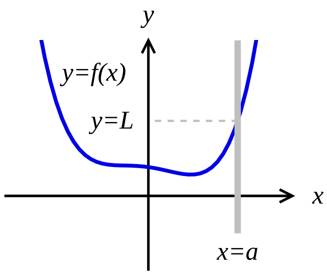
11. If $p(x)$ is any polynomial and a is any number, then $\lim_{x\to a} p(x) = p(a)$.

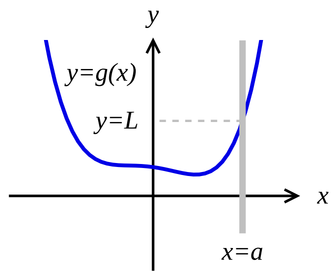
12. If $r(x)$ is any rational function (i.e., the ratio of two polynomials) and a is any number, then $\lim_{x\to a} r(x) = r(a)$, provided $r(a)$ exists.

Next, we see the most commonly used limit law of all.

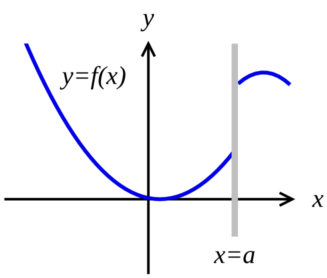
13. If $f(x) = g(x)$ for all $x \neq a$, then $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ are the same.

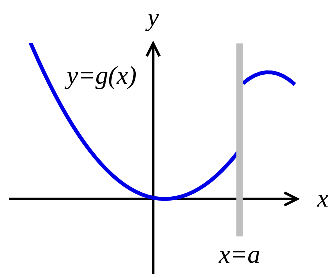
2.3.e2.





2.3.e3.





2.3.e4. Evaluate the limits:

a. $\displaystyle\lim_{x\to 2} \frac{x^2+x-6}{3x^2-6x}$ b. $\displaystyle\lim_{x\to 3} \frac{x^2-9}{x^3-27}$ c. $\displaystyle\lim_{x\to 1} \frac{\frac{1}{x+3}-\frac{1}{4}}{x-1}$ d. $\displaystyle\lim_{x\to -3} \frac{2-\sqrt{x+7}}{x+3}$

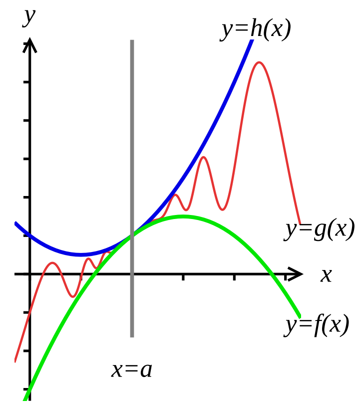
The Squeeze Theorem 2.3.1. *If*

$$f(x) \leq g(x) \leq h(x)$$

for all $x \neq a$, and if

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L,$$

then $\lim_{x\to a} g(x)$ also exists and equals L.



2.3.e5. $\lim_{x\to 0} |x| \sin\left(\frac{1}{x}\right)$

2.4: The Precise Definition of Limit

If the definition of limit in section 2.2 seems vague, it's because it turned out to be rather difficult to say exactly what we mean by limit. In fact, the unambiguous definition of limit that we use today didn't appear until 1821, about 130-200 years after the first work in modern calculus. We'll work our way up to the definition with an example.

2.4.e1. If $f(x) = \frac{(x-3)(2x+1)}{x-3}$, then $\lim_{x\to 3} f(x) =$

When we say $\lim_{x\to 3} f(x) = 7$, we don't mean that $f(x)$ ever equals 7. We mean that $f(x)$ can be made *arbitrarily close* to 7 by taking x *sufficiently close* to 3. Naturally, the closeness of two numbers a and b is measured by the distance between them $|a - b|$.

Suppose we wanted $f(x)$ to be within $\frac{1}{10}$ units of 7:

Suppose we now wanted $f(x)$ to be within $\frac{1}{50}$ units of 7:

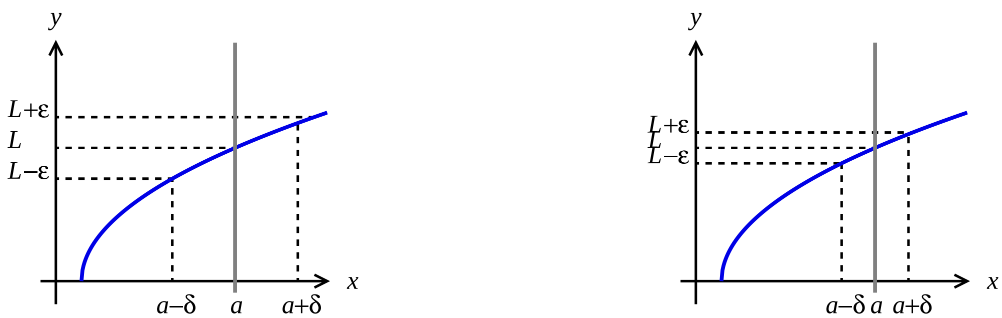
Suppose we now wanted $f(x)$ to be within $\frac{1}{150}$ units of 7:

When we say that the limit is seven, we mean that if $\frac{1}{10}$, $\frac{1}{50}$, $\frac{1}{150}$ is replaced by *any* positive number ε, then the corresponding numbers $\frac{1}{20}$, $\frac{1}{100}$, $\frac{1}{300}$ can be replaced by some number δ for which

$$|f(x) - 7| < \varepsilon \text{ whenever } 0 < |x - 3| < \delta.$$

Definition 2.4.1. *We say $\lim_{x\to a} f(x) = L$ if, for every positive number ε, there is a corresponding positive number δ for which*

$$|f(x) - L| < \varepsilon \text{ whenever } 0 < |x - a| < \delta.$$



There's an interactive graph illustrating this definition at
https://www.desmos.com/calculator/qrgrxymepu

2.4.e2. As a numerical example, consider the statement $\lim_{x\to 2} x^2 = 4$ (as promised by Limit Law 10). Find δ when $\varepsilon = 0.1$, and again when $\varepsilon = 0.03$.
As the picture above indicates, it would be helpful to know when $y = x^2$ reaches the heights $4 + \varepsilon$ and $4 - \varepsilon$:

y	4	4.1	3.9	4.03	3.97
$\sqrt{y}$	2	2.0248	1.9748	2.0074	1.99248

2.4.e3. Write an $\varepsilon - \delta$ proof of the following limits:

a. $\lim_{x\to 3} \dfrac{(x-3)(2x+1)}{x-3} = 7$

b. $\lim_{x\to -1} \dfrac{3x^2+x-2}{x+1} = -5$

c. $\lim_{x\to 2}(7-3x) = 1$

d. $\lim_{x\to a} x = a$

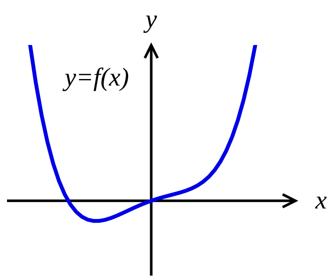
2.5: Continuity

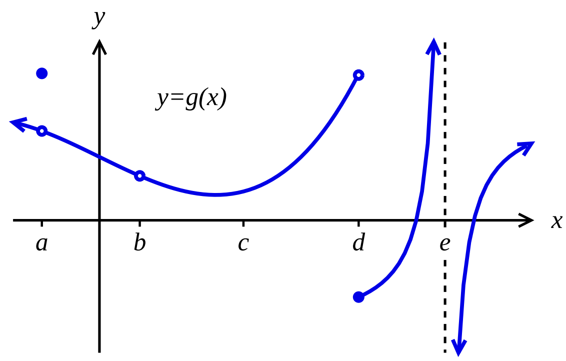
Definition 2.5.1. *The function $f(x)$ is said to be* **continuous** *at the number a if*

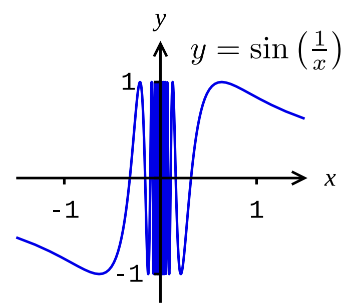
$$\lim_{x \to a} f(x) = f(a).$$

For f to be continuous at a means:

1. the limit $\lim_{x \to a} f(x)$ exists,
2. the function value $f(a)$ exists, and
3. the two are equal.







Definition 2.5.2. *The function $f(x)$ is said to be* **continuous from the right** *at the number a if $\lim_{x \to a^+} f(x) = f(a)$, and* **continuous from the left** *at a if $\lim_{x \to a^-} f(x) = f(a)$.*

Definition 2.5.3. *The function $f(x)$ is said to be* **continuous** *if it is continuous at every point a in its domain.*

(If its domain includes an endpoint, then "continuous at every point a in its domain" is interpreted to mean continuous from the left or right at the endpoint, whichever is appropriate.)

2.5.e1. When we say $\sqrt{x}$ is continuous on its domain $[0, \infty)$, we mean that

a. $\sqrt{x}$ is continuous at every $a > 0$, and
b. $\sqrt{x}$ is continuous from the right at 0.

(This is true because Limit Law 10 says that the limit of $\sqrt{x}$ is the same as its value wherever it is defined.)

The limit laws also tell us that every rational function is continuous at every point in its domain, and that polynomials are continuous at all real numbers. The next three facts tell us that, almost every function we can write down is continuous.

Fact 2.5.4. *These functions are all continuous wherever they are defined:*

polynomials *rational functions* *power functions* *absolute value*
trig functions *inverse trig functions* *exponentials* *logarithms*

Fact 2.5.5. *If f and g are continuous at a, then so are*

$$f + g \qquad f - g \qquad f \cdot g \qquad f \div g \text{ (assuming } g(a) \neq 0)$$

Fact 2.5.6. *If f is continuous at a, and g is continuous at $f(a)$, then $g \circ f$ is continuous at a.*

Fact 2.5.4 says most functions or families of function we've already given names to are continuous, and Facts 2.5.5 and 2.5.6 imply that when we combine any of those by addition, subtraction, multiplication, division or composition, the result is continuous at all points in their domain.

2.5.e2. Where is the given function continuous?

a. $3x^2 + x\sqrt{2} - \pi$ b. $\ln\left(\dfrac{x-2}{x^2-9}\right)$ c. $\dfrac{x}{|x|}$

See 2.2.e2.

A piecewise-defined function needn't be continuous, even if it's made of continuous pieces.

2.5.e3. Where is the given function continuous?

a. $\alpha(x) = \begin{cases} e^x & \text{if } x \le 0 \\ \ln(x+1) & \text{if } x > 0 \end{cases}$

b. $\beta(x) = \begin{cases} x^2 + 4x & \text{if } x \le 0 \\ 1 - x & \text{if } 0 < x < 1 \\ x^4 - 1 & \text{if } 1 \ge x \end{cases}$

2.5.e4. Find the values of a and b that make $\gamma(x) = \begin{cases} 1 + x^2 & \text{if } x \le -1 \\ ax + b & \text{if } -1 < x \le 1 \\ 3 - x^2 & \text{if } 1 < x \end{cases}$ continuous.

Here's an important property of continuous functions that we use frequently. Though it may appear obvious, its truth relies on a fundamental properties of the real numbers that we won't cover in this class.

Intermediate Value Theorem (IVT) 2.5.7. *If f is continuous on $[a, b]$, and if M is a number between $f(a)$ and $f(b)$, then there's at least one number c in $[a, b]$ for which $f(c) = M$.*

The IVT is what allows us to build a sign chart for a continuous function by finding its zeros and testing points. For instance, our solution to Example 2.5.e2b depended on the fact that $\frac{x-2}{x^2-9}$ can change sign only where it is zero ($x = 2$) or where it is discontinuous ($x = \pm 3$).

Similarly, IVT can tells us that an equation has a solution, even if we're unable to produce it exactly.

2.5.e5. Show that $2 - x^{19} = 2x$ has a solution in the interval $(0, 1)$.

2.6: Limits at infinity; Asymptotes

Definition 2.2.2 *The line $x = a$ is a* **vertical asymptote** *of the graph of $f(x)$ if either*

$$\lim_{x \to a^-} f(x) \text{ or } \lim_{x \to a^+} f(x)$$

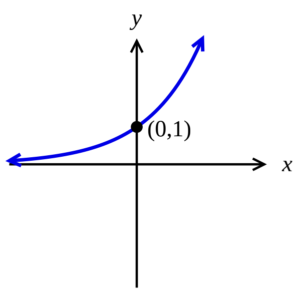
is infinite.

Definition 2.6.1. *The line $y = a$ is a* **horizontal asymptote** *of the graph of $f(x)$ if either*

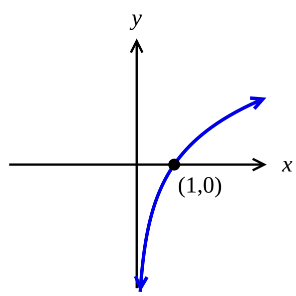
$$\lim_{x \to -\infty} f(x) = a \text{ or } \lim_{x \to \infty} f(x) = a.$$

In short, a curve has a horizontal or vertical asymptote if one variable approaches a finite number while the other blows up.

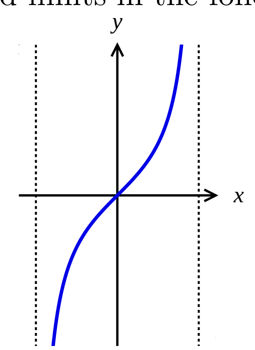
2.6.e1. Identify the asymptotes and associated limits in the following graphs.



