

### 0.1: The Binomial Theorem and Pascal's Triangle.

The formulas

$$(x + y)^2 = x^2 + 2xy + y^2, \text{ and}$$

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3.$$

are special instances of the **Binomial Theorem**, which says that the coefficients in the expansion

$$(x + y)^n = x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \dots + \binom{n}{n-2}x^2y^{n-2} + \binom{n}{n-1}xy^{n-1} + y^n$$

are found in **Pascal's Triangle**:

$$\begin{array}{cccccc} & & & & & 1 \\ & & & & & & 1 \\ & & & & & 1 & & 1 \\ & & & & & 1 & & 2 & & 1 \\ & & & & & 1 & & 3 & & 3 & & 1 \\ & & & & & 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

Details can be found in <http://kunklet.people.cofc.edu/MATH111/pascal.pdf>

0.1.re.e1.

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$

$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

0.1.re.e2. Generate the next three rows of Pascal's Triangle.

0.1.re.e3. Expand the following.

a.	$(x + 3)^4$	b.	$(u + v)^5$	c.	$(u - v)^6$
d.	$(x^3 + y)^4$	e.	$(x - x^{-1})^5$	f.	$(\xi - 2)^6$

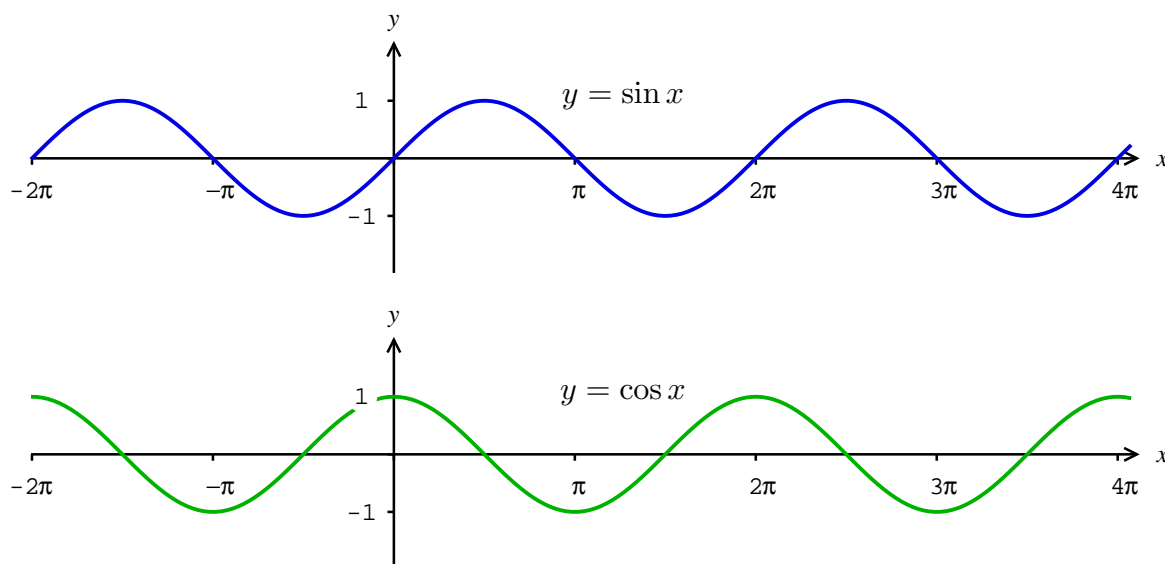
#### Answers

- 0.1.re.e2. 5th row: 1 5 10 10 5 1. 6th row: 1 6 15 20 15 6 1. 7th row: 1 7 21 35 35 21 7 1.
- 0.1.re.e3a.  $x^4 + 12x^3 + 54x^2 + 108x + 81$     0.1.re.e3b.  $u^5 + 5u^4v + 10u^3v^2 + 10u^2v^3 + 5uv^4 + v^5$
- 0.1.re.e3c.  $u^6 - 6u^5v + 15u^4v^2 - 20u^3v^3 + 15u^2v^4 - 6uv^5 + v^6$     0.1.re.e3d.  $x^{12} + 4x^9y + 6x^6y^2 + 4x^3y^3 + y^4$
- 0.1.re.e3e.  $x^5 - 5x^3 + 10x - 10x^{-1} + 5x^{-3} + x^{-5}$     0.1.re.e3f.  $\xi^6 - 12\xi^5 + 60\xi^4 - 160\xi^3 + 240\xi^2 - 192\xi + 64$

### Ap.D: Trigonometry

For a more complete review of trigonometry, see Appendix D of our text.

The two basic functions in trigonometry are the sine and cosine, graphed here:

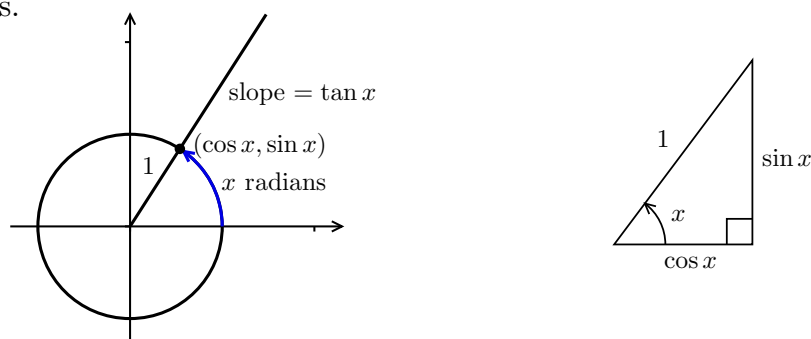


The other four trig functions are defined using sine and cosine:

$\tan x = \frac{\sin x}{\cos x}$	$\cot x = \frac{\cos x}{\sin x}$
$\sec x = \frac{1}{\cos x}$	$\csc x = \frac{1}{\sin x}$

$\sin x$  and  $\cos x$  are defined for all real numbers  $x$ , but  $\tan x$  and  $\sec x$  are undefined whenever  $\cos x = 0$ , and  $\cot x$  and  $\csc x$  are undefined whenever  $\sin x = 0$ .

By definition,  $\cos x$  and  $\sin x$  are the coordinates of the point on the **unit circle** (i.e., the circle of radius one centered at the origin)  $x$  radians counterclockwise from the positive horizontal axis.



Consequently, the ray through the origin  $x$  radians from the positive horizontal axis has slope  $\tan x$ , and, when  $x$  is an acute angle,  $\cos x$  and  $\sin x$  are the legs of this right triangle with hypotenuse 1 and interior angle  $x$

These basic trigonometric identities follow from the definitions of sine and cosine.

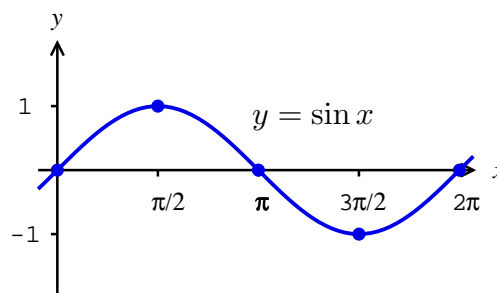
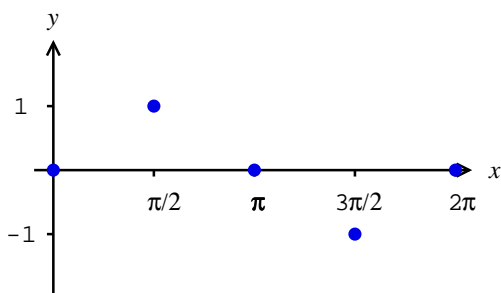
PYTHAGOREAN IDENTITIES
$\sin^2 x + \cos^2 x = 1$
$\tan^2 x + 1 = \sec^2 x$
$1 + \cot^2 x = \csc^2 x$
EVEN & ODD IDENTITIES
$\cos(-x) = \cos x$
$\sin(-x) = -\sin x$

### Sketching the sine and cosine

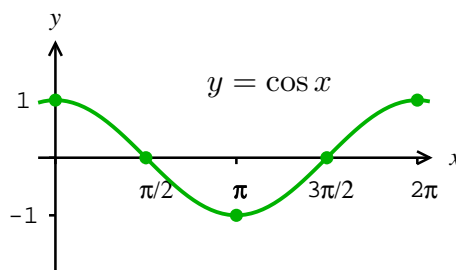
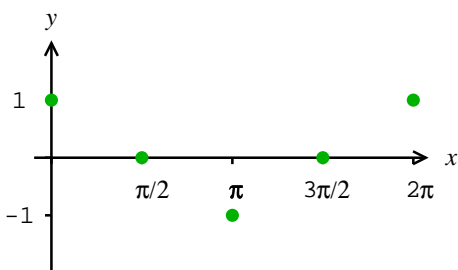
The sine and cosine are periodic functions having period  $2\pi$ , meaning

$$\sin(x + 2\pi) = \sin x \quad \text{and} \quad \cos(x + 2\pi) = \cos x.$$

To sketch one cycle of the sine, plot the vertical coordinates on the unit circle at the angles  $0, \pi/2, \pi, 3\pi/2,$  and  $2\pi$ . Be careful to make these five points equally spaced horizontally and vertically. Then connect them with a smooth curve.



To sketch the cosine, start by plotting the horizontal coordinates on the unit circle at the same five angles.



*Ap.D.re.e1.* Sketch the graphs of the sine and cosine on the given interval. Label hashmarks so as to clearly indicate all points where  $y = -1, 0, 1$  along your curve. (You can check your answers using Desmos.com.)

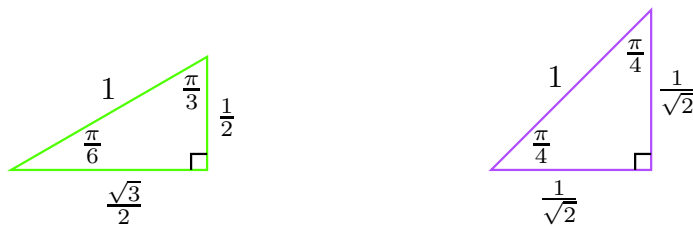
a.  $[0, 2\pi]$

b.  $[0, 3\pi]$

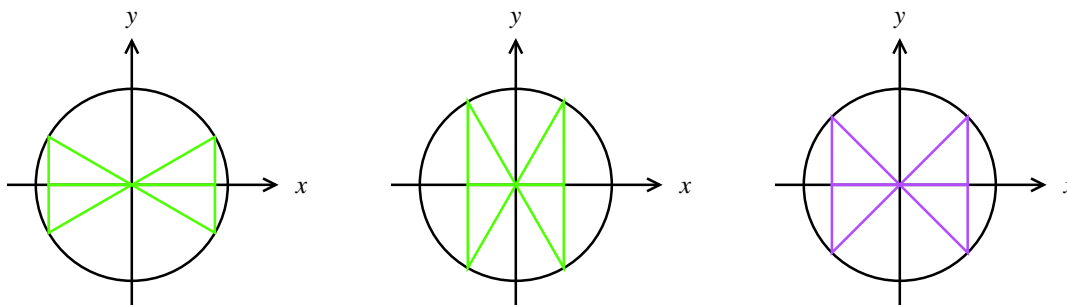
c.  $[-\pi, \pi]$

*Known values of sine and cosine*

We already know the values of sine and cosine at the four points where the unit circle intersects the  $x$  and  $y$  axes. By placing these two triangles:



around the unit circle like this:



we find the sines and cosines at 12 more points on the unit circle (and the infinitely many angles that reach those points).

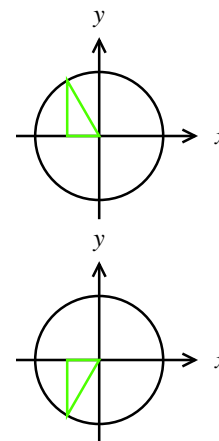
*Ap.D.re.e2.* Find all angles whose cosine is  $-\frac{1}{2}$ .

We recognize  $1/2$  as the short side of the 30-60-90 triangle, so for the cosine to be  $-1/2$ , the angle must be one of the two pictured at right. Find one angle that matches each drawing, for instance,

$$x = \pi - \pi/3 = 2\pi/3 \quad \text{and} \quad x = \pi + \pi/3 = 4\pi/3.$$

Then add all multiples of  $2\pi$  to describe *all* angles that fit the drawings:

$$x = 2\pi/3 + 2\pi n \quad \text{and} \quad x = 4\pi/3 + 2\pi n \quad (\text{where } n \text{ is any integer}).$$



*Ap.D.re.e3.* Find all solutions  $x$  to the given equation.

- |                           |                          |                        |
|---------------------------|--------------------------|------------------------|
| a. $\sin x = -1/\sqrt{2}$ | b. $\cos x = 0$          | c. $\tan x = -1$       |
| d. $\sec x = -2$          | e. $\csc x = 2/\sqrt{3}$ | f. $\cot x = \sqrt{3}$ |

*Solving trigonometric equations*

When the variable of the equation appears inside a trig function, first solve for the function, and then solve for the variable.

Ap.D.re.e4. Solve for  $x$  in the equation  $4\sin^2 x - 8\sin x + 3 = 0$

Solution: This is a quadratic equation in  $\sin x$  which can be solved by factoring:

$$(2\sin x - 3)(2\sin x - 1) = 0$$

which implies either  $\sin x = \frac{3}{2}$  or  $\sin x = \frac{1}{2}$ . The first of these has no real solutions, so the solution set of the original equation is the same as the solution set of  $\sin x = \frac{1}{2}$ , namely  $x = \pi/6 + 2\pi n$  or  $x = 5\pi/6 + 2\pi n$  (for any  $n \in \mathbb{Z}$ ).

It sometimes help to use the Pythagorean identities to rewrite the equation entirely in terms of one trig function.

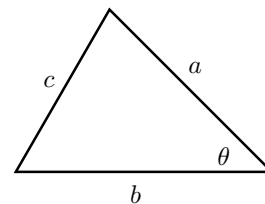
Ap.D.re.e5. Find all solutions  $x$  to the given equation.

- a.  $1 - \sin t - 2\cos^2 t = 0$    b.  $3\cos t - 2\sin^2 t = 0$    c.  $\sin^2 t + 3\sin t + 2 - \cos^2 t = 0$   
 d.  $1 - \cos^2 t = 0$    e.  $\cos^2 t - 3 = 0$    f.  $2\sin^2 t - 2\cos^2 t = 1$

*The Law of Cosines*

When we label the sides and any one angle of a triangle as shown in the figure, the Law of Cosines states that

$$c^2 = a^2 + b^2 - 2ab \cos \theta.$$



In case  $\theta$  is a right angle, the Law of Cosines reduces to the Pythagorean identity.

*The Area of a Triangle*

Since the height of the triangle pictured above is  $a \sin \theta$ , its area is

$$A = \frac{1}{2}ab \sin \theta$$

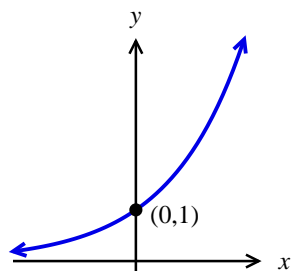
Ap.D.re.e6. Two ships leave the port of Charleston. One sails due east at a speed of 5 knots (nautical miles per hour) while the other sails in a direction  $30^\circ$  north of due east at 6 knots. What is the distance (in nautical miles) between the ships after two hours?

Answers

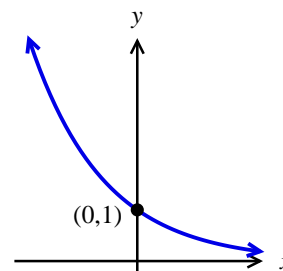
Ap.D.re.e3a.  $x = -\pi/4 + 2\pi n$  or  $x = -3\pi/4 + 2\pi n$    Ap.D.re.e3b.  $x = \pi/2 + 2\pi n$  or  $x = -\pi/2 + 2\pi n$   
 Ap.D.re.e3c.  $x = 3\pi/4 + 2\pi n$  or  $x = 11\pi/4 + 2\pi n$    Ap.D.re.e3d. (same as in example Ap.D.re.e2)  
 Ap.D.re.e3e.  $x = \pi/3 + 2\pi n$  or  $x = 2\pi/3 + 2\pi n$    Ap.D.re.e3f.  $x = \pi/3 + 2\pi n$  or  $x = 4\pi/3 + 2\pi n$   
 Ap.D.re.e5a.  $\sin t = -1/2$  or  $1$ ,  $t = \pi/2 + 2\pi n, 7\pi/6 + 2\pi n, 11\pi/6 + 2\pi n$    Ap.D.re.e5b.  $\cos t = -2$  (no sol'ns)  
 or  $1/2$ ,  $t = \pi/3 + 2\pi n, 5\pi/3 + 2\pi n$    Ap.D.re.e5c.  $\sin t = -1/2$  or  $-1$ ,  $t = 3\pi/2 + 2\pi n, 7\pi/6 + 2\pi n, 11\pi/6 + 2\pi n$   
 Ap.D.re.e5d.  $\cos t = \pm 1$ ,  $t = n\pi$    Ap.D.re.e5e.  $\cos t = \pm\sqrt{3}$  no real sol'ns   Ap.D.re.e5f.  $\sin t = \pm\sqrt{3}/2$ ,  $t = \pi/3 + 2\pi n, 2\pi/3 + 2\pi n, 4\pi/3 + 2\pi n, 5\pi/3 + 2\pi n$    Ap.D.re.e6.  $\sqrt{100 + 144 - 2 \cdot 10 \cdot 12 \cos \pi/6} = \sqrt{244 - 120\sqrt{3}}$

### 1.4: Exponential Functions

An **exponential function** is one of the form  $f(x) = a^x$  for some positive number  $a$ . All exponential functions have the same graph, in the sense that the graph of any one can be obtained from that of any other by rescaling in the  $x$ -direction and, possibly, reflecting about the  $y$ -axis:



$$y = a^x \\ a > 1$$



$$y = a^x \\ 0 < a < 1$$

The symbol  $e$  represents a mathematical constant, roughly equal to 2.71828. For reasons we'll go into in section 3.1,  $e^x$  is the most commonly occurring exponential function in calculus.