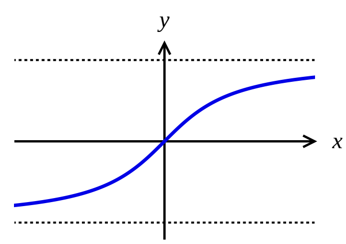
$y = e^x$



$y = \ln x$



$y = \tan x$



$y = \tan^{-1} x$

Some useful facts about limits at infinity

1. Something goes to zero if and only if its reciprocal goes to $\pm\infty$.

2. If p is a constant,
$$\lim_{x \to \infty} x^p = \begin{cases} \infty & \text{if } p > 0, \text{ and} \\ 0 & \text{if } p < 0. \end{cases}$$

3. If $p(x)$ is a polynomial and $\ell t_p(x)$ is its lead term, then
$$\lim_{x \to \infty} p(x) = \lim_{x \to \infty} \ell t_p(x).$$

4. If $p(x)$ and $q(x)$ are polynomials then
$$\lim_{x \to \infty} \frac{p(x)}{q(x)} = \lim_{x \to \infty} \frac{\ell t_p(x)}{\ell t_q(x)}.$$

2.6.e2. Evaluate the limit.

a. $\lim_{x\to\infty} (x^{\frac{1}{3}} - x^{-4})$

b. $\lim_{x\to\infty} (4x^5 - 2x^4 + 3x + 1)$

c. $\lim_{x\to\infty} \frac{2x-1}{x+3}$

d. $\lim_{x\to\infty} \frac{3x-1}{3-x^2}$

e. $\lim_{x\to\infty} \frac{2x^2+3x-1}{3-x}$

f. $\lim_{x\to-\infty} \frac{2x^2+3x-1}{3-x}$

Tip: If the rules for polynomials and rational functions don't apply, try to factor out and cancel the dominant term in the top and bottom.

2.6.e3. Evaluate the limit.

a. $\lim_{x \to \infty} \frac{\sqrt{2x^2 - x}}{-3x + 2}$

b. $\lim_{x \to -\infty} \frac{\sqrt{2x^2 - x}}{-3x + 2}$

c. $\lim_{x \to \infty} \frac{2e^{3x} - e^{2x}}{4e^{3x} + 7e^{2x}}$

d. $\lim_{x \to -\infty} \frac{2e^{3x} - e^{2x}}{4e^{3x} + 7e^{2x}}$

e. $\lim_{x \to \infty} \left(x - \sqrt{x^2 - 2x}\right)$

2.7: Derivatives and Rates of Change

Here's the conclusion of section 2.1, restated with the limit notation we've developed in this chapter:
If $f(t)$ is an object's position, then the difference quotient

$$\frac{f(a+h)-f(a)}{h}$$

is *both* the object's average velocity between times a and $a+h$ *and* the slope of secant line passing through the graph of f at times a and $a+h$. The limit of the difference quotient,

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

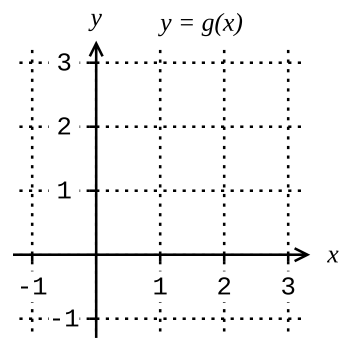
is both the object's velocity at time a *and* the slope of the line tangent to the graph of f at the point $\big(a, f(a)\big)$.

Definition 2.7.1. *The* **derivative** *of the function $f(x)$ at $x=a$, denoted $f'(a)$ is*

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

2.7.e1. Sketch the graph of a function g satisfying

$g(0)=1 \qquad g'(0)=-2 \qquad g(2)=0 \qquad g'(2)=1$



2.7.e2. Find $p(2)$ and $p'(2)$ if the line tangent to $y=p(x)$ at the point $(2,5)$ passes through $(4,-8)$.

2.7.e3. Find the derivative at a:

a. $f(x) = x^3 - 2x^2$ b. $g(x) = \sqrt{x+1}$ c. $k(x) = \dfrac{2x}{x-1}$ d. $\ell(x) = |x|$

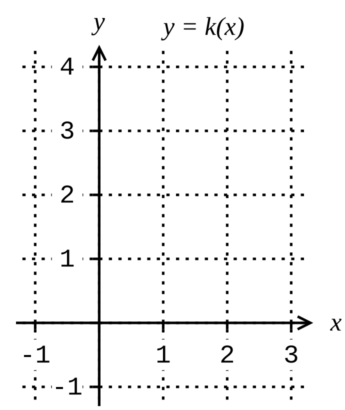
2.7.e4. A particle moving along a coordinate axis is at position $f(t) = t^3 - 2t^2$ (meters) at time t (seconds). Find the particle's velocities at times $t = 1$ and $t = 2$. What are the units in your answer?

2.7.e5. *Whose Derivative am I?* Find $f(x)$ and a if the following limit is $f'(a)$.

a. $\lim_{h \to 0} \dfrac{\sqrt{16+h} - 4}{h}$

b. $\lim_{x \to -3} \dfrac{2x^2 + 4x - 6}{x + 3}$

2.7.e6. Sketch $y = k(x)$ if $k(1) = -1$ and $k'(x) = \begin{cases} 2 & \text{if } x > 1, \\ -1 & \text{if } x < 1. \end{cases}$



2.8: The Derivative as a Function

Definition 2.7.1, restated. The derivative of the function $f(x)$ is **the function**

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{t \to x} \frac{f(t) - f(x)}{t - x}$$

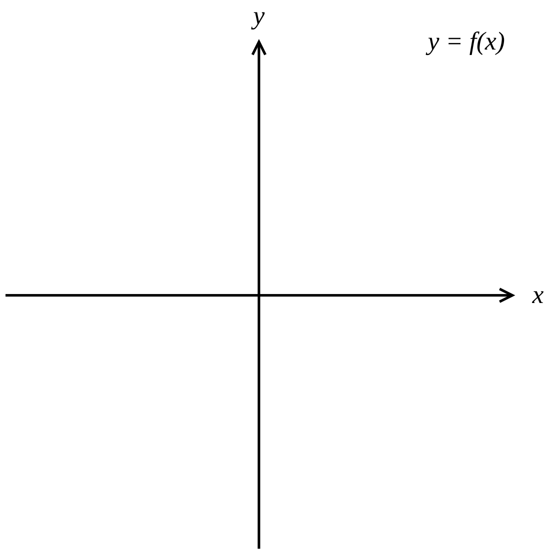
2.8.e1. Find $g'(x)$ if $g(x) = \sqrt{x+1}$

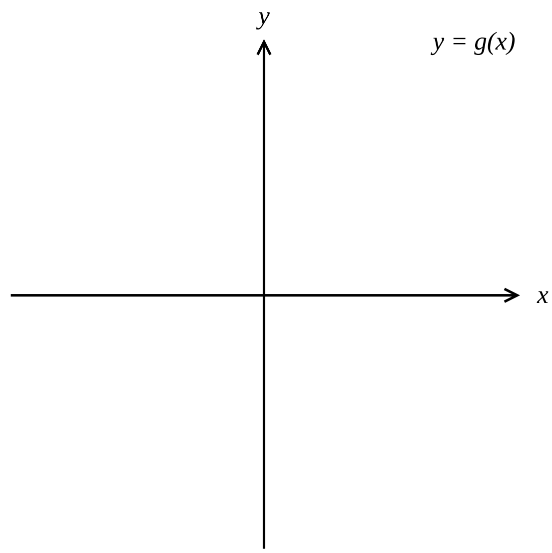
Notice that finding the derivative with the formula in 2.7.1 in the previous section, and finding it with the formula in 2.7.1, restated above is the same process but with a different variable name.

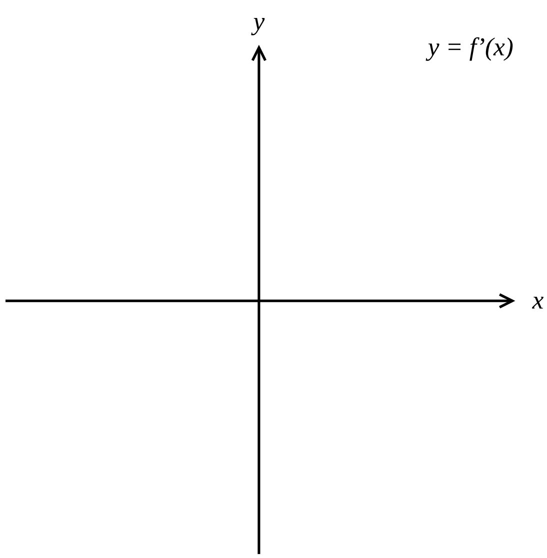
For more practice taking the derivative using the definition, see
http://kunklet.people.cofc.edu/MATH120/derivative_prac.pdf

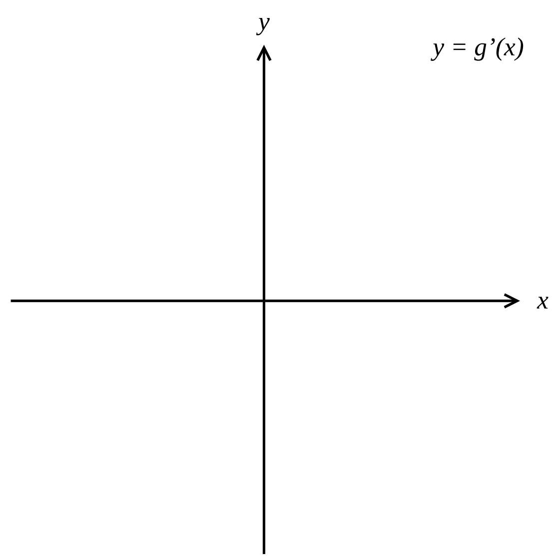
2.8.e2. Sketch the functions and their derivatives found in 2.7.e3.

a. $f(x) = x^3 - 2x^2$ b. $g(x) = \sqrt{x+1}$





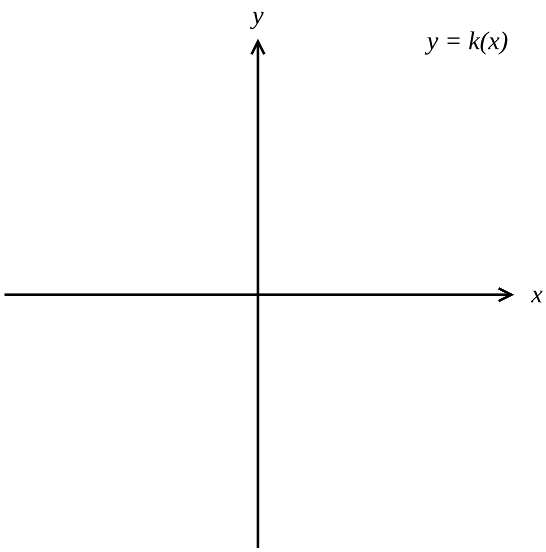


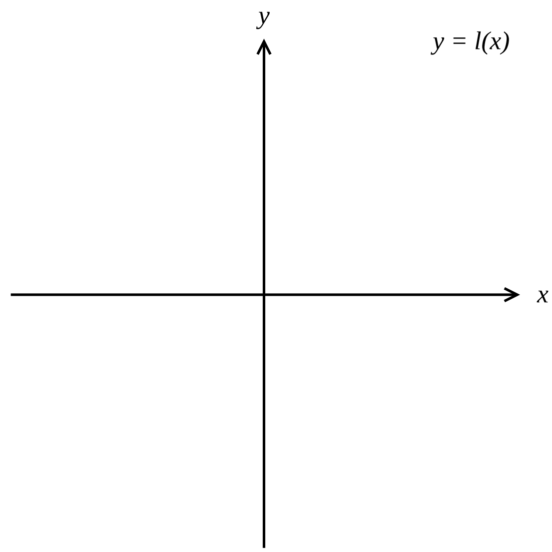


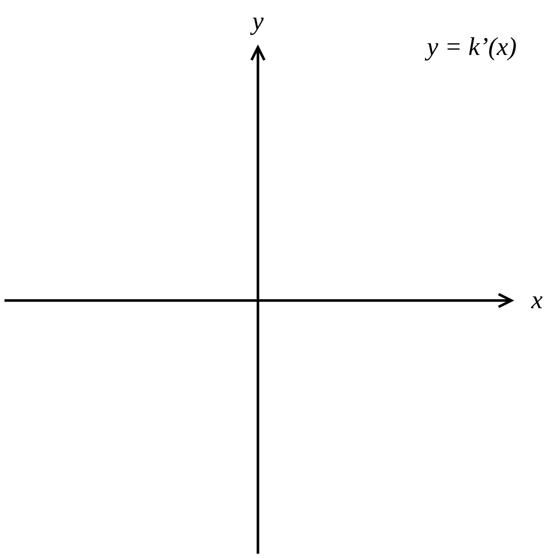
2.8.e2, continued. Sketch the functions and their derivatives found in 2.7.e4.

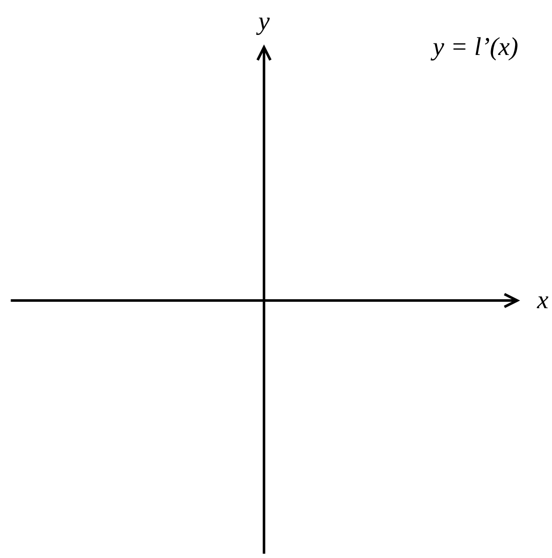
c. $k(x) = \dfrac{2x}{x-1}$

d. $\ell(x) = |x|$

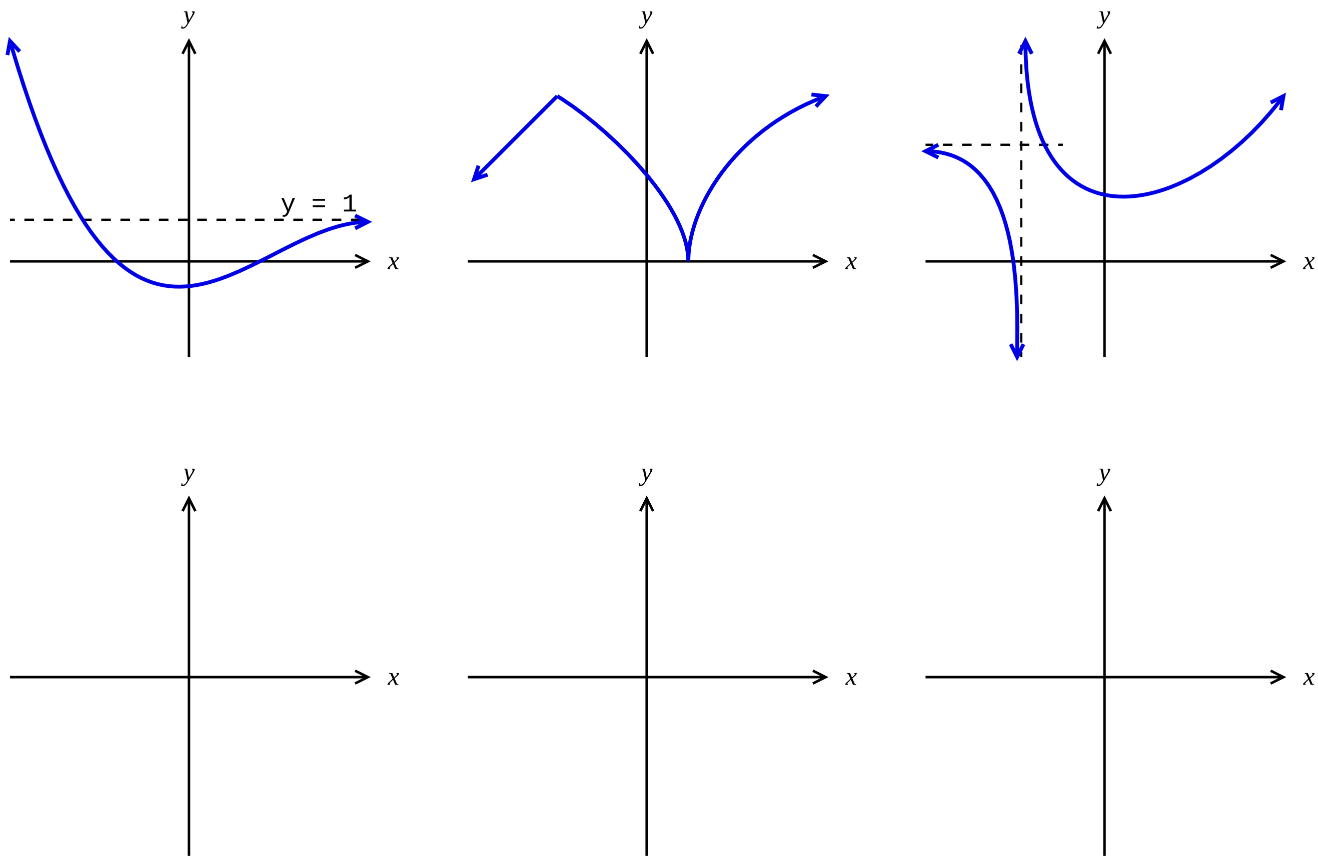








2.8.e3. Sketch the derivative of the function graphed.



Definition 2.8.1. *The function $f(x)$ is said to be* **differentiable** *at a if $f'(a)$ exists.*

Fact 2.8.2. *If $f(x)$ is differentiable at a, then it must be continuous at a.*

Reasons why $f(x)$ might fail to be differentiable at a:

1.
2.
3.
4.

Higher order derivatives

The derivative of a function is also called its **first derivative**.
The **second derivative** of a function is the derivative of its first derivative.
The **third derivative** of a function is the derivative of its second derivative, and so on.

2.8.e4. Find the second and third derivative of $f(x) = x^3 - 2x^2$.

Various notation for derivatives.

The symbol $\frac{d}{dx}$ is read "the derivative of" (just as $\sqrt{}$ is read "the square root of").

First derivative of $y = f(x)$	$f'(x)$	$\frac{dy}{dx}$	$\frac{d}{dx}y$	$\frac{d}{dx}f(x)$
Second derivative of $y = f(x)$	$f''(x)$	$\frac{d^2y}{dx^2}$	$\frac{d}{dx}\left(\frac{dy}{dx}\right)$	$\left(\frac{d}{dx}\right)^2 f(x)$
Third derivative of $y = f(x)$	$f'''(x)$	$f^{(3)}(x)$	$\frac{d^3y}{dx^3}$	$\left(\frac{d}{dx}\right)^3 f(x)$
Fourth derivative of $y = f(x)$	$f^{(4)}(x)$	$\frac{d^4y}{dx^4}$		

2.8.e4, continued. The position of a particle on an axis is $f(t) = t^3 - 2t^2$. If position f is measured in meters and time t in seconds, what are the units of its first three derivatives?

3.1: Derivatives of Polynomials and Exponentials

In this chapter we learn Differentiation Laws that allow us to find derivatives more easily than by the definition in Section 2.7. They come in two types.
Below, $f(x)$ and $g(x)$ can be any differentiable functions, and c and n can be any constants, and e is the mathematical constant mentioned in Section 1.4.

I. A catalog of elementary functions and their derivatives.

$(c)' = 0$	$(x^n)' = nx^{n-1}$	$(e^x)' = e^x$

II. Combination laws.

$\big(cf(x)\big)' = cf'(x)$
$\big(f(x) + g(x)\big)' = f'(x) + g'(x)$

As a consequence of the above two rules,

$\big(f(x) - g(x)\big)' = f'(x) - g'(x)$

3.1.e1. Find the indicated derivative of the given function.

a. $f''(x)$ where $f(x) = 7x^3 - \dfrac{3}{\sqrt[3]{x^2}} + 2x + 5^2$

b. $g^{(4)}(x)$ where $g(x) = 7x^3 + 4x^2 + 2x + 25$

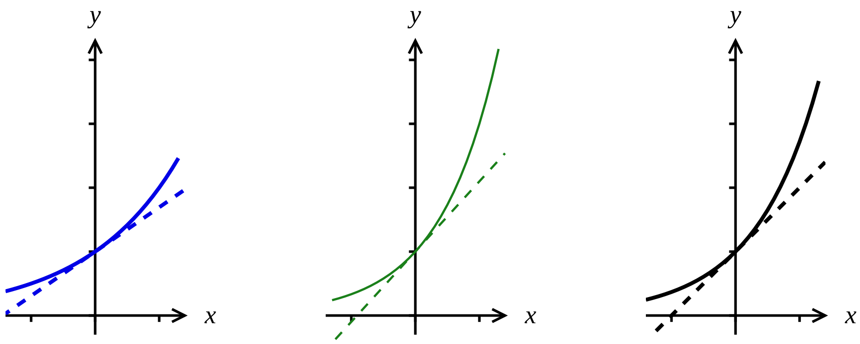
As a matter of fact,

The function f if a polynomial if and only if $f^{(n)} = 0$ for some integer n.

The definition of the constant e

If $a > 0$, then the derivative of the function a^x is

$$\frac{d}{dx}a^x = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} =$$



Calculations indicate that this limit, which is the derivative of a^x at $x = 0$, is roughly 0.7 when $a = 2$, and 1.1 when $a = 3$. It stands to reason that there must be a number, which we call e, between 2 and 3 that causes this limit to equal 1, and therefore

$$\frac{d}{dx}e^x = e^x,$$

as stated in derivative rule 3 on the previous page

An exercise in **curve-fitting**: designing a function to take a set of prescribed values.

3.1.e2. Find numbers a, b, and c so that $k(x) = ax^2 + bx + c + e^x$ so that

$$k(0) = 1 \qquad k'(0) = -1 \qquad k''(0) = 3.$$

3.2: The Product and Quotient Rules

II. Combination laws (continued).

Product Rule: If $f(x)$ and $g(x)$ are differentiable, then so is their product, and

$$\big(f(x)g(x)\big)' = f'(x)g(x) + f(x)g'(x).$$

Quotient Rule: If $f(x)$ and $g(x)$ are differentiable, then so is their quotient, and

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} \qquad \text{(provided } g(x) \neq 0\text{)}.$$

Proof of the product rule

$$\big(f(x)g(x)\big)' = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} =$$

3.2.e1. Find the first derivative of the given function.

a. x^2e^x b. $x^2e^x(2x-3)$ c. $\dfrac{3x^3+2}{x^2}$ d. $\dfrac{x}{3x^2+1}$

3.2.e2. Find the equation of the line tangent to $y = \dfrac{xe^x}{3x^2 - 1}$ at $x = 1$.

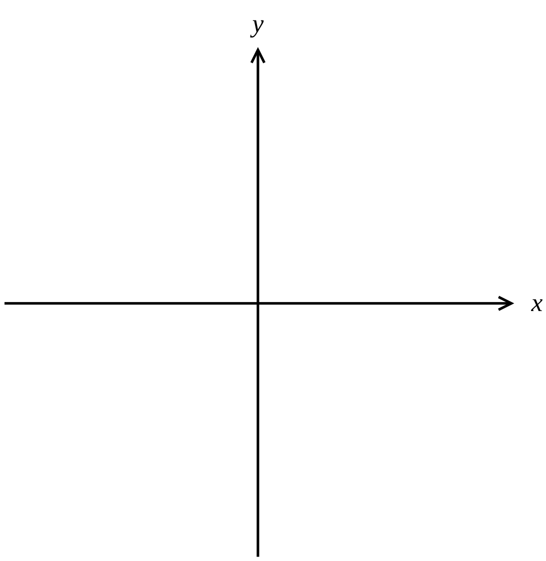
There are two ways to take the derivative in this example; either will require us to use two differentiation rules in the correct order. Remember,

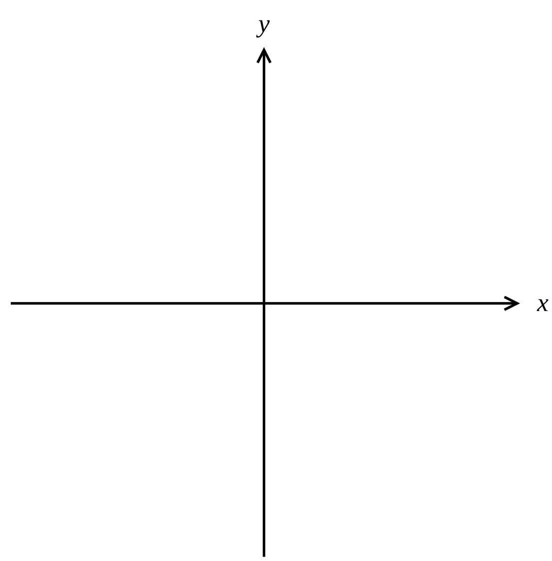
> The last operation determines the first differentiation rule we must use.

3.2.e3. Find all x-values at which the slope of $y = \dfrac{x-1}{x+3}$ is 6.

3.2.e3, continued. Find the asymptotes to $y = \dfrac{x-1}{x+3}$.

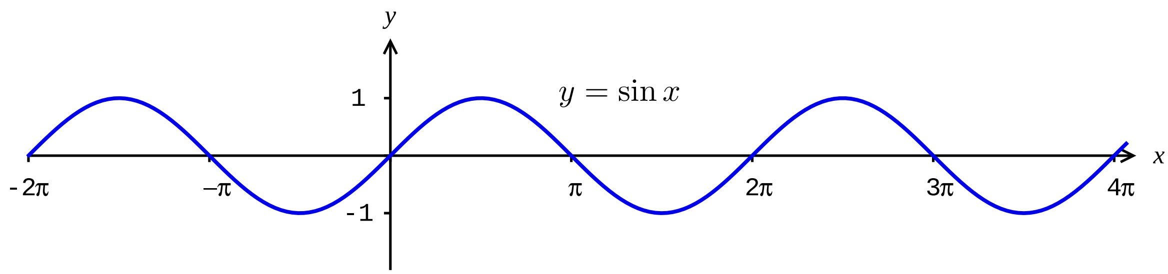
3.2.e3, continued. Make a rough sketch of the graphs of $y = \dfrac{x-1}{x+3}$ and its derivative.

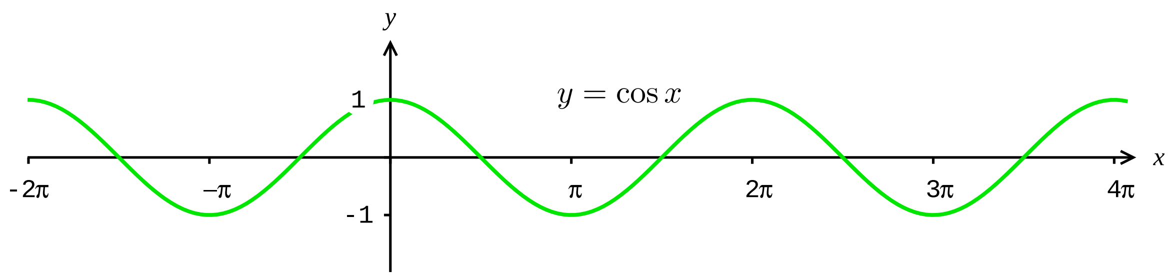


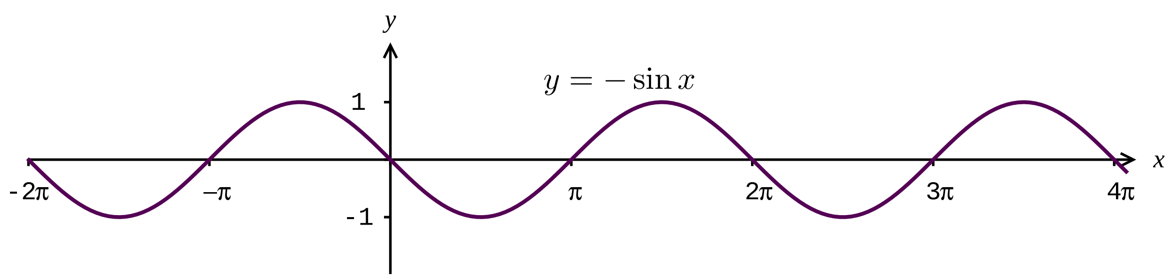


3.3: The Derivatives of the Trig Functions

In this section we add 6 more functions to our derivative catalog.







I. Catalog.

$(\sin x)' = \cos x$	$(\cos x)' = -\sin x$
$(\tan x)' = \sec^2 x$	$(\cot x)' = -\csc^2 x$
$(\sec x)' = \sec x \tan x$	$(\csc x)' = -\csc x \cot x$

Once we establish the first two rules, the other four follow from the quotient rule.