1.4.re.e1. Make a rough sketch of the graphs of the following. Label all intercepts and asymptotes. How do the four graphs compare?

- a.  $y = 2^x$       b.  $y = e^x$       c.  $y = (\frac{1}{2})^x$       d.  $e^{-x}$       e.  $y = e^{2x}$

(In the sciences, exponential functions are useful for modeling population growth, asset appreciation/depreciation, radioactive decay, and other quantities that grow or decay **exponentially** over time, that is, at a constant percent per unit of time. For example, radioactive isotopes exhibit exponential decay. A sample of Fe-59 (iron 59) will lose about 1.55% of its mass per day.

A quantity  $y$  that grows/decays exponentially can be expressed as

$$y = C \cdot a^t$$

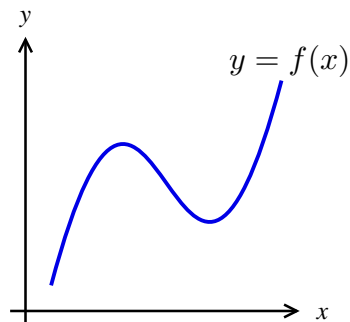
for some constants  $C$  and  $a$ .)

#### Answers

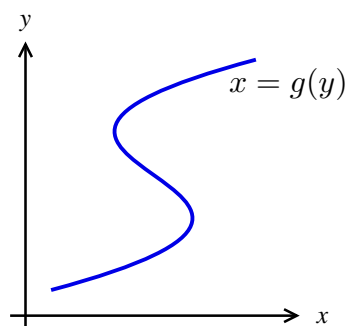
1.4.re.e1. All five curves have the same  $y$ -intercept  $(0, 1)$  and no  $x$ -intercept. All five lie above the  $x$ -axis.  $y = 0$  is a horizontal asymptote of  $2^x$ ,  $e^x$ , and  $e^{2x}$  as  $x \rightarrow -\infty$  and of  $e^{-x}$  and  $(\frac{1}{2})^x$  as  $x \rightarrow +\infty$ .  $y = e^x$  rises more steeply than  $y = 2^x$ ;  $y = e^{2x}$  rises more steeply than  $y = e^x$ .  $y = e^{-x}$  is the reflection of  $y = e^x$  across the  $y$ -axis;  $y = (\frac{1}{2})^x = 2^{-x}$  is the reflection of  $y = 2^x$  across the  $y$ -axis.

### 1.5: Inverse Functions, including logarithms and inverse trig functions

A curve that passes the **vertical line test** has an equation the form  $y = f(x)$  for some function  $f$ .

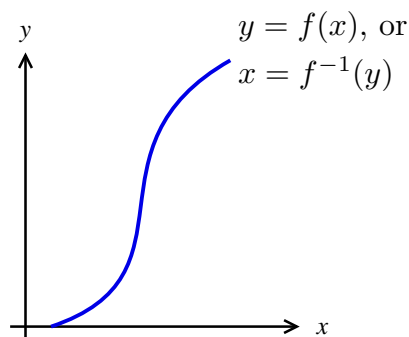


A curve that passes the **horizontal line test** has an equation the form  $x = g(y)$  for some function  $g$ .



And a curve that passes both has both types of equations. In this case, we say that  $f$  is **invertible**, and we name the other function  $f^{-1}$  (read “ $f$ -inverse”) due to the relation

$$\begin{aligned} x &= f^{-1}(f(x)) && \text{if } x \in \text{domain } f = \text{range } f^{-1} \\ y &= f(f^{-1}(y)) && \text{if } y \in \text{domain } f^{-1} = \text{range } f \end{aligned}$$



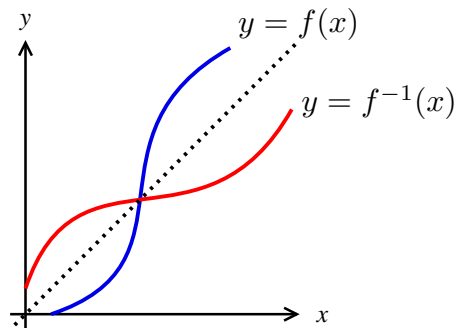
Since the equation

$$y = f^{-1}(x)$$

is obtained from

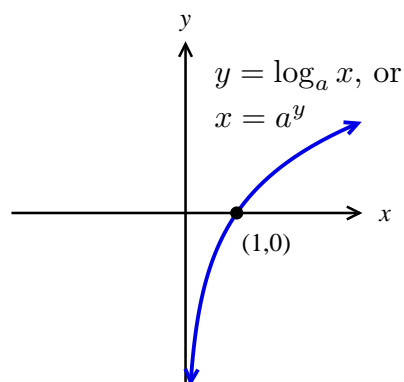
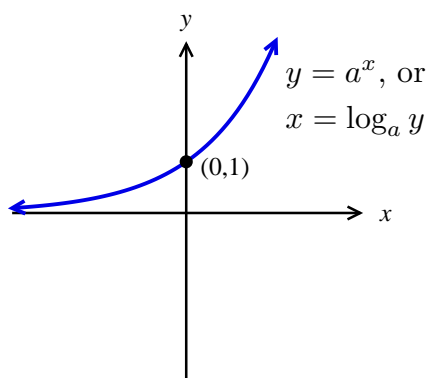
$$x = f^{-1}(y)$$

by interchanging  $x$  and  $y$ , the graph of  $f^{-1}$  is obtained by reflecting the graph of  $f$  across the line  $y = x$ .



## Logarithms

The inverse of the exponential function  $a^x$  is called  $\log_a x$  (read “log base  $a$  of  $x$ ”).



By convention, “ln” stands for  $\log_e$ , the **natural logarithm**. In calculus, we work with the natural log almost exclusively.

1.5.re.e1. Sketch graphs of the following functions. Find all intercepts and asymptotes. (You can check your answers using Desmos.com.)

a.  $\ln x$

b.  $\ln(x + 2)$

c.  $-\ln x$

d.  $-2 \ln x$

Some important properties of the natural logarithm:

1. domain  $\ln x = (0, \infty) = \text{range } e^x$
2. range  $\ln x = (-\infty, \infty) = \text{domain } e^x$
3.  $x = e^{\ln x}$  for all  $x > 0$ .
4.  $x = \ln(e^x)$  for all real numbers  $x$ .
5.  $\ln(AB) = \ln A + \ln B$  for all  $A$  and  $B > 0$ .
6.  $\ln\left(\frac{A}{B}\right) = \ln A - \ln B$  for all  $A$  and  $B > 0$ .
7.  $\ln(A^t) = t \ln A$  for all  $A > 0$  and for any real  $t$ .