3.3.e1. Show that $(\sec x)' = \sec x \tan x$ using the derivatives of $\sin x$ and of $\cos x$.

3.3.e2. Find the derivative of the given function.

a. $e^x\sqrt{x}\sin x$ b. $\dfrac{e^x \sin x}{e^x \tan x}$ c. $\dfrac{e^x \sin x + 3\cos x}{1 - \tan x}$

3.3.e3. An object moving along a number line is at position $s(t) = 3\sin t + 2t - 1$ at time t. Find the object's velocity and acceleration. When on the interval of time $[-\pi, \pi]$ is the object moving forward?

3.3.e4. Find the 123rd derivative of $\sin x$.

Why is $\frac{d}{dx} \sin x = \cos x$*?*

3.4: The Chain Rule

Our final derivative rule will allow us to differentiate the *composition* of two functions. Suppose that y is a differentiable function of u, which in turn is a differentiable function of x. How are the three derivatives

$$\frac{dy}{du}, \quad \frac{du}{dx}, \quad \frac{dy}{dx}$$

related? The answer is obvious in the right notation:

$$\frac{dy}{du} = \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u}$$
$$\frac{du}{dx} = \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}$$

And therefore

$$\begin{aligned} \frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u}\frac{\Delta u}{\Delta x} \\ &= \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} = \frac{dy}{du}\frac{du}{dx} \end{aligned}$$

II. Combination laws (continued).

Chain Rule: If y is a differentiable function of u, and u is a differentiable function of x, then y is a differentiable function of x, and

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

3.4.e1. Find the derivative of the given function.

a. $\sin x^2$ b. $\sin^2 x$ c. $\sqrt{x^2 - 5x + 1}$

In the last example, we saw that the derivative of a composition was always the derivative of the outer function evaluted at the inner function, times the derivative of the inner function:

Chain Rule, in function notation: If f and g are differentiable functions, then their composition is differentiable (where it exists), and

$$\Big[f\big(g(x)\big)\Big]' = f'\big(g(x)\big)g'(x)$$

3.4.e2. Find the derivative of the given function.

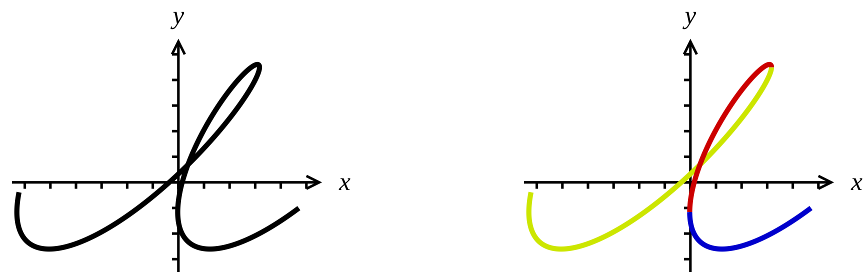
a. $e^{x \sin x}$ b. 2^x c. $e^{\tan^2 x}$

d. $x^2 \cos^2 x^3$ e. $(\sin^2 x)e^{\cos x}$ f. $\dfrac{e^x}{\sec\sqrt{x}}$

In part c, at what x-values is the tangent line horizontal?

3.5: Implicit Differentiation and the Inverse Trig Functions

Implicit differentiation is a method to find $\frac{dy}{dx}$ along the graph of an equation. It's based on the fact that, even if the graph of an equation fails to be the graph of a function, it can



still consist of the graphs of *several* functions. So, if we differentiate both sides of an x-y equation, it's safe to assume that y is some unspecified function of x. The result is an equation in x, y, and $\frac{dy}{dx}$ that, regardless of the original x-y equation, is **linear** in $\frac{dy}{dx}$.

3.5.e1. Find the equation of the line tangent to the curve $x^2 - y^4 = 2x^4y - 2$ at the point $(1, 1)$.

3.5.e1, continued. Find $\frac{dy}{dx}$ along the curve $x^2 - y^4 = 2x^4y - 2$

3.5.e2. Find $\frac{dy}{dx}$ along the curve $xy^2 = e^y(x^2 + y^2)$

3.5.e3. Find $\frac{dy}{dx}$ along the curve $x^{2/3} + y^{2/3} = 4$

Here's the graph: https://www.desmos.com/calculator/y4nvy6epy8

3.5.e3, continued. Find $\frac{d^2y}{dx^2}$ along the curve in 3.5.e3.

3.5.e4. Find $\frac{dy}{dx}$ along the curve $x^2 = y^2(y+1)$

Here's the graph: https://www.desmos.com/calculator/y4nvy6epy8

Derivatives of the inverse trig functions

More entries in our catalog of functions and their derivatives.

I. Catalog.

$(\sin^{-1} x)' = \dfrac{1}{\sqrt{1-x^2}}$	$(\cos^{-1} x)' = -\dfrac{1}{\sqrt{1-x^2}}$
$(\tan^{-1} x)' = \dfrac{1}{1+x^2}$	$(\cot^{-1} x)' = -\dfrac{1}{1+x^2}$
$(\sec^{-1} x)' = \dfrac{1}{x\sqrt{x^2-1}}$	$(\csc^{-1} x)' = -\dfrac{1}{x\sqrt{x^2-1}}$

3.5.e5. Show that $\frac{d}{dx} \sec^{-1} x = \frac{1}{x\sqrt{x^2-1}}$.

3.5.e6. Find the derivative of the given function.

a. $(\tan^{-1} x)(\sin^{-1} x)$ b. $e^x \arccos x^3$ c. $\sec(\sec^{-1} x)$ d. $\cot(\arctan e^x)$

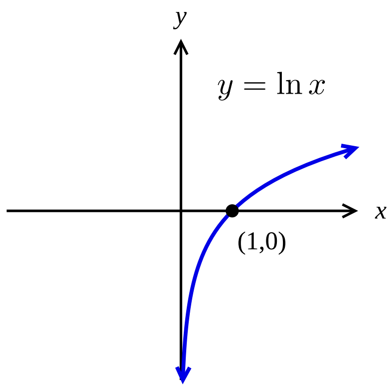
3.6: Derivatives of Logarithmic Functions

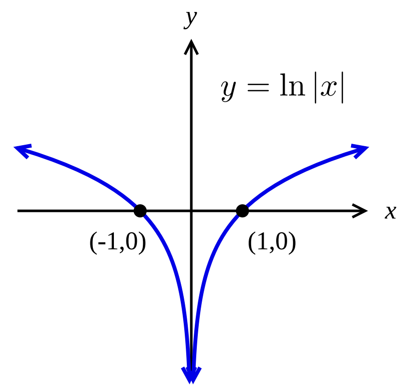
Two more entries for our catalog of functions and their derivatives.

I. Catalog.

$(\ln x)' = \dfrac{1}{x}$	$(\ln \lvert x \rvert)' = \dfrac{1}{x}$

The functions $\ln x$ and $\ln |x|$ agree on $(0, \infty)$, but, of the two, only $\ln |x|$ is defined on $(-\infty, 0)$. We sometimes say that $\ln |x|$ is the even extension of $\ln x$.





Interestingly, the derivative of $\ln |x|$ has exactly the same algebraic formula as the derivative of $\ln x$. This fact will be of great importance in Chapter 5.

3.6.e1. Show that $\frac{d}{dx} \ln x$ and $\frac{d}{dx} \ln |x|$ both $= \frac{1}{x}$.

3.6.e2. Find the derivative of the given function.

a. $\ln 2$ b. $\ln(2x)$ c. $\ln|\sec x|$ d. $\ln(e^{-x}5x^2 \sin x)$

e. $\log_3 x$ f. $e^{\frac{1}{2}\ln x}$ g. $x^{x/2}$ h. $(x^2+1)^{\sin x}$

3.7: Rates of change in the sciences

Recall that $\frac{dy}{dx}$ is the rate of change of the function y with respect to the variable x. If $\frac{dy}{dx}$ were constant, it would equal the change in y corresponding to a 1 unit increase in x. The units of $\frac{dy}{dx}$ are the units of y per units of x.

3.7.e1. A ball launched upward is at altitude $s = 64t - 16t^2$ meters at time t seconds after launch. Find the following.

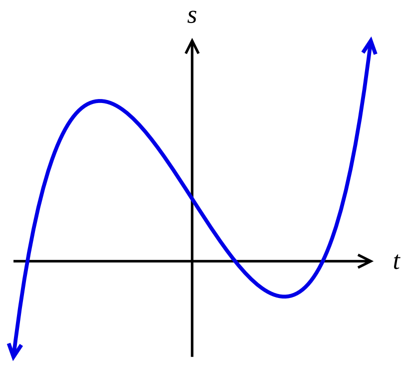
a. The ball's velocity and acceleration at time t, and the units of each.
b. When the ball is traveling upward and when it is traveling downward.
c. A **schematic diagram** of the ball's motion
d. When the ball's velocity is increasing.
e. The ball's **speed** = the absolute value of velocity.
f. The graph of the ball's velocity and speed.
g. When the ball is speeding up and when it is slowing down.

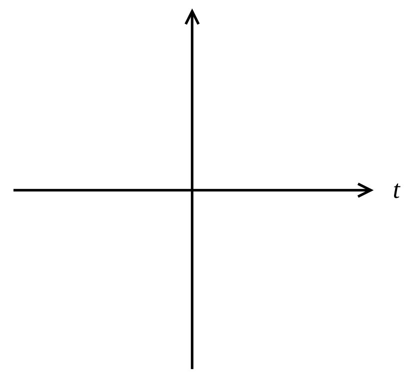
3.7.e2. An object traveling along a number line as it position $s = \frac{1}{4}t^4 - 2t^2$ m at time t sec, for $-\infty < t < \infty$.

a. Find the object's velocity, acceleration, and speed.
b. When is the object moving in the positive direction? Draw a schematic graph.
c. Find the **net distance** traveled and **total distance** traveled by the object ...
  i. ...from $t = 0$ to $t = 2\sqrt{2}$.
  ii. ...from $t = 0$ to $t = 3$.

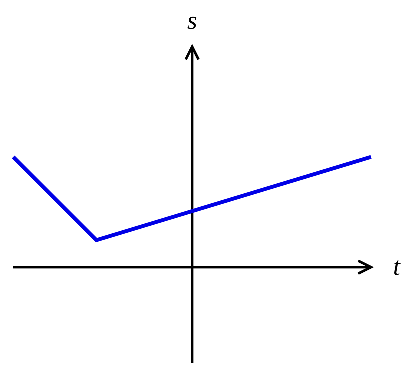
3.7.e3. Graph velocity and speed for the given position function.

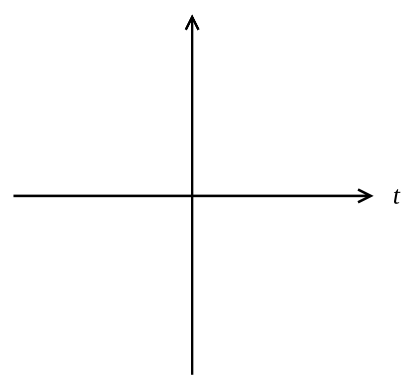
a.





b.





3.9: Related Rates

3.9.e1. A 10-ft long ladder is leaning against a vertical wall on horizontal ground. If the bottom of the ladder is pulled away from the wall at a rate of 1 ft/sec, how fast is the ladder sliding down the wall when the bottom is 6 ft from the wall?

See an animation of this at https://www.desmos.com/calculator/akotvavges

Tip: Identify all rates referred to in the problem as derivatives of quantities in your picture (lengths, areas, volumes, angle measurements, etc.) and make those quantities your variables. Find an equation that relates just those variables and no others.

3.9.e2. A kite flying at altitude 50 m is moving horizontally 2 m/sec away from the kite flyer on the ground.
a. How fast is the kite flyer letting out string when the kite is 100 m away from her?
b. At that moment, how fast is the angle of elevation from the flyer to the kite decreasing in radians/sec?

3.9.e3. A conical paper cup has height 3 in and radius 1in. Water is leaking from the bottom of the cup $\frac{1}{18}$ in^3/sec. Find how fast the depth of water in the cup is falling when it is 2 in deep.

Tip: The most common sources of equations in these problems is the pythagorean theorem, similar triangles, right-triangle trigonometry, 3-D geometry formulas, and, occasionally, the law of cosines.

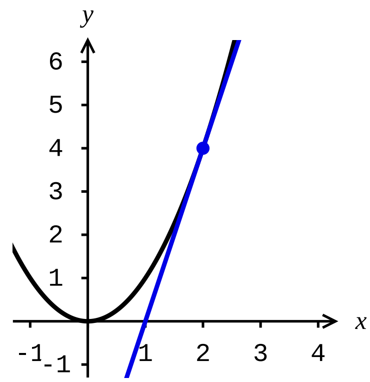
3.9.e4. One ship leaves harbor and travels at the speed 2 nautical miles per hour, or **knots**. One hour later, a second ship leaves harbour traveling 3 knots. If the two ships travel in straight lines meeting at a $60°$ angle, how fast is the distance changing when the second ship has been under sail for 1 hour?
Animation at https://www.desmos.com/calculator/fngaejjixm

3.9.e5. A lighthouse stands on a small island 1 mi from a point P on a straight shoreline. If the light atop the light house rotates 3π radians/min, how fast is the beam of light moving along the shore when it's shining on a point $\frac{1}{2}$ mile from P?
Animation at https://www.desmos.com/calculator/d09fdiiwah

3.10: Linear Approximation

We often say that a tangent line is a good approximation to a curve near the point of tangency. In this section, we discuss what exactly we mean by this.

3.10.e1. Find the line tangent to $y = x^2$ at $x = 2$ and compare its y-values with those of the curve near $x = 2$.



x	x^2	$4 + 4(x - 2)$	error	$x - 2$
2.1	4.41	4.4	.01	.1
2.01	4.0401	4.04	.0001	.01
2.001	4.004001	4.004	.000001	.001

Definition 3.10.1. *If f is differentiable at a, then the* **linearization** *of f at a is the function*

$$L(x) = f(a) + f'(a)(x - a).$$

Generally, **Linear Approximation** refers to the approximation of the function $f(x)$ by the function $L(x)$ for x near a, which we vaguely indicate by

$$f(x) \approx L(x) \qquad (x \approx a).$$

That much could said for *any* linear function, if it equals $f(a)$ at $x = a$. What makes $L(x)$ special among all linear functions is the property that the error of the approximation

$$f(x) - L(x) = (x - a)\varepsilon,$$

where ε is a quantity which goes to zero as $x \to a$. This makes the error $(x - a)\varepsilon$ small relative to $x - a$. Compare, for example, the last two columns in the table in 3.10.e1.

3.10.e2. Find the linearization of the function at the given point.

a. $e^x,\ a = 0$

b. $e^x,\ a = 1$

c. $\sin x,\ a = 0$

d. $\sin x,\ a = \pi/4$

e. $\sqrt{x},\ a = 9$

3.10.e3. Use linear approximation to approximate the given number

a. $\ln(1.05)$

b. $\cos(89^\circ)$

Definition 3.10.2. *The* **differential** *of y is $\frac{dy}{dx}\,dx$, which we write as dy.*

3.10.e4. Find the differential of the function, a

a. $\tan^{-1} x$ b. $\ln|\sin x|$ c. $e^{-x}\cos x$

The differential is of great importance in Chapter 5, but it is natural to introduce it here, since it is the change in the linearization between a and x, and linear approximation could be phrased as saying

$$f(x) - f(a) \approx f'(a)(x - a),$$

that is, the change in y along the curve $y = f(x)$ is approximately the change in y along the line $y = L(x)$, which we write as

$$\Delta y \approx dy.$$

4.1: Extrema

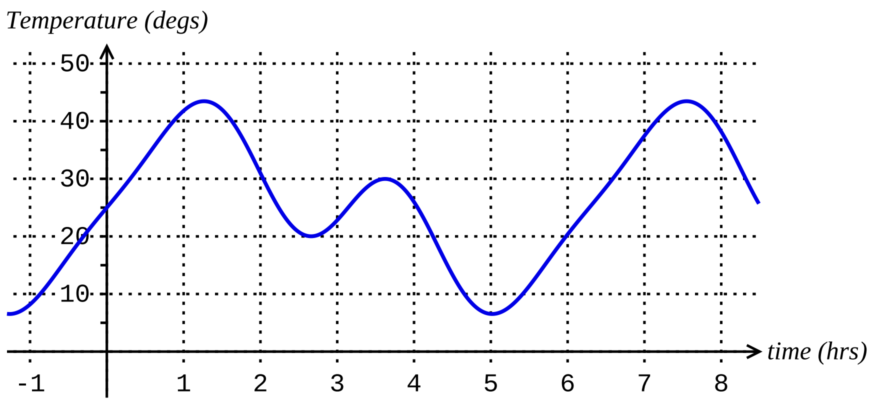
Definition 4.1.1. *Suppose a function f is defined on some interval D. The* **absolute maximum** *of f on D is the largest value that f attains on D, if such a value exists.*

The **absolute minimum** *of f on D is its smallest value on D, if it exists.*

Vocabulary:

maxima/minima: plural of maximum/minimum
extremum (*pl* **extrema)**: a maximum or minimum

4.1.e1. In the graph below, temperature T is a function of time t.



Find the absolute extrema of T on the given intervals of t.

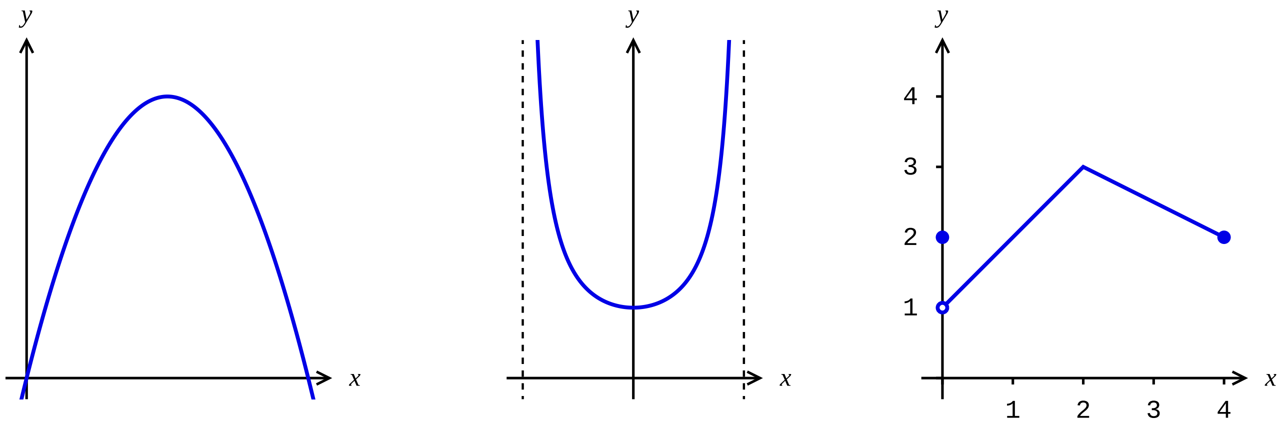
	$[0,3]$	$[2,4]$	$[0,8]$	$[6,8]$	$(6,8]$
abs max T					
abs min T					

The extrema are function values, not input values.

The absolute extrema depend on the function *and* the interval.

On any given interval, the absolute maximum, if it exists, is unique.
Similarly, the absolute minimum is unique.

4.1.e2.



$y = 4 - (x-2)^2$ $\qquad$ $y = \sec x$ $\qquad$ $y = \alpha(x)$

Theorem 4.1.2. *If the function f is continuous on $[a, b]$, then f attains an absolute maximum and an absolute minimum on $[a, b]$.*

Definition 4.1.3. *The function f has a* **local maximum** *at $x = c$ if $f(c)$ is the absolute maximum of f on some (possibly small) open interval containing c.*

f has a **local maximum** *at $x = c$ if $f(c)$ is the absolute maximum of f on some open interval containing c.*

4.1.e1 and 4.1.e2, continued. Identify the local extrema on the given graphs and where they occur.

Definition 4.1.4. *c is a* **critical number** *for f if $f'(c)$ either $= 0$ or does not exist*

Theorem 4.1.5. *A function can attain local extrema only at its critical numbers.*

4.1.e3. Find the critical numbers of the function.

a. $4x^3 + 9x^2 - 12x + 1$ b. x^2e^x c. $e^x \sin x$ d. $x^{2/3} - x^{8/3}$

Finding absolute extrema on a closed interval

Theorem 4.1.6. *The absolute extrema of $f(x)$ on $[a, b]$ can only occur at the endpoints (a and b) or critical points in the interior (a, b).*

4.1.e4. Find the absolute extrema of the function on the given interval.

a. $4x^3 + 9x^2 - 12x + 1 \quad [-1, 2]$

b. $x^{2/3} - x^{8/3} \quad [-1, 1]$

4.1.e4, continued. Find the absolute extrema of the function on the given interval.

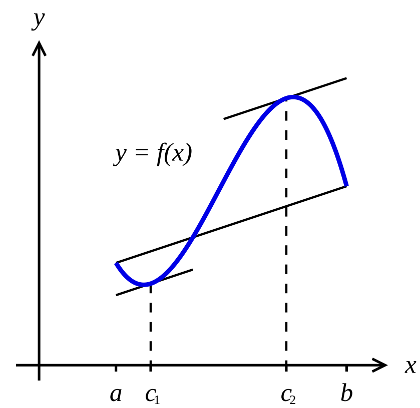
c. $\frac{1}{4}x + x^{-1} \quad [1,3]$

d. $\cos x + \cos^2 x \quad [0, 2\pi]$

4.2: The Mean Value Theorem

Mean Value Theorem (MVT) 4.2.1. *If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a number c in (a, b) for which*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



In the special case that $f(a) = f(b)$, the MVT guarantees a c in (a, b) at which $f'(c) = 0$. This is known as **Rolle's Theorem**.

4.2.e1. Check whether the function satisfies the hypothesis of the MVT on the given interval. If it does, find the number(s) c as in the conclusion.

a. $f(x) = x^3 + x + 2 \quad [0, 2]$

b. $g(x) = 2x^{2/3} - x^{5/3} \quad [0, 2]$

c. $h(x) = x^{2/3} \quad [-1, 1]$

Definition 4.2.2.

$$f \textbf{ increasing } \textit{means:} \quad a < b \implies f(a) < f(b)$$
$$f \textbf{ decreasing } \textit{means:} \quad a < b \implies f(a) > f(b)$$

The MVT implies this useful fact.

Theorem 4.2.3. *Suppose f is differentiable on an interval I. Then:*

a. $f' > 0$ *on* $I \implies f$ *is increasing on* $I \implies f' \geq 0$ *on* I.
b. $f' < 0$ *on* $I \implies f$ *is decreasing on* $I \implies f' \leq 0$ *on* I.
c. $f' = 0$ *on* $I \iff f$ *is constant on* I.

If these results appear obvious, that means that the derivative agrees with our intuitive expectations of "slope," and that our definition of derivative is a good one.

4.2.3 part c has this consequence.

Corollary 4.2.4. *If the functions f and g have the same derivative on an interval I, then $f - g$ must be a constant on I.*

4.2.e2. Find all functions whose derivative on $(-\infty, \infty)$ equals $2x$.

The MVT 4.2.1 and the Intermediate Value Theorem 2.5.7 are often combined in problems like the next one.

4.2.e3. How many solutions are there to $x^5 + 5x = e^{-x}$?

4.3: A Function's Derivative and its Graph

By Theorem 4.2.3, the sign of the first derivative determines a function's **monotonicity** (increasingness and decreasingness) and, consequently, where it attains local extrema.

First Derivative Test for Local Extrema 4.3.1. *Suppose c is a critical point of f. If f' changes sign at c:*

$$\begin{array}{r|l} f'(x): & - - - - \; + + + + \\ \hline x: & \qquad c \end{array} \quad \Longrightarrow \quad f \text{ has a local minimum at } c.$$

$$\begin{array}{r|l} f'(x): & + + + + \; - - - - \\ \hline x: & \qquad c \end{array} \quad \Longrightarrow \quad f \text{ has a local maximum at } c.$$

If f' does not change sign at c, then f does not have a local extremum at c.

In general, the First Derivative Test (and Second Derivative Test to follow) are not useful in determining *absolute* extrema.

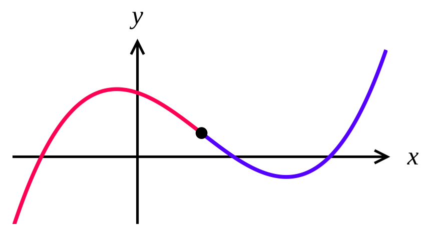
4.3.e1. Make a rough sketch of the graph of the function, showing intervals of increase and decrease.

a. $f(x) = x(x-2)^2$

b. $g(x) = \dfrac{1}{2}x + \cos x$, $[0, 2\pi]$ only

Next, we'll see how to smooth out these graphs using **concavity**.

Definition 4.3.2. *The graph of f is* **concave up** *when f' is increasing (or $f'' > 0$), and* **concave down** *when f' is decreasing (or $f'' < 0$).*
An **inflection point** *is a point on the graph of f where f changes concavity (and is continuous).*



4.3.e2. Go to https://www.desmos.com/calculator/necjpvhc6j for an interactive graph of $f(x) = \frac{1}{3}(x+1)(x-1)(x-2)$, including its concavity and inflection point.

Monotonicity and concavity are independent, in that a function can be any of

increasing and concave up,	increasing and concave down,
decreasing and concave up, or	decreasing and concave down.

Second Derivative Test for Local Extrema 4.3.3. *Suppose $f('c) = 0$.*

$$\text{If } f''(c) > 0, \text{ then } f(c) \text{ is a local minimum.}$$

$$\text{If } f''(c) < 0, \text{ then } f(c) \text{ is a local maximum.}$$

Observe that the Second Derivative Test is inconclusive if $f''(c) = 0$.

4.3.e3. Sketch of the graph of the function, showing intervals of monotonicity and concavity and local extrema and inflection points.

a. $f(x) = x(x-2)^2$

b. $g(x) = \dfrac{1}{2}x + \cos x$ $[0, 2\pi]$ only

4.3.e3, continued. Sketch of the graph of the function, showing intervals of monotonicity and concavity and local extrema and inflection points.

c. $k(x) = x^{1/2} - x^{3/2}$

d. $\ell(x) = \dfrac{1}{x^2 - 4}$

4.3.e4. Sketch the graph of function g satisfying

$g'(x) < 0$ if $x > 1$ | $g''(x) > 0$ if $x \neq 1$
$g'(x) > 0$ if $x < 1$ | $g''(x)$ dne at $x = 1$
$g'(x) \to \infty$ as $x \to 1^-$
$g'(x) \to -\infty$ as $x \to 1^+$

4.3.e5. Sketch the graph of function f satisfying

$f'(x) < 0$ if $x < 0$ | $f''(x) < 0$ if $x < -2$ or $1 < x < 2$
$f'(x) > 0$ if $0 < x < 2$ or $2 < x$ | $f''(x) > 0$ if $-2 < x < 1$ or $2 < x$
$f'(x) = 0$ at $x = 0, 2$ | $f''(x) = 0$ at $x = -2, 1, 2$

4.4: l'Hospital's Rule and indeterminant forms.

When taking limits,

$$\frac{\text{nonzero}}{0} \implies \lim = \pm\infty \qquad\qquad \frac{\text{finite}}{\infty} \implies \lim = 0.$$

But

$$\frac{0}{0} \quad \text{and} \quad \frac{\infty}{\infty}$$

are called **indeterminate forms** because they tell us nothing about the actual value of the limit. Fortunately, there's a tool for these situations.

l'Hôpital's Rule 4.4.1. *If*

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \quad \textit{or} \quad \lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \pm\infty$$

and if

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = L,$$

then

$$\lim_{x\to a} \frac{f(x)}{g(x)} \textit{ exists and also equals } L.$$

Notes on l'Hospital's Rule

1. $x \to a$ can be replaced throughout by $x \to \infty$, $x \to -\infty$, $x \to a^-$, or $x \to a^+$.
2. L can be ∞, $-\infty$, or any real number.
3. l'Hôpital's Rule never allows us to conclude that $\lim_{x\to a} \frac{f(x)}{g(x)}$ does not exist.