The **change of base** formula allows us to rewrite logs of any base in terms of ln:

$$8. \log_b x = \frac{\ln x}{\ln b} \text{ for all } b \text{ and } x > 0.$$



1.5.re.e2. Combine into a single log.

a.  $\ln(2x - 1) + \ln(x + 1)$

b.  $\ln(x + 3) - \ln(x^2 - 4x - 21)$

c.  $2 \ln x - \frac{1}{2} \ln(x + 3) + 3 \ln(x - 1)$

1.5.re.e3. Do the opposite of what you did in example 1.5.re.e2; break into multiple logs with simple arguments.

a.  $\ln(12x^3 - 40x^2 + 12x)$

b.  $\ln((3x^2 - 13x + 4)^4)$

c.  $\ln\left(\frac{x^2 - 4}{x^3 - 8}\right)$

1.5.re.e4. Solve for x.

a.  $e^{x-4} = 2$

b.  $\ln(3x - 4) = -1$

c.  $3^{2x+1} = 2$

#### Answers

1.5.re.e2a.  $\ln(2x - 1)(x + 1)$ , or  $\ln(2x^2 + x - 1)$  1.5.re.e2b.  $\ln\left(\frac{x+3}{x^2-4x-21}\right) = \ln\left(\frac{1}{x-7}\right)$

1.5.re.e2c.  $\ln\left(\frac{x^2(x-1)^3}{\sqrt{x+3}}\right)$  1.5.re.e3a.  $\ln(3x - 1) + \ln(x - 3) + \ln x + 2 \ln 2$  1.5.re.e3b.  $4 \ln(3x - 1) + 4 \ln(x - 4)$

1.5.re.e3c. Hint:  $x^2 - 4$  and  $x^3 - 8$  have a common factor. Answer =  $\ln(x + 2) - \ln(x^2 + 2x + 4)$ .

1.5.re.e4a.  $x = 4 + \ln 2$  1.5.re.e4b.  $x = \frac{1}{3}(4 + e^{-1})$  1.5.re.e4c. Hint: take ln of both sides.  $x = \frac{1}{2}\left(\frac{\ln 2}{\ln 3} - 1\right)$

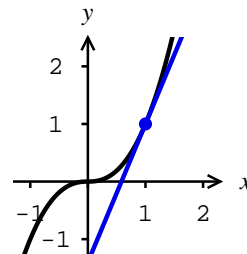
## 2.1: The Tangent Line and Velocity Problems

This section is a motivation for what follows in Chapter 2. Its calculator-based problems are good exercises but would have to be carefully recrafted before they could appear on our no-calculator exams.

Two problems gave rise to differential calculus (that is, the study of derivatives).

The **Tangent Line Problem**: what is the slope of the line tangent to the graph of a function  $y = f(x)$  at a given point?

(In calculus when we refer to the slope of a curve at a point, we mean the slope of the line tangent to the curve at that point.)



The **Velocity Problem**: what is the (instantaneous) velocity of an object at a given time if its position at time  $x$  is  $f(x)$ ?

(In calculus, unless we specify otherwise, “velocity” always means “instantaneous velocity,” not “average velocity.”)

The solution to both problems uses the **difference quotient**

$$\frac{f(a+h) - f(a)}{h},$$

which can be interpreted both as

1. the slope of the secant line passing through  $y = f(x)$  at  $x = a$  and  $x = a + h$ , and
2. the average velocity between times  $x = a$  and  $x = a + h$  of an object whose position at time  $x$  is  $f(x)$ .

When  $h$  is very close to zero, we expect the difference quotient to approximate both

- 1\*. the slope of the line tangent to  $y = f(x)$  at  $x = a$ , and
- 2\*. the velocity at time  $x = a$  of an object whose position at time  $x$  is  $f(x)$ .

Consequently, the velocity of an object at time  $a$  is the same as the slope of the line tangent to the graph of its position function at that time.

2.1.re.e1. Approximate the slope of the line tangent to  $y = \cos x$  at  $x = 1$  with slopes of secant lines. What appears to be the true slope of the tangent line?

Using a spreadsheet, I calculated some difference quotients for  $\cos x$  at  $a = 1$ :

$h$	$\frac{\cos(1+h) - \cos 1}{h}$	$h$	$\frac{\cos(1+h) - \cos 1}{h}$
0.1	-0.8671	-0.1	-0.8131
0.08	-0.8622	-0.08	-0.8190
0.06	-0.8572	-0.06	-0.8248
0.04	-0.8521	-0.04	-0.8304
0.02	-0.8468	-0.02	-0.8360
0.01	-0.8442	-0.01	-0.8388
0.005	-0.8428	-0.005	-0.8401

Based on these calculations, the slope of the line tangent to  $y = \cos x$  at  $x = 1$  appears to be roughly  $-0.841$ .

This interactive graph will allow you to see the secant lines and their slopes along  $y = \cos x$  or any other function you input: <https://www.desmos.com/calculator/qrvbhvz3wa>

2.1.re.e2. Use the interactive graph above to approximate the slope of the line tangent to the graph of  $f(x)$  at the given point. The actual slope appears in the answers.

a.  $f(x) = x^4 - 2x^2$  at  $a = 0.5$    b.  $f(x) = \sqrt{x+1}$  at  $a = 2$    c.  $f(x) = 2\sin^2 x$  at  $a = 1$

2.1.re.e3. The table below gives the position  $s$  at several times of an object moving along an axis (e.g., the real number line). Estimate the object's velocity at time  $t = 4$  seconds.

$t$ (sec)	3.5	3.75	4.0	4.25	4.75
$s$ (m)	4.1	4.5	4.9	5.2	5.4

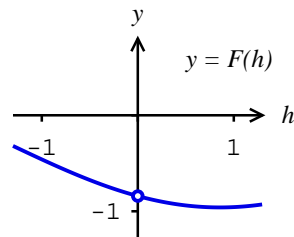
#### Answers

2.1.re.e2a. -1.5   2.1.re.e2b. 0.28868   2.1.re.e2c. 1.81859   2.1.re.e3. The object's average velocity from time 3.75 to time 4.0 is  $(4.9-4.5)/0.25 = 1.6$  m/sec, and its average velocity from time 4.0 to time 4.25 is  $(5.2-4.9)/0.25 = 1.2$  m/sec. We would guess the velocity at time 4 to be the average of these, or 1.4 m/sec. A better approximation would require finer data.

## 2.2: Limits

In Example 2.1.re.e1, we saw that the slope of the secant line to  $y = \cos x$  at the points 1 and  $1 + h$  is

$$F(h) = \frac{\cos(1+h) - \cos 1}{h}$$



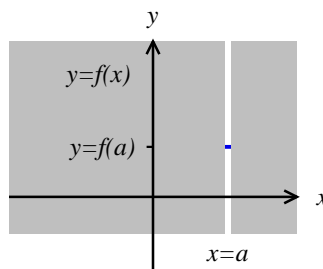
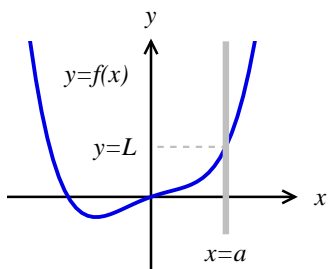
The graph of  $F(h)$  appears to the right. We concluded that the slope of the tangent line is roughly  $-0.841$ , not because  $F(0)$  equals that number (it doesn't), but because  $-0.841$  seems to be the natural altitude of the graph of  $F$  at  $h = 0$ , based on its altitudes at  $h$ 's near 0. The accepted way to indicate this (assuming our guess is correct) is to write

$$\lim_{h \rightarrow 0} \frac{\cos(1+h) - \cos 1}{h} = -0.841,$$

which is read, “the limit as  $h$  goes to 0 of  $\frac{\cos(1+h) - \cos 1}{h}$  equals  $-0.841$ .”

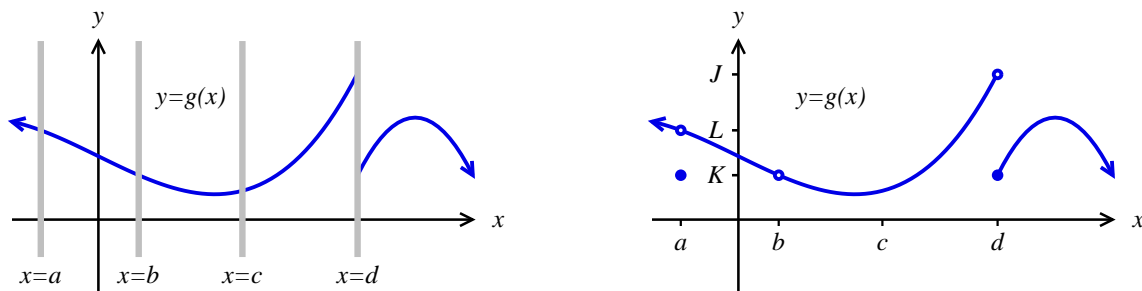
*Limit at  $a$  vs. function value at  $a$*

The  $\lim_{x \rightarrow a} f(x)$  is based on the values of  $f$  at  $x$ 's near  $a$ , ignoring what the function does at  $a$  itself. The function value  $f(a)$  is the opposite. It's based solely on what  $f$  does at  $a$ , and ignores all other values:



Most of the time, these two are the same thing.