Indeterminate forms and examples of each

$$\frac{0}{0} \qquad \frac{\infty}{\infty} \qquad 0\cdot\infty \qquad \infty-\infty$$

l'Hospital's Rule applies only to the first two of these. Any others would have to be rewritten as a quotient in order to use l'Hôpital's Rule.

4.4.e1. $\displaystyle\lim_{x\to 0} \frac{e^x - e^{-2x}}{x}$

4.4.e2. $\lim_{x \to 0} \frac{\sqrt{1+x} - 1 - \frac{1}{2}x}{x^2}$

4.4.e3. $\lim_{x \to 0} \frac{x}{1 - \cos x}$

4.4.e4. a. $\lim_{x\to\infty} \frac{e^x}{x^2}$ b. $\lim_{x\to\infty} \frac{e^x}{x^{99}}$ c. $\lim_{x\to\infty} \frac{e^x}{x^n}$ (n any positive integer)

4.4.e5. $\lim_{x\to 0^+} x^p \ln x$ $(p > 0)$

4.4.e6. $\lim_{x\to\infty}\big(\ln(3x-1)-\ln(2x+1)\big)$

4.4.e7. a. $\lim_{x\to\infty}\big(3x-\sqrt{9x^2-2x}\big)$ b. $\lim_{x\to-\infty}\big(3x-\sqrt{9x^2-2x}\big)$

When l'Hospital's Rule fails to produce an answer

4.4.e8. $\lim_{x\to\infty} \frac{\sqrt{x^2+1}}{x}$

4.4.e9. a. $\lim_{x\to\infty} \frac{2e^x - e^{-x}}{e^x + 3e^{-x}}$

b. $\lim_{x\to-\infty} \frac{2e^x - e^{-x}}{e^x + 3e^{-x}}$

4.5: Summary of Curve Sketching

When a computer produces the graph of a function, it does so by calculating and precisely plotting a hundred or so points on the curve. In this section, we'll sketch the graph without calculating more than a couple of points, using ideas of calculus and precalculus:

- Domain (the set of all x-values at which $f(x)$ exists)
- Intercepts
  - y-axis: $f(0) =?$
  - x-axis: $f(?) = 0$
- Symmetry of even and odd functions
  - $f(x)$ is even if $f(-x) = f(x)$.
  - $f(x)$ is odd if $f(-x) = -f(x)$.
  - The graph of of an even function is symmetric across the y-axis.
  - The graph of of an odd function is symmetric through the origin.
- Asymptotes
  - $x = c$ is a Vertical Asymptote if $|y| \to \infty$ as $x \to c$.
  - $y = c$ is a Horizontal Asymptote if $y \to c$ as either $x \to \infty$ or $x \to -\infty$.
- Monotonicity
  - $f(x)$ is increasing when $f'(x)$ is positive.
  - $f(x)$ is decreasing when $f'(x)$ is negative.
- Concavity
  - The graph of $f(x)$ is concave up when $f''(x)$ is positive .
  - The graph of $f(x)$ is concave down when $f''(x)$ is negative.

4.5.e1. Sketch the graph of $y = \dfrac{x}{x^2+4}$.

4.5.e2. Sketch: $y = x^2 + x - \ln|x|$

4.5.e3. $y = xe^{-x}$

4.5.e4. $y = x + \sin x$ for $-2\pi \leq x \leq 2\pi$.

4.5.e5. $y = \dfrac{x^2}{x^2 - 4}$

4.5.e6. $y = x + \dfrac{27}{x^3}$

4.7: Optimization Problems

In this section, we look for absolute maxima or minima of functions arising in story problems. We'll start with two simple examples illustrating what we typically find in these problems.

4.7.e1. Maximize xy^2 subject to $x + y = 10$, $x, y \geq 0$.

4.7.e2. Minimize $x + y$ subject to the **constraint** $xy^2 = 256$, $x, y > 0$.

Tip: Questions in 4.7 usually ask for either an absolute max or an absolute min, but not both, because one of these is trivial or does not exist. If a question asks only for a max (or a min), it is safe to assume that one exists, and if you find only one critical point, it's safe to assume that you've located the max (or min) in question.

4.7.e3. A rectangular box with a square base is to have volume 1. Find the dimensions of such a box with minimum surface area.

4.7.e4. A box will be constructed from a 4 ft by 6 ft piece of cardboard by cutting a square from each corner and folding up the sides. Find the dimensions of such a box with largest possible volume.

4.7.e5. A rectangular box with a square base and an open top is constructed from two different grades of cardboard. The material for the sides costs 1 ¢/m^2 and the material for the base costs 4 ¢/m^2. If the cost of materials of the box is limited to \$1.08, find the dimensions that will maximize the volume of the box.

4.7.e6. Find the point on $y = x^2$ closest to the point $(0, 1)$.

Tip: The author was fond of problems that, like the next example, contain an unspecified parameter. I suggest you simplify the problem by assigning it some sensible value.

4.7.e7. A soup can in the shape of a right circular cylinder is to be inscribed in a sphere of radius R. Find the dimensions of the can that will maximize the area of its label.

4.7.e8. A right circular cone is inscribed in a sphere of radius 1. Find the maximum possible volume of the cone.

4.9: Antiderivatives

An **antiderivative** for $f(x)$ is a function whose derivative is $f(x)$.

4.9.e1. $2x$ is the deritative of x^2, so x^2 is *an* antiderivative of $2x$. Are there others?

According to Corollary 2 of the Mean Value Theorem, if two functions have the same derivative on an interval, then they must differ by a constant on that interval. So, *every* antiderivative of $2x$ on $(-\infty, \infty)$ has the form $x^2 + C$ for some constant C.

Theorem/Definition 4.9.1. *If $F(x)$ is an antiderivative of $f(x)$ on an interval, than every antiderivative of f on that interval equals $F(x) + C$ for some constant C, and so we say that the $F(x) + C$ is the* **General antiderivative** *of f on that interval. The general antiderivative of $f(x)$ is denoted $\int f(x)\,dx$.*

4.9.e2. Find the general antiderivative of the function:

a. $3x^2 - \frac{15}{2}\sqrt{x} + 7$

b. $2e^x - 4\sin x + 4\cos(4x)$

c. $\sec^2 x - \dfrac{1}{x}$.

A table of antiderivatives is practically the same as a table of derivatives:

$f(x)$	derivative of f
$f(x)$	$f'(x)$
x^n	nx^{n-1}
$\ln\|x\|$	x^{-1}
e^x	e^x
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\csc^2 x$
$\sec x$	$\sec x \tan x$
$\csc x$	$-\csc x \cot x$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\sec^{-1} x$	$\frac{1}{x\sqrt{x^2-1}}$

$f(x)$	antiderivative of f
$f(x)$	$\int f(x)\,dx$
x^n	$\begin{cases} \frac{1}{n+1}x^{n+1} & \text{if } n \neq -1 \\ \ln\|x\| & \text{if } n = -1 \end{cases}$
e^x	e^x
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\sec^2 x$	$\tan x$
$\csc^2 x$	$-\cot x$
$\sec x \tan x$	$\sec x$
$\csc x \cot x$	$-\csc x$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$
$\frac{1}{1+x^2}$	$\tan^{-1} x$
$\frac{1}{x\sqrt{x^2-1}}$	$\sec^{-1} x$

There are fewer combination rules for antiderivatives than for derivatives.

Fact 4.9.2. *If F and G are antiderivatives for f and g, and if c is a constant, then*

1. *$F + G$ is an antiderivative for $f + g$.*
2. *$F - G$ is an antiderivative for $f - g$.*
3. *cF is an antiderivative for cf.*

It is important to note that FG is **not** an antiderivative for fg, and F/G is not an antiderivative for f/g.

4.9.e3. x^2 is an antiderivative for $2x$, and e^x is an antiderivative for itself, but x^2e^x is **not** an antiderivative for $2xe^x$, and $\dfrac{e^x}{x^2}$ is **not** an antiderivative for $\dfrac{e^x}{2x}$

4.9.e4. Find an antiderivative for $(x^2 - 3)\sqrt{x}$.

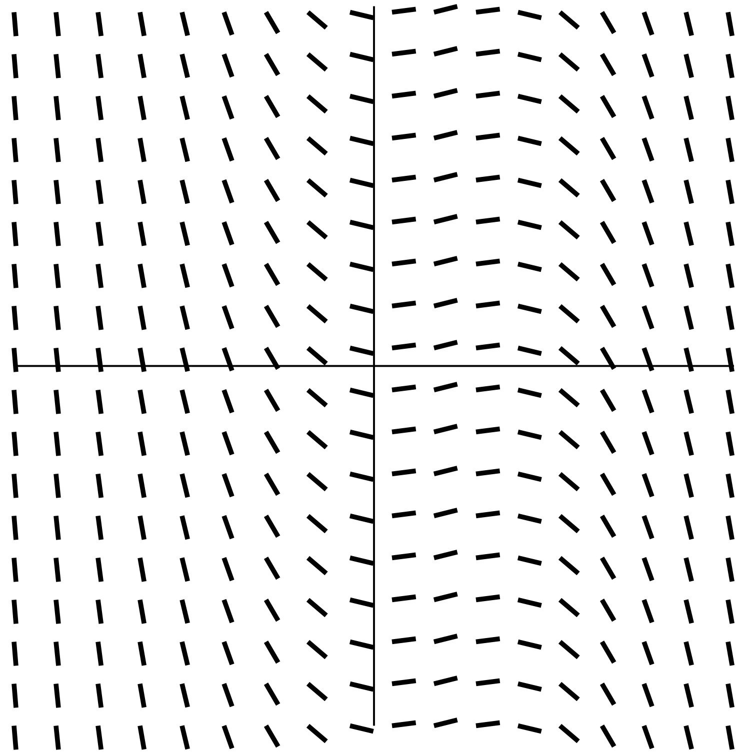
4.9.e5. Find an antiderivative for $\dfrac{x^2 - 3}{\sqrt{x}}$.

Initial Value Problems Find $f(x)$, given $f'(x)$ and $f(x_0)$ for some x_0.

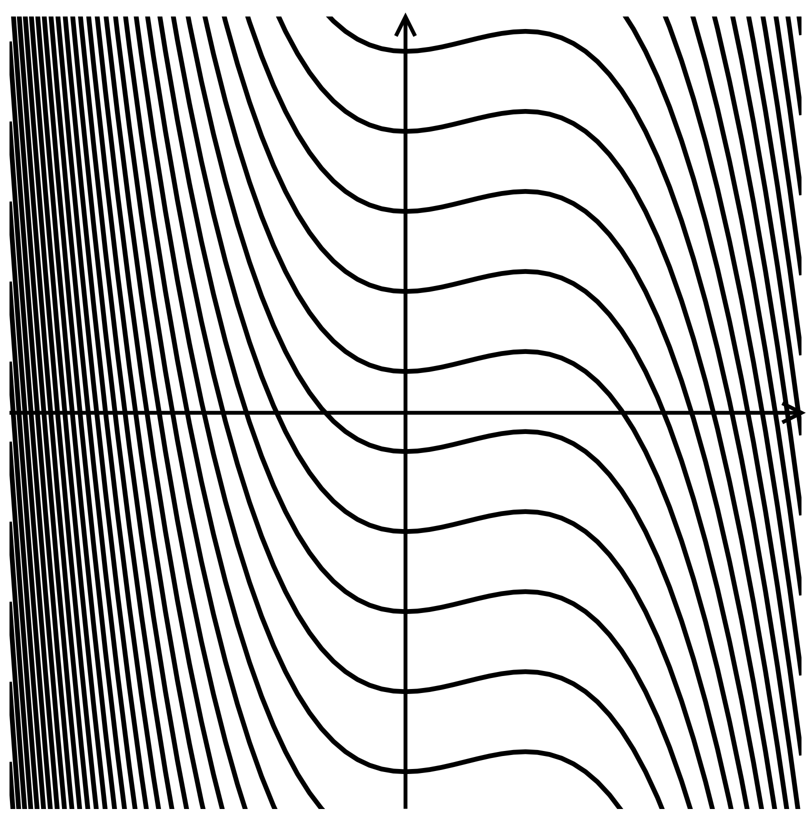
4.9.e6. Find $f(x)$ if $f'(x) = x - x^2$ and $f(0) = 1$.

4.9.e7. Find $f(x)$ if $f'(x) = x - x^2$ and $f(-) = \frac{1}{3}$.

$$\frac{dy}{dx} = x - x^2$$



$$y = \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 + C$$



4.9.e8. Find $f(x)$ if $f'(x) = \dfrac{1}{x}$ and $f(-2) = 1$.

4.9.e9. Find $f(x)$ if $f''(x) = x - x^2$ and $f'(1) = 2$ and $f(1) = -1$

Classic application.

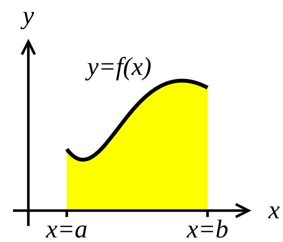
Ignoring friction, the downward acceleration due to gravity of a falling object is constant.

4.9.e10. At time $t = 0$, a marble is tossed from a 150 ft tower with an upward velocity of 20 ft/sec. Assuming its acceleration due to gravity is $a(t) = -32$ ft/sec^2, find the marble's altitude at time t. (In other words, express the marble's height as a function of time t.)

5.1: The Area and Distance Problems

Calculus can be divided into two parts, the study of derivatives, or *differential calculus*, and the study of integrals, or *integral calculus*. Just as differential calculus came about as a solution to the tangent line and velocity problems, as we saw in section 2.1, integral calculus was created to address the area and distance problems, which are introduced in this section.

The **Area Problem** is this: how can we find the area trapped between the the graph of a function $f(x)$ and the x-axis, between $x = a$ and $x = b$?



5.1.e1. Approximate the area between $y = x^2$ and the x-axis over the interval $[0, 2]$.