2.2.re.e1. Compare function values and limits values at  $a$ ,  $b$ ,  $c$ , and  $d$  for the function  $g(x)$  graphed below.



In Example 2.2.re.e1, we write

$$\lim_{x \rightarrow d^-} g(x) = J$$

$$\lim_{x \rightarrow d^+} g(x) = K$$

These are read

“the limit as  $x$  approaches  $d$  from the left of  $g(x)$  equals  $J$ ,”

and

“the limit as  $x$  approaches  $d$  from the right of  $g(x)$  equals  $K$ .”

As a rule, the (two-sided) limit

$$\lim_{x \rightarrow a} f(x)$$

exists if and only if both one-sided limits

$$\lim_{x \rightarrow a^-} f(x) \text{ and } \lim_{x \rightarrow a^+} f(x)$$

exist and are equal. In that case, all three limits are the same.

2.2.re.e2. Let  $f(x) = \begin{cases} -x - 1 & \text{if } x < -1, \\ \frac{1}{2}(x + 1) & \text{if } -1 \leq x \leq 2, \text{ and} \\ 4 - x & \text{if } x > 2. \end{cases}$

Evaluate the limit, or explain why it does not exist:

a.  $\lim_{x \rightarrow -1} f(x)$

b.  $\lim_{x \rightarrow 2^-} f(x)$

c.  $\lim_{x \rightarrow 2^+} f(x)$

d.  $\lim_{x \rightarrow 2} f(x)$

*Infinite limits*

The symbol  $\infty$  always means positive infinity. Negative infinity must be written  $-\infty$ . You should be able to answer problems like the next one without a graphing device.

2.2.re.e3. Evaluate the limit, or explain why it does not exist:

a.  $\lim_{x \rightarrow 0^+} \ln x$

b.  $\lim_{x \rightarrow \pi/2^-} \tan x$

2.2.re.e4. Use the graph of  $p(x)$  to find the following. Your response to each part should be a number, infinity, -infinity, or DNE.

a.  $\lim_{x \rightarrow 0} p(x)$

b.  $\lim_{x \rightarrow 3} p(x)$

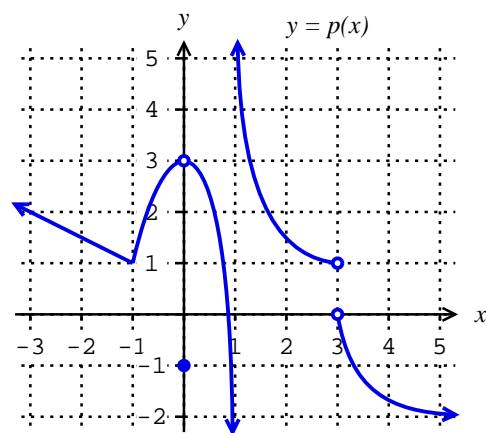
c.  $\lim_{x \rightarrow 3^+} p(x)$

d.  $p(0)$

e.  $p(3)$

f.  $\lim_{x \rightarrow -1} p(x)$

g.  $\lim_{x \rightarrow 1^-} p(x)$



$\frac{\text{nonzero}}{0}$  indicates a quantity that's blowing up to either  $\pm\infty$ . To decide between these two, determine the sign of the quotient.

2.2.re.e5. Evaluate the limit, or explain why it does not exist:

a.  $\lim_{x \rightarrow 0} \frac{1}{x^2}$

b.  $\lim_{x \rightarrow 3} \frac{x-4}{(x-3)^4}$

c.  $\lim_{x \rightarrow 0} \frac{1}{x^3}$

d.  $\lim_{x \rightarrow -1^+} \frac{x}{x^2 - x - 2}$

## Answers

2.2.re.e1.  $g(a) = K$ , but  $\lim_{x \rightarrow a} g(x) = L$ .  $g(b)$  DNE;  $\lim_{x \rightarrow b} g(x) = K$ .  $g(c)$  and  $\lim_{x \rightarrow c} g(x)$  are equal (to some number less than  $K$ ).  $g(d) = K$ , but  $\lim_{x \rightarrow d} g(x)$  DNE. 2.2.re.e2a. 0 2.2.re.e2b.  $3/2$  2.2.re.e2c. 2 2.2.re.e2d. the one-sided limits disagree at  $x = 2$ , so the two-sided limit does not exist. 2.2.re.e3a.  $-\infty$  2.2.re.e3b.  $\infty$  2.2.re.e4a. 3 2.2.re.e4b. DNE 2.2.re.e4c. 0 2.2.re.e4d.  $-1$  2.2.re.e4e. DNE 2.2.re.e4f. 1 2.2.re.e4g.  $-\infty$  2.2.re.e5a.  $\infty$  2.2.re.e5b.  $-\infty$  2.2.re.e5c. The limit from the right is  $\infty$ , but the limit from the left is  $-\infty$ , so the two-sided limit DNE. 2.2.re.e5d.  $\infty$

### 2.3: Limit Laws

The Limit Laws are rules that allow us to calculate limit more easily. There are two types:

#### I. Examples of some elementary continuous functions.

$$1. \lim_{x \rightarrow a} x = a \qquad 2. \lim_{x \rightarrow a} c = c \qquad 3. \lim_{x \rightarrow a} e^x = e^a$$

**II. Combination laws.** If  $\lim_{x \rightarrow a} f(x)$  and  $\lim_{x \rightarrow a} g(x)$  both exist and are finite, then:

$$4. \lim_{x \rightarrow a} (f(x) + g(x)) = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$$

$$5. \lim_{x \rightarrow a} (f(x) - g(x)) = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$$

$$6. \lim_{x \rightarrow a} (f(x) \times g(x)) = \lim_{x \rightarrow a} f(x) \times \lim_{x \rightarrow a} g(x)$$

$$7. \lim_{x \rightarrow a} (f(x) \div g(x)) = \lim_{x \rightarrow a} f(x) \div \lim_{x \rightarrow a} g(x) \text{ (except in the case of division by zero).}$$

If  $c$  is a constant, then

$$8. \lim_{x \rightarrow a} cf(x) = c \lim_{x \rightarrow a} f(x)$$

If  $r$  is a rational number, then, except in case of division by zero or roots that do not exist,

$$9. \lim_{x \rightarrow a} (f(x))^r = \left( \lim_{x \rightarrow a} f(x) \right)^r$$

As a consequence of rules 1. and 9.,

$$10. \lim_{x \rightarrow a} x^r = a^r, \text{ (except in the case of division by zero or roots that do not exist)}$$

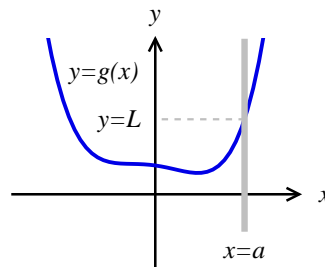
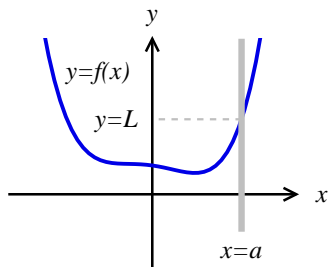
$$11. \text{ If } p(x) \text{ is any polynomial and } a \text{ is any number, then } \lim_{x \rightarrow a} p(x) = p(a).$$

12. If  $r(x)$  is any rational function (i.e., the ratio of two polynomials) and  $a$  is any number, then  $\lim_{x \rightarrow a} r(x) = r(a)$ , provided  $r(a)$  exists.

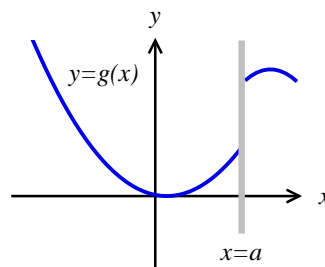
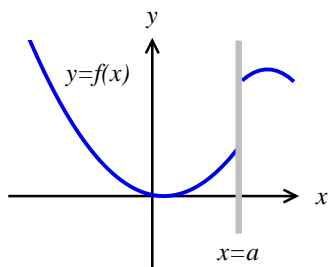
The most commonly used limit law of all:

13. If  $f(x) = g(x)$  for all  $x \neq a$ , then  $\lim_{x \rightarrow a} f(x)$  and  $\lim_{x \rightarrow a} g(x)$  are the same, whether both exist (as in 2.3.re.e1) or both fail to exist (2.3.re.e2).

2.3.re.e1.



2.3.re.e2.



2.3.re.e3. Evaluate the limits:

a.  $\lim_{x \rightarrow 2} \frac{x^2 + x - 6}{3x^2 - 6x}$     b.  $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x^3 - 27}$     c.  $\lim_{x \rightarrow 1} \frac{\frac{1}{x+3} - \frac{1}{4}}{x - 1}$     d.  $\lim_{x \rightarrow -3} \frac{2 - \sqrt{x+7}}{x+3}$

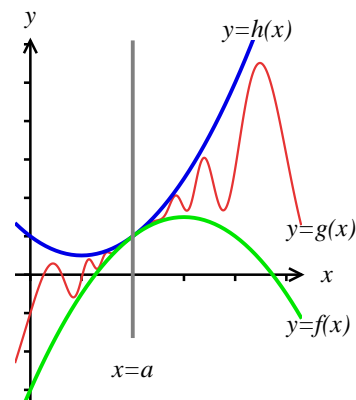
**The Squeeze Theorem 2.3.re.1.** If

$$f(x) \leq g(x) \leq h(x)$$

for all  $x \neq a$ , and if

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} h(x) = L,$$

then  $\lim_{x \rightarrow a} g(x)$  also exists and equals  $L$ .



2.3.re.e4.  $\lim_{x \rightarrow 0} x^2 \sin\left(\frac{1}{x^2}\right)$

Answers

2.3.re.e3a. 5/6    2.3.re.e3b. 2/9    2.3.re.e3c. -1/16    2.3.re.e3d. -1/4    2.3.re.e4. Because  $-1 \leq \sin\left(\frac{1}{x^2}\right) \leq 1$  for all  $x \neq 0$ ,  $-x^2 \leq x^2 \sin\left(\frac{1}{x^2}\right) \leq x^2$ . Since  $\pm x^2 \rightarrow 0$  as  $x \rightarrow 0$ , the limit of  $x^2 \sin\left(\frac{1}{x^2}\right)$  is also zero, by the squeeze theorem.

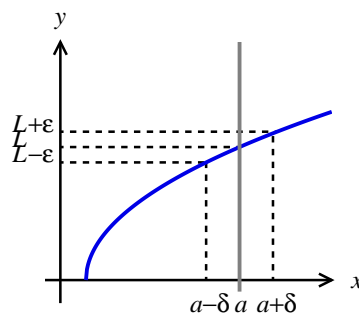
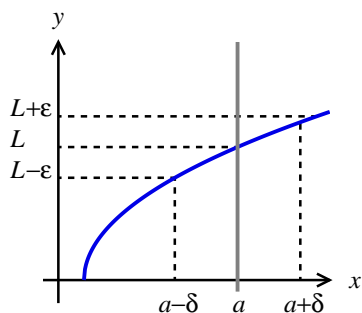


### 2.4: The Precise Definition of Limit

You must be able to state the following definition.

**Definition 2.4.re.1.** We say  $\lim_{x \rightarrow a} f(x) = L$  if, for every positive number  $\varepsilon$ , there is a corresponding positive number  $\delta$  for which

$$|f(x) - L| < \varepsilon \text{ whenever } 0 < |x - a| < \delta.$$



There's an interactive graph illustrating this definition at <https://www.desmos.com/calculator/qrgrxymepu>

2.4.re.e1. Limit Law 10 promises that  $\lim_{x \rightarrow 2} x^2 = 4$ . Use the table of values to find the positive number  $\delta$  corresponding to the given  $\varepsilon$  for this limit.

a.  $\varepsilon = 0.1$

b.  $\varepsilon = 0.03$

$x^2$	3.9	3.97	4	4.03	4.1
$x$	1.9748	1.9925	2	2.0074	2.0248

*Writing a limit proof*

To write an  $\varepsilon$ - $\delta$  proof of a simple limit requires us to figure out how  $\delta$  depends on the given  $\varepsilon$  and explain to the reader why our choice of  $\delta$  works. The good news is that proofs of limits of linear functions all look the same.

2.4.re.e2. Write an  $\varepsilon$ - $\delta$  proof of the fact that  $\lim_{x \rightarrow 4} (3 - 5x) = -17$ .

First some analysis. Given  $\varepsilon > 0$ , we're looking for a number  $\delta$  so that  $0 < |x - 4| < \delta$  will imply that  $|(3 - 5x) - (-17)| < \varepsilon$ . Start by simplifying:

$$|(3 - 5x) - (-17)| = |3 - 5x + 17| = |20 - 5x| = |-5(x - 4)| = |-5||x - 4| = 5|x - 4|,$$

so to make  $5|x - 4| < \varepsilon$ , just make sure that  $|x - 4| < \frac{1}{5}\varepsilon$ .

Now we're ready to write a proof.

**Proof:** Suppose that  $\varepsilon > 0$ . Choose  $\delta = \frac{1}{5}\varepsilon$ . Then, whenever

$$|x - 4| < \delta = \frac{1}{5}\varepsilon,$$

$$|(3 - 5x) - (-17)| = |3 - 5x + 17| = |20 - 5x| = |-5(x - 4)| = |-5||x - 4| = 5|x - 4| < 5 \cdot \frac{1}{5}\varepsilon = \varepsilon,$$

as desired.

2.4.re.e3. Write an  $\varepsilon - \delta$  proof of the following limits:

a.  $\lim_{x \rightarrow 2} (7 - 4x) = -1$

b.  $\lim_{x \rightarrow -1} \frac{3x^2 + x - 2}{x + 1} = -5$

## Answers

2.4.re.e1a.  $\delta = 0.0248$  2.4.re.e1b.  $\delta = 0.0074$  2.4.re.e3a. Suppose  $\varepsilon > 0$ . Choose  $\delta = \frac{1}{4}\varepsilon$ . Then, whenever  $|x - 2| < \delta$ ,  $|(7 - 4x) - (-1)| = |8 - 4x| = |-4||x - 2| = 4|x - 2| < 4\delta = \varepsilon$ , as desired. 2.4.re.e3b. Suppose  $\varepsilon > 0$ . Choose  $\delta = \frac{1}{3}\varepsilon$ . Then, whenever  $0 < |x - (-1)| < \delta$ ,  $|\frac{3x^2 + x - 2}{x + 1} - (-5)| = |\frac{(3x - 2)(x + 1)}{x + 1} + 5| = |(3x - 2) + 5| = |3x + 3| = |3||x + 1| = 3|x + 1| < 3\delta = \varepsilon$ , as desired.

## 2.5: Continuity

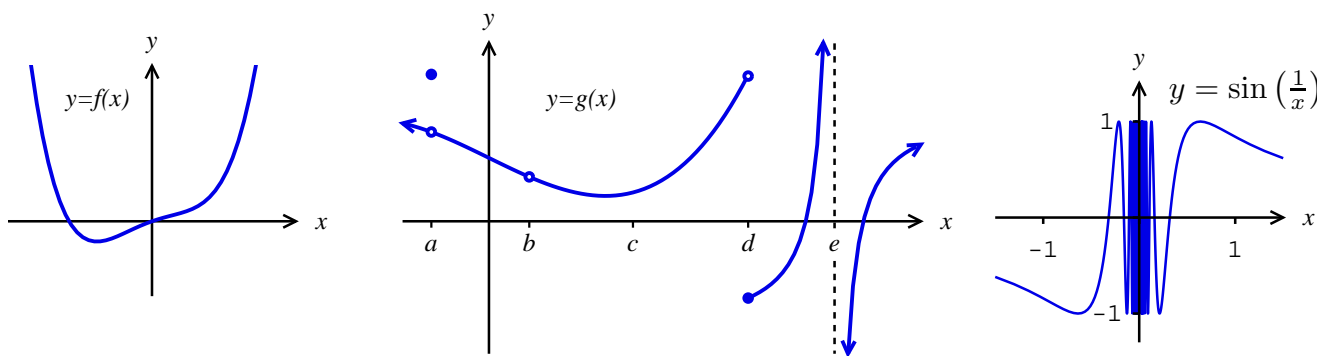
**Definition 2.5.re.1.** The function  $f(x)$  is said to be **continuous** at the number  $a$  if

$$\lim_{x \rightarrow a} f(x) = f(a).$$

For  $f$  to be continuous at  $a$  means:

1. the limit  $\lim_{x \rightarrow a} f(x)$  exists,
2. the function value  $f(a)$  exists, and
3. the two are equal.

Informally, if you were drawing the graph of a function from left to right, the function is discontinuous at those  $x$ -values where you have to lift your pencil off the paper.



2.5.re.e1. Where are the above functions continuous? Discontinuous?