Approximating sums like these are called Riemann sums:

Definition 5.1.1. *Divide the interval $[a, b]$ into n subintervals of equal length Δx, and select a number x_i^* in each $(1 \leq i \leq n)$. If the function $f(x)$ is defined on $[a, b]$, then*

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + f(x_3^*)\Delta x + \cdots + f(x_n^*)\Delta x = \sum_{i=1}^{n} f(x_i^*)\Delta x$$

is called a **Riemann Sum** *for $f(x)$ on $[a, b]$.*

The symbol $\sum$ may be used to denote any kind of sum, not just a Riemann sum in calculus. Generally, $\sum_{i=1}^{n}$ means "the sum of values of the following, for i equal $1, 2, \ldots, n$." For instance,

$$\sum_{i=1}^{5}(i^2+1) =$$

As in 5.1.e1, when evaluating a Riemann sum, it's best to take out the common factor:

$$\sum_{i=1}^{n} f(x_i^*)\Delta x = \Delta x\big(f(x_1^*) + f(x_2^*) + f(x_3^*) + \cdots + f(x_n^*)\big) = \Delta x \sum_{i=1}^{n} f(x_i^*)$$

5.1.e1, continued. When n was 4, we chose the sample points to be the midpoints of the subintervals:

$$x_1^* = \frac{1}{4}, \quad x_2^* = \frac{3}{4}, \quad x_3^* = \frac{5}{4}, \quad x_4^* = \frac{7}{4},$$

and when $n = 6$ our sample points were the right endpoints of the subintervals:

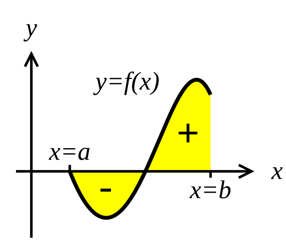
$$x_1^* = \frac{1}{3}, \quad x_2^* = \frac{2}{3}, \quad x_3^* = \frac{3}{3} = 1, \quad x_4^* = \frac{4}{3}, \quad x_5^* = \frac{5}{3}, \quad x_6^* = \frac{6}{3} = 2,$$

You can see some Riemann sums for x^2 on $[0, 2]$ at http://kunklet.people.cofc.edu/MATH120/riemannsuma.pdf. There are also many online Riemann sum calculators, e.g., https://mathworld.wolfram.com/RiemannSum.html.

There are two things to notice about these graphs and the accompanying calculations. First, the calculations suggest that the Riemann sums approach a limit as $n \to \infty$, and the pictures suggest that that limit is the area under the curve. Second, the limit seems to be the same regardless of how we choose the sample point in each subinterval.

5.1.e2. See http://kunklet.people.cofc.edu/MATH120/riemannsumb.pdf for some Riemann sums for the function $x^2 - \frac{3}{2}$ on the interval $[0, 2]$. Why are the Riemann sums negative, and what does this mean?

Conclusions The limit as $n \to \infty$ of the Riemann sums for $f(x)$ on $a \leq x \leq b$ is the **net signed area** between the graph of f and the x-axis between $x = a$ and $x = b$. Areas above the x-axis are counted as positive and areas below the x-axis are counted as negative.



The pre-calculus formula distance = rate × time applies only to object moving at a constant velocity. The **Distance Problem** is this natural question: how can we determine the distance traveled by an object with nonconstant velocity?

5.1.e3. Approximate the distance traveled by the object whose velocity is recorded in the following table.

t(sec)	0	60	120
v(m/sec)	10	36	46

5.1.e3, continued. Repeat, using more data:

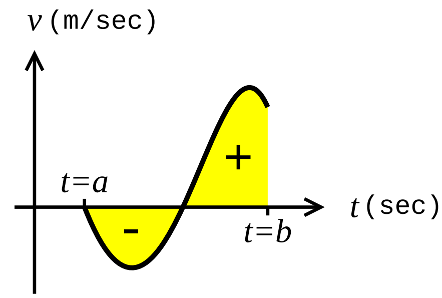
t(sec)	0	30	60	90	120
v(m/sec)	10	25	36	43	46

5.1.e3, continued. Repeat, using more data:

t(sec)	0	15	30	45	60	75	90	105	120
v(m/sec)	10	18	25	31	36	40	43	45	46

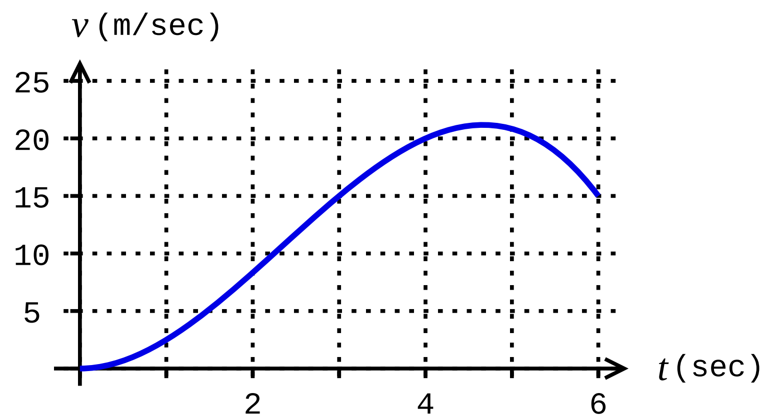
Note that these approximations are the same Riemann sums we saw in the area problem, and the true distance traveled by the object between $t = 0$ and $t = 120$ is the limit of these approximations as the number of t-subintervals $n \to \infty$, just as area was the limit of Riemann sums. (To continue this process, we need to know the object's velocity not just at finitely may times, but at *every* time t in $[0, 120]$. That is, we need v expressed as a function of t.)

Conclusions: The limit as $n \to \infty$ of the Riemann sums for $v(t)$ on $a \leq t \leq b$ is the **net** distance traveled by the object whose velocity at time t is $v(t)$. This means that, since the limit of Riemann sums also represents area, the net distance traveled is the *same* as the net signed area between the graph of velocity $y = v(t)$ and the t-axis between $t = a$ and $t = b$. Areas above the t-axis, where velocity is positive, measure motion in the positive direction and areas below the t-axis, when velocity is negative, measure motion in the negative direction.



5.1.e4. The velocity of a car for the first six seconds after it starts moving is graphed in the figure below.

a. When (between times 0 and 6 sec) is the car moving forward and when is it moving backward?



b. When (between times 0 and 6 sec) is the car speeding up and when is it slowing down?

c. Approximate the distance traveled by the car between time $t = 0$ and time $t = 6$ sec.

5.2: The Definite Integral

In the last section, we saw that the area problem and the distance problem were both solved by a limit of Riemann sums. In this section, we give a name to the limit and discover some of its properties.

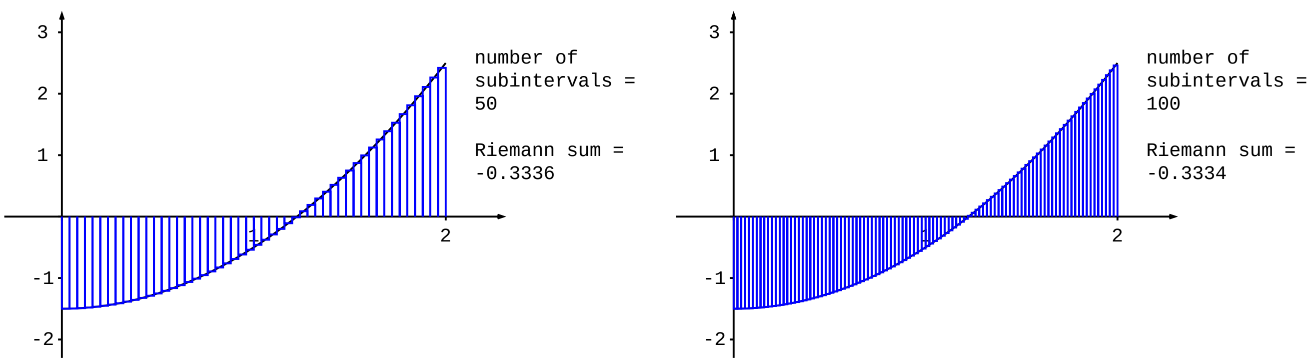
Definition 5.2.1. *If $f(x)$ is defined on the interval $[a, b]$, then the limit of its Riemann sums on $[a, b]$ as $n \to \infty$, if it exists, is called the* **definite integral** *(or* **Riemann integral***)* **from a to b of** $f(x)\,dx$*, written $\int_a^b f(x)\,dx$. That is,*

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x,$$

When this limit exists we say that $f(x)$ is **integrable** *on $[a, b]$.*

Note that $\int_a^b f(x)\,dx$, when it exists, is a number.

5.2.e1. Here are two Riemann sums for the function $x^2 - \frac{3}{2}$ on the interval $[0, 2]$ (which we saw in example 5.1.e2):



Based on these two Riemann sums, $\int_0^2 (x^2 - \frac{3}{2})\,dx$ appears to equal about -0.3333.

Not every function in integrable, but a great many are:

Theorem 5.2.2. *If $f(x)$ is piecewise continuous with finitely many jump or removable discontinuities on $[a, b]$, then $\int_a^b f(x)\,dx$ exists. In particular, if $f(x)$ in continuous on $[a, b]$, then f is integrable.*

In section 5.1, we concluded that the definite integral calculates net signed area *and* net distance travelled:

Fact 5.2.3. *The definite integral* $\int_a^b f(x)\,dx$ *can be interpreted as the:*

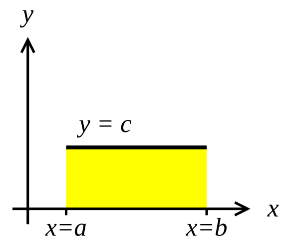
1) net signed area between the graph of $f(x)$ *and the* x*-axis between* $x = a$ *an* $x = b$*, and*
2) net distance traveled from time a *to* b *by an object whose velocity at time* x *is* $f(x)$*.*

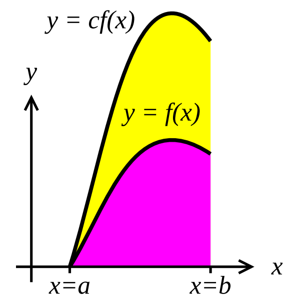
5.2.e2. Calculate the Riemann integral $\int_0^1 e^x\,dx$ from Definition 5.2.1.

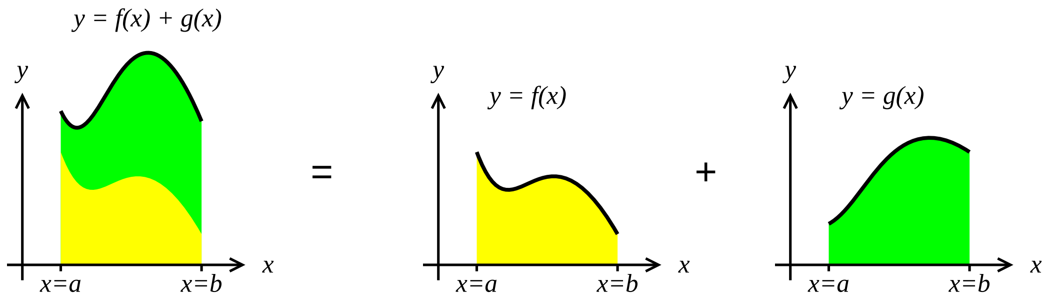
Next, we'll see some useful algebraic properties of the definite integral.

Fact 5.2.4. *If $f(x)$ and $g(x)$ are integrable on $[a, b]$ and c is a constant, then:*

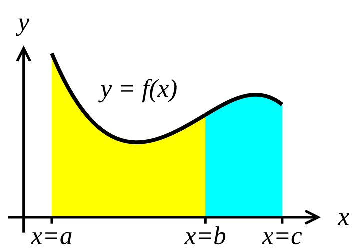
1. $\int_a^b c\,dx = c(b-a)$
2. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$

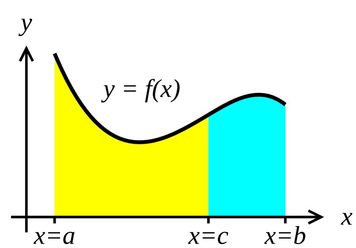






3. $\int_a^b \big(f(x)+g(x)\big)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
4. $\int_b^a f(x)\,dx \overset{\text{def}}{=} -\int_a^b f(x)\,dx$
5. $\int_a^a f(x)\,dx \overset{\text{def}}{=} 0$

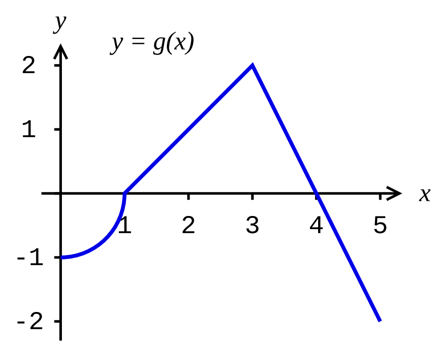




6. $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$
(assuming f is integrable on the largest of these three intervals).
7. *If $f(x) \geq 0$ for all x in $[a, b]$, then* $\int_a^b f(x)\,dx \geq 0$.
8. *If $f(x) \geq g(x)$ for all x in $[a, b]$, then* $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

5.2.e3. Find $\int_5^1 \big(2f(x) + \pi\big)\,dx$ if $\int_1^3 f(x)\,dx = 5$ and $\int_5^3 f(x)\,dx = 7$.

5.2.e4. Let $g(x)$ be the function graphed here (its first piece is a quarter circle), and find the following.



a. $\int_0^4 g(x)\,dx$

b. $\int_0^5 g(x)\,dx$

c. $\int_3^5 g(x)\,dx$

5.2.e5. Suppose now that $g(x)$ from 5.2.e4 is the velocity at time x of an object moving along a number line. Find the following.

a. The displacement (net distance traveled) of the object between time 0 and time 4.

b. The displacement of the object between time 0 and time 1.

c. When the object is traveling forward, and when it's traveling backward.

d. When the object is speeding up, and when it's slowing down.
(For more on c and d, see section 3.7.)

5.2.e6. Find the following.