- a.  $f(x)$                                       b.  $g(x)$                                       c.  $\sin(\frac{1}{x})$

**Definition 2.5.re.2.** The function  $f(x)$  is said to be **continuous from the right** at  $a$  if  $\lim_{x \rightarrow a^+} f(x) = f(a)$ , and **continuous from the left** at  $a$  if  $\lim_{x \rightarrow a^-} f(x) = f(a)$ .

2.5.re.e2. At what points, if any, is  $g(x)$  from example 2.5.re.e1 discontinuous but

- a. continuous from the left                      b. continuous from the right

**Definition 2.5.re.3.** The function  $f(x)$  is said to be **continuous** if it is continuous at every point  $a$  in its domain.

The next three facts tell us that, almost every function we can write down is continuous.

**Fact 2.5.re.4.** *These functions are all continuous wherever they are defined:*

<i>polynomials</i>	<i>rational functions</i>	<i>power functions</i>	<i>absolute value</i>
<i>trig functions</i>	<i>inverse trig functions</i>	<i>exponentials</i>	<i>logarithms</i>

**Fact 2.5.re.5.** *If  $f$  and  $g$  are continuous at  $a$ , then so are*

$$f + g \quad f - g \quad f \cdot g \quad f \div g \text{ (assuming } g(a) \neq 0\text{)}$$

**Fact 2.5.re.6.** *If  $f$  is continuous at  $a$ , and  $g$  is continuous at  $f(a)$ , then  $g \circ f$  is continuous at  $a$ .*

Fact 2.5.re.4 says most functions or families of function we've already given names to are continuous, and Facts 2.5.re.5 and 2.5.re.6 imply that when we combine any of those by addition, subtraction, multiplication, division or composition, the result is continuous at all points in their domain.

2.5.re.e3. Where is the given function continuous? (Hint: what is its domain?)

a.  $3x^2 + x\sqrt{2} - \pi$                       b.  $\frac{x}{|x|}$                       c.  $\ln\left(\frac{x-2}{x+3}\right)$

A piecewise-defined function needn't be continuous at its "knots" (where its pieces join), even if it's made of continuous pieces.

2.5.re.e4. Where is the given function discontinuous?

a.  $\alpha(x) = \begin{cases} e^x & \text{if } x \leq 0 \\ \ln(x+1) & \text{if } x > 0 \end{cases}$                       b.  $\beta(x) = \begin{cases} x^2 + 4x & \text{if } x \leq 0 \\ 1 - x & \text{if } 0 < x < 1 \\ x^4 - 1 & \text{if } 1 \geq x \end{cases}$

2.5.re.e5. Find the values of  $a$  and  $b$  that make  $\gamma(x) = \begin{cases} 1 - 2x^2 & \text{if } x \leq -1 \\ ax + b & \text{if } -1 < x \leq 1 \\ 3 + x^2 & \text{if } 1 < x \end{cases}$  continuous.

**Intermediate Value Theorem (IVT) 2.5.re.7.** *If  $f$  is continuous on  $[a, b]$ , and if  $M$  is a number between  $f(a)$  and  $f(b)$ , then there's at least one number  $c$  in  $[a, b]$  for which  $f(c) = M$ .*

The IVT is what allows us to build a sign chart for a continuous function by finding its zeros and testing points. For instance, our solution to Example 2.5.re.e3b depended on the fact that  $\frac{x-2}{x+3}$  can change sign only where it is zero ( $x = 2$ ) or where it is discontinuous ( $x = -3$ ).

IVT can often tell us that an equation has a solution, even if we're unable to produce it exactly.

2.5.re.e6. Show that  $x \cos x = 3$  has a solution in the interval  $(0, 2\pi)$ .

#### Answers

2.5.re.e1a.  $f$  appears to be continuous everywhere. 2.5.re.e1b.  $g$  is discontinuous at the  $x$ -values  $a$  (limit value and function value exist but are unequal),  $b$  (limit exists but function does not),  $d$  (function exists but limit does not), and  $e$  (neither limit nor function exist).  $g$  is continuous everywhere else.  
 2.5.re.e1c.  $\sin(\frac{1}{x})$  is discontinuous only at  $x = 0$ , where neither the function nor its limit exist, and it continuous at all other real numbers  $x$ . 2.5.re.e2a. Nowhere. 2.5.re.e2b. At  $x = d$  only. 2.5.re.e3a. As a polynomial, this function is continuous at all real numbers. 2.5.re.e3b. At all numbers except  $x = 0$ .  
 2.5.re.e3c. Everywhere that  $\frac{x-2}{x+3} > 0$ , that is on the interval  $(-\infty, -3)$  and on  $(2, \infty)$ . 2.5.re.e4a. only at  $x = 0$  (where the right- and left-hand limits disagree). 2.5.re.e4b. only at  $x = 0$  (where the right- and left-hand limits disagree). 2.5.re.e5.  $b = 3/2$  and  $a = 5/2$ , so that the line goes through the points  $(-1, -1)$  and  $(1, 4)$ . 2.5.re.e6. The function  $g(x) = x \cos x$  is continuous everywhere. Since  $g(0) = 0$  and  $g(2\pi) = 2\pi$  and  $0 < 3 < 2\pi$ , by the IVT, there must be at least one  $x$  in  $(0, 2\pi)$  at which  $g(x) = 3$ .

## 2.6: Limits at infinity; Asymptotes

**Definition 2.6.re.1.** The line  $x = a$  is a **vertical asymptote** of the graph of  $f(x)$  if either

$$\lim_{x \rightarrow a^-} f(x) \text{ or } \lim_{x \rightarrow a^+} f(x)$$

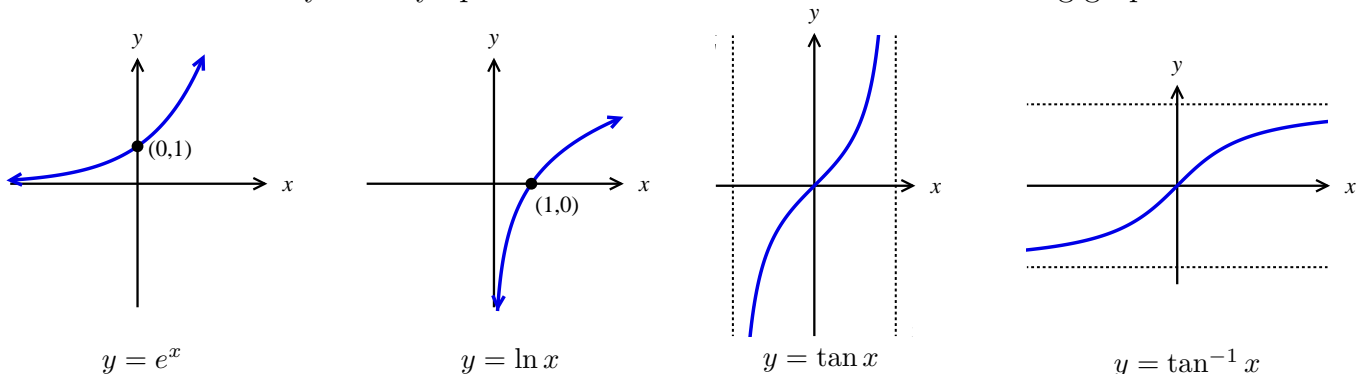
is infinite.

**Definition 2.6.re.2.** The line  $y = a$  is a **horizontal asymptote** of the graph of  $f(x)$  if either

$$\lim_{x \rightarrow -\infty} f(x) = a \text{ or } \lim_{x \rightarrow \infty} f(x) = a.$$

In short, a curve has a horizontal or vertical asymptote if one variable approaches a finite number while the other blows up.

2.6.re.e1. Identify the asymptotes and associated limits in the following graphs.