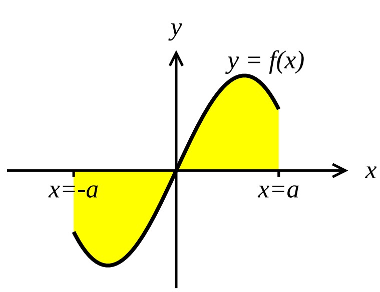
a. $\int_0^4 |x-3|\,dx$

b. $\int_{-1}^2 2x\,dx$

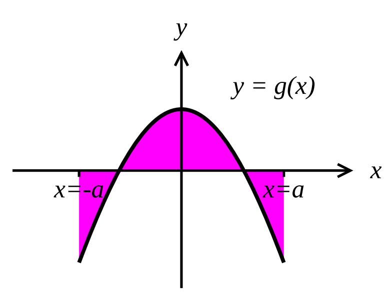
c. $\int_{-3}^0 \sqrt{9-x^2}\,dx$

Finally, here's 7 of section 5.4, that seems to go well in this section:

Fact 5.2.5. *If f is an odd function and g is an even function, and if $a > 0$, then*



$$\int_{-a}^{a} f(x)\,dx = 0$$



$$\int_{-a}^{a} g(x)\,dx = 2\int_{0}^{a} g(x)\,dx$$

5.3: The Fundamental Theorem of Calculus

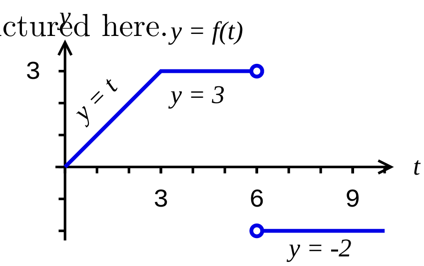
The Fundamental Theorem of Calculus is the statement of exactly how the two subjects of calculus, the derivative and the integral, fit together. When we think of the derivative and integral from the point of view of geometry, it is a completely surprising result, while from the point of view of velocity and displacement, it seems so obvious that a student wonders why it even requires proof.
The first half of the FTC says that the derivative of an integral with respect to its upper endpoint is the integrand.

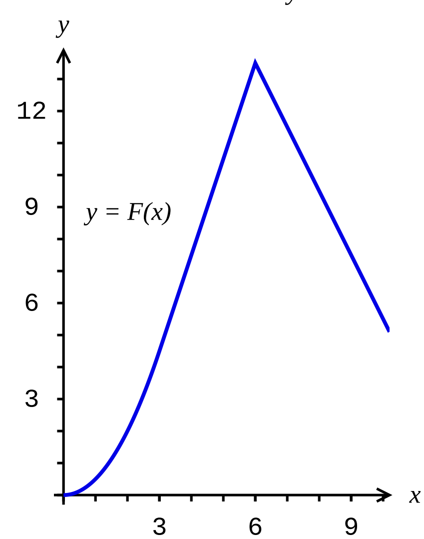
Fundamental Theorem of Calculus I (FTC.I) 5.3.1. *If the function f is integrable on the interval $[a, b]$, and if $F(x)$ is defined as*

$$F(x) \stackrel{\text{def}}{=} \int_a^x f(t)\,dt,$$

then, wherever $f(x)$ is continuous, $F(x)$ is differentiable and $F'(x) = f(x)$.

5.3.e1. Verify that the FTC.I is true for the function f pictured here.





Here's a graph of the function $F(x)$ we found.

From what we saw in section 5.1 and 5.2, we expect definite integral $\int_a^x f(t)\,dt$ to be the displacement between times a and x of an object whose velocity at time t is $f(t)$. But that displacement is the object's position at time x relative to its position at time a. Since we expect the derivative of position to be velocity, the FTC.I is a verification that the definite integral does what it was designed to do.

Next we see some typical problems that require us to use FTC.I

5.3.e2. Find the derivative.

a. $\dfrac{d}{dx}\displaystyle\int_0^x \sec t\,dt$

b. $\dfrac{d}{dx}\displaystyle\int_x^0 \ln(t^4+2)\,dt$

One consequence of the FTC.I is that every continuous function has an antiderivative which can be expressed as the function's definite integral. Even though an antiderivative of $\sec x$ doesn't appear in the table of antiderivatives in section 4.9, we see one in 5.3.e2a.

c. $\dfrac{d}{dx}\displaystyle\int_1^{e^x} \ln u\,du$

d. $\dfrac{d}{dx}\displaystyle\int_{x^2}^{x^3} \sin^2\theta\,d\theta$

5.3.e3. How are $\int_1^x \frac{1}{1+t^2}\,dt$ and $\tan^{-1} x$ related? Are they the same?

Generally, if F is an antiderivative for f, then

$$\int_a^x f(t)\,dt = F(x) + C$$

When we find the value of C by evaluating both sides of this equation at $x = a$, we have a formula for evaluating the definite integral!

Fundamental Theorem of Calculus II (FTC.II) 5.3.2. *If $F(x)$ is any antiderivative of $f(x)$ and it f is continuous on $[a, b]$, then*

$$\int_a^b f(t)\,dt = F(b) - F(a) \stackrel{\text{def}}{=} F(x)\Big|_a^b$$

5.3.e4. Evaluate the definite integral.

a. $\int_0^1 e^x\,dx$

b. $\int_0^{\pi/2} \sin x\,dx$

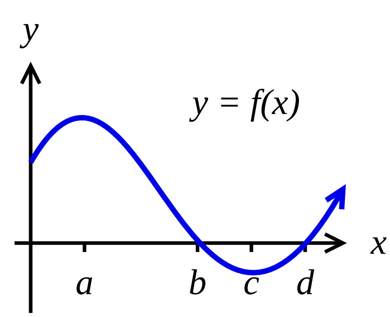
c. $\int_{-6}^{-2} \frac{dt}{t}$

d. $\int_1^4 \left(x\sqrt{x} + \frac{2+x^3}{x^2} \right) dx$

5.3.e5. Find the area trapped between the curve $y = x^2 - 5x - 6$ and $y = 0$.

5.3.e6. A particle moving along a number line has velocity $v(t) = e^t - 1$ at time t. Find the displacement of the object between times $t = -2$ and $t = 2$.

5.3.e7. Let $f(x)$ be the function graphed here, and define $g(x) = \int_0^x f(t)\,dt$.
a. Find the critical numbers of $g(x)$. Where does $g(x)$ have a local max or min?
b. Find the inflection points of $g(x)$. Where is the graph of $g(x)$ concave up and concave down?



5.4: Indefinite integrals and net change

Earlier, we defined the *definite* integral of a function to be the limit of its Riemann sums. There's another way the word "integral" and the integral sign $\int$ is used. To be clear, we'll state them both now.

Definition 5.4.1. *The* **definite integral** *from a to b of $f(x)\,dx$, written*

$$\int_a^b f(x)\,dx,$$

is the limit of the Riemann sums of f on the interval $[a, b]$ as the number of subintervals goes to infinity.

The **indefinite integral** *of $f(x)\,dx$, written*

$$\int f(x)\,dx,$$

is the collection of all antiderivatives of $f(x)$.

That is,

$$\begin{aligned}\int_a^b f(x)\,dx &= \text{a number.}\\ &= \text{the net signed area trapped between the graph of } f(x) \\ &\quad\ \text{and the } x\text{-axis from } x = a \text{ to } x = b.\\ &= \text{the net distance traveled from time } x = a \text{ to time } x = b\\ &\quad\ \text{by an object whose velocity at time } x \text{ is } f(x).\end{aligned}$$

and

$$\begin{aligned}\int f(x)\,dx &= \text{a collection of functions.}\\ &= \text{the set of all antiderivatives of } f(x).\\ &= F(x) + C, \text{ where } F'(x) = f(x)\end{aligned}$$

5.4.e1. Find the following.

a. $\displaystyle\int \frac{1}{1+x^2}\,dx$

b. $\displaystyle\int_0^1 \frac{1}{1+x^2}\,dx$

To restate the results from section 4.9, page 91, here's a list of the indefinite integrals we know so far, and the combination rules for integration:

$f(x)$	$\int f(x)\,dx$
x^n	$\begin{cases} \frac{1}{n+1}x^{n+1} + C & \text{if } n \neq -1, \text{ and} \\ \ln\lvert x\rvert + C & \text{if } n = -1. \end{cases}$
e^x	$e^x + C$
$\cos x$	$\sin x + C$
$\sin x$	$-\cos x + C$
$\sec^2 x$	$\tan x + C$
$\csc^2 x$	$-\cot x + C$
$\sec x \tan x$	$\sec x + C$
$\csc x \cot x$	$-\csc x + C$
$\dfrac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x + C$
$\dfrac{1}{1+x^2}$	$\tan^{-1} x + C$
$\dfrac{1}{x\sqrt{x^2-1}}$	$\sec^{-1} x + C$

If f and g are functions and c is a constant, then

$$\int \big(f(x) + g(x)\big)\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int cf(x)\,dx = c\int f(x)\,dx$$

5.4.e2. Evaluate the definite or indefinite integrals:

a. $\int (x^2 - 1)^3 \, dx$

b. $\int_0^{\pi/4} \frac{1}{\cos^2 t} \, dt$

c. $\int \frac{r^3 - 7r^2 + 1}{2r} \, dr$

d. $\int \frac{x^3 + 8}{x + 2} \, dx$

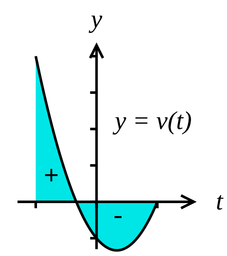
e. $\int_{-2}^{2} |x| \, dx$

f. $\int_0^{1/2} \frac{du}{\sqrt{1 - u^2}}$

5.4.e3. The velocity of an object moving along a number line is $v(t) = 3t^2 - 2t - 1$.

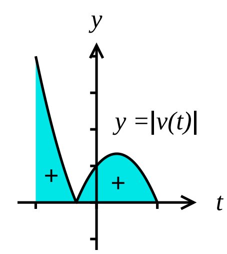
a. Find the object's position at time t (to the extent possible).

b. Find the *net* distance it travels between times -1 and 1.



net distance traveled
= net signed area

c. Find the *total* distance it travels between times -1 and 1.



total distance traveled
= total area

5.4.e4. Suppose an object's acceleration at time t is $2t$ and that is velocity at time 1 is 0. Find the net and total distances traveled between times -1 and 2.

The definite integral of a rate of change

The basic fact we've used in the last two examples,

$$\int_a^b (\text{velocity})\, dt = \text{displacement from time } a \text{ to time } b$$

could be rephrased

$$\int_a^b (\text{rate of change of } \textbf{position})\, dt = \text{change in } \textbf{position} \text{ from time } a \text{ to time } b.$$

So it must also be true that, for any quantity **X** that might change with time,

$$\int_a^b (\text{rate of change of } \mathbf{X})\, dt = \text{change in } \mathbf{X} \text{ from time } a \text{ to time } b.$$

For example

$$\int_a^b (\text{rate of change of } \textbf{temperature})\, dt = \text{change in } \textbf{temperature} \text{ from time } a \text{ to time } b.$$

$$\int_a^b (\text{rate of change of } \textbf{altitude})\, dt = \text{change in } \textbf{altitude} \text{ from time } a \text{ to time } b.$$

$$\int_a^b (\text{rate of change of } \textbf{wealth})\, dt = \text{change in } \textbf{wealth} \text{ from time } a \text{ to time } b.$$

$$\int_a^b (\text{rate of change of } \textbf{population})\, dt = \text{change in } \textbf{population} \text{ from time } a \text{ to time } b.$$

5.4.e5. Suppose $v(t)$ is the rate of change (in \$/day) of the balance in my bank account. What do $\int_0^{30} v(t)\, dt$ and $\int_0^{30} |v(t)|\, dt$ mean?

5.4.e6. Suppose $r(t)$ is the rate of change (in m/hour) of the height of an elevator. What do $\int_0^{24} r(t)\, dt$ and $\int_0^{24} |r(t)|\, dt$ mean?

5.5: Substitution

Products of functions are generally hard to integrate. But one time when a product occurs in a derivative is when the derivative was obtained by the chain rule.

5.5.e1. $\int 3x^2 e^{x^3}\,dx = e^{x^3} + C$, because

5.5.e2. $\int x^3 \sin(x^4)\,dx = -\frac{1}{4}\cos(x^4) + C$, because

The **Substitution Rule** is a mechanism using the chain rule that helps us to simplify an integral. It says that an integral of the form $\int f\big(u(x)\big)u'(x)\,dx$ (for some function u of x) can be rewritten

$$\int f(u)\frac{du}{dx}\,dx = \int f(u)\,du$$

and thenceforth be treated as if u were the variable of integration. For this reason, substitution is also called a **change of variable**.

5.5.e2, continued. $\int x^3 \sin(x^4)\,dx$

5.5.e3. $\int (2x+1)\sqrt{x^2+x}\,dx$

5.5.e4. $\displaystyle\int \frac{\sqrt{\ln x}}{x}\,dx$

5.5.e5. $\displaystyle\int \frac{e^x}{e^{2x}+1}\,dx$

One nice thing about substitution is that it allows us to take a step in the problem without having to know the entire solution first.
Another is that you can always try a substitution, even if you're not sure it will work, as long as you can recognize when a substitution fails. In general, guessing a wrong substitution won't lead you to a wrong answer or waste a lot of time.

5.5.e6. a. $\displaystyle\int x^2(x^3+1)^{19}\,dx$ b. $\displaystyle\int x(x^3+1)^{19}\,dx$

You can always make a linear substitution $u = mx + b$:

5.5.e7. $\displaystyle\int (2x+1)^{19}\,dx$

5.5.e8. $\displaystyle\int 5e^{4-3x}\,dx$

5.5.e9. $\displaystyle\int x\sqrt{x+2}\,dx$

5.5.e10. $\displaystyle\int \tan x\, dx$

You can make more than one substitution in the same problem:

5.5.e11. $\displaystyle\int 2^{x+3}\, dx$

(Make two simple substitutions in a row is always equivalent to making just one more complicated substitution.)

5.5.e12. $\displaystyle\int \cos x \cot(\sin x) \sec^2(\sin x)\, dx$

$v = \sqrt{x}$ and $u = e^x$ are special substitutions that can always be competed, as seen in the next examples.

5.5.e13. $\displaystyle\int \frac{2\sqrt{x}}{x+1}\,dx$

5.5.e14. $\displaystyle\int \sqrt{e^x - 1}\,dx$

Substitutions in definite integrals

When making a change of variable in a definite integral, there are two ways to finish the problem. I'll demonstrate both in the next example.

5.5.e15. $\displaystyle\int_1^{\pi} \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

Changing the limits to the new variable isn't necessary, but it's guaranteed not to cost more work, and sometimes saves work.

5.5.e16. $\displaystyle\int_0^{\pi} e^{1+\sin x}\cos x\,dx$