*Some useful facts about limits at infinity*

1. Something goes to zero if and only if its reciprocal goes to  $\pm\infty$ .
2. If  $p$  is a constant,

$$\lim_{x \rightarrow \infty} x^p = \begin{cases} \infty & \text{if } p > 0, \text{ and} \\ 0 & \text{if } p < 0. \end{cases}$$

3. If  $p(x)$  is a polynomial and  $\ell t_p(x)$  is its lead term, then

$$\lim_{x \rightarrow \infty} p(x) = \lim_{x \rightarrow \infty} \ell t_p(x).$$

4. If  $p(x)$  and  $q(x)$  are polynomials then

$$\lim_{x \rightarrow \infty} \frac{p(x)}{q(x)} = \lim_{x \rightarrow \infty} \frac{\ell t_p(x)}{\ell t_q(x)}.$$

2.6.re.e2. Evaluate the limit.

$$\begin{array}{lll} \text{a. } \lim_{x \rightarrow \infty} (x^{\frac{1}{3}} - x^{-4}) & \text{b. } \lim_{x \rightarrow \infty} (4x^5 - 2x^4 + 3x + 1) & \text{c. } \lim_{x \rightarrow \infty} \frac{2x - 1}{x + 3} \\ \text{d. } \lim_{x \rightarrow \infty} \frac{3x - 1}{3 - x^2} & \text{e. } \lim_{x \rightarrow \infty} \frac{2x^2 + 3x - 1}{3 - x} & \text{f. } \lim_{x \rightarrow -\infty} \frac{2x^2 + 3x - 1}{3 - x} \end{array}$$

**Tip:** If the rules for polynomials and rational functions don't apply, try to factor out and cancel the dominant term in the top and bottom.

2.6.re.e3. Evaluate the limit.

$$\begin{array}{ll} \text{a. } \lim_{x \rightarrow \infty} \frac{\sqrt{2x^2 - 5x}}{-3x + 2} & \text{b. } \lim_{x \rightarrow -\infty} \frac{\sqrt{2x^2 - 5x}}{-3x + 2} \\ \text{c. } \lim_{x \rightarrow \infty} \frac{2e^x - e^{-x}}{4e^x + 7e^{-x}} & \text{d. } \lim_{x \rightarrow -\infty} \frac{2e^x - e^{-x}}{4e^x + 7e^{-x}} \\ \text{e. } \lim_{x \rightarrow \infty} \left( x - \sqrt{x^2 - 3x} \right) & \end{array}$$

### Answers

2.6.re.e1.  $y = 0$  is a HA to  $y = e^x$  because  $\lim_{x \rightarrow -\infty} e^x = 0$ .  $x = 0$  is a VA to  $y = \ln x$  b.c.  $\lim_{x \rightarrow 0^+} \ln x = -\infty$ . Both  $x = \frac{\pi}{2}$  and  $x = -\frac{\pi}{2}$  are VAs to the graph of  $y = \tan x$ , b.c.  $\lim_{x \rightarrow -(\pi/2)^+} \tan x = -\infty$  and  $\lim_{x \rightarrow (\pi/2)^-} \tan x = \infty$ .  $y = \frac{\pi}{2}$  and  $y = -\frac{\pi}{2}$  are HAs to the graph of  $y = \tan^{-1} x$ , b.c.  $\lim_{x \rightarrow -\infty} \tan^{-1} x = -\frac{\pi}{2}$  and  $\lim_{x \rightarrow \infty} \tan^{-1} x = \frac{\pi}{2}$ . 2.6.re.e2a. " $\infty - 0$ ," so the limit  $= \infty$ . 2.6.re.e2b.  $= \lim_{x \rightarrow \infty} 4x^5 = \infty$ . 2.6.re.e2c.  $= \lim_{x \rightarrow \infty} \frac{2x}{x} = 2$ . 2.6.re.e2d.  $= \lim_{x \rightarrow \infty} \frac{3x}{-x^2} = \lim_{x \rightarrow \infty} \frac{3}{-x} = 0$ . 2.6.re.e2e.  $= \lim_{x \rightarrow \infty} \frac{2x^2}{-x} = \lim_{x \rightarrow \infty} -2x = -\infty$ . 2.6.re.e2f.  $= \lim_{x \rightarrow -\infty} -2x = \infty$ . 2.6.re.e3a.  $\lim_{x \rightarrow \infty} \frac{\sqrt{x^2} \sqrt{2 - \frac{5}{x}}}{x(-3 + \frac{2}{x})} = \lim_{x \rightarrow \infty} \frac{|x| \sqrt{2 - \frac{5}{x}}}{x(-3 + \frac{2}{x})}$   
 $= \lim_{x \rightarrow \infty} \frac{x \sqrt{2 - \frac{5}{x}}}{x(-3 + \frac{2}{x})} = \lim_{x \rightarrow \infty} \frac{\sqrt{2 - \frac{5}{x}}}{(-3 + \frac{2}{x})} = -\frac{\sqrt{2}}{3}$  2.6.re.e3b.  $\lim_{x \rightarrow -\infty} \frac{\sqrt{x^2} \sqrt{2 - \frac{5}{x}}}{x(-3 + \frac{2}{x})} = \lim_{x \rightarrow -\infty} \frac{|x| \sqrt{2 - \frac{5}{x}}}{x(-3 + \frac{2}{x})} =$   
 $\lim_{x \rightarrow -\infty} \frac{-x \sqrt{2 - \frac{5}{x}}}{x(-3 + \frac{2}{x})} = \lim_{x \rightarrow -\infty} \frac{-\sqrt{2 - \frac{5}{x}}}{(-3 + \frac{2}{x})} = \frac{\sqrt{2}}{3}$  2.6.re.e3c. factor out and cancel  $e^x$ .  $\lim_{x \rightarrow \infty} \frac{2 - e^{-2x}}{4 + 7e^{-2x}} = \frac{2-0}{4+0} = \frac{1}{2}$ . 2.6.re.e3d. factor out and cancel  $e^{-x}$ .  $\lim_{x \rightarrow \infty} \frac{2e^{2x} - 1}{4e^{2x} + 7} = \frac{0-1}{0+7} = -\frac{1}{7}$ . 2.6.re.e3e. Multiply by the conjugate, then factor out and cancel  $x$ . Limit  $= 3/2$ .

## 2.7: Derivatives and Rates of Change

Here's the conclusion of section 2.1, restated with the limit notation we've developed in this chapter:

If  $f(t)$  is an object's position at time  $t$ , then the difference quotient

$$\frac{f(a+h) - f(a)}{h}$$

is *both* the object's average velocity between times  $a$  and  $a+h$  *and* the slope of secant line passing through the graph of  $f$  at times  $a$  and  $a+h$ . The limit of the difference quotient,

$$\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

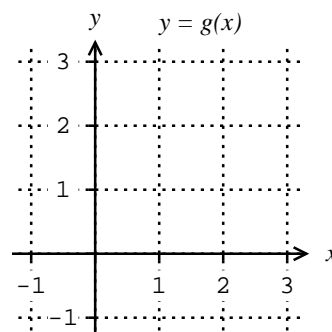
is *both* the object's velocity at time  $a$  *and* the slope of the line tangent to the graph of  $f$  at the point  $(a, f(a))$ .

**Definition 2.7.re.1.** The **derivative** of the function  $f(x)$  at  $x = a$ , denoted  $f'(a)$  is

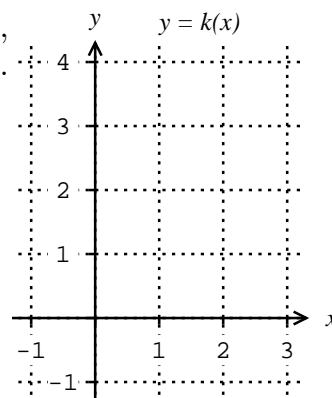
$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

2.7.re.e1. Sketch the graph of a function  $g$  satisfying

$$g(0) = 0 \quad g'(0) = 3 \quad g(2) = 2 \quad g'(2) = -1$$



2.7.re.e2. Sketch  $y = k(x)$  if  $k(1) = -1$  and  $k'(x) = \begin{cases} 2 & \text{if } x > 1, \\ -1 & \text{if } x < 1. \end{cases}$



2.7.re.e3. Find  $p(2)$  and  $p'(2)$  if the line tangent to  $y = p(x)$  at the point  $(2, 5)$  passes through  $(4, -7)$ .



On exams and quizzes, you may be asked to find the derivative of a function using Definition 2.7.re.1, as in this next example.

2.7.re.e4. Use the definition of derivative to find  $f'(a)$  if  $f(x) = \frac{2}{3-x}$ :

Solution:

$$\begin{aligned} f'(a) &= \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \rightarrow 0} \left( \frac{\frac{2}{3-(a+h)} - \frac{2}{3-a}}{h} \right) \cdot \frac{(3-a-h)(3-a)}{(3-a-h)(3-a)} \\ &= \lim_{h \rightarrow 0} \frac{2(3-a) - 2(3-a-h)}{h(3-a-h)(3-a)} = \lim_{h \rightarrow 0} \frac{6-2a-6+2a+2h}{h(3-a-h)(3-a)} \\ &= \lim_{h \rightarrow 0} \frac{2h}{h(3-a-h)(3-a)} = \lim_{h \rightarrow 0} \frac{2}{(3-a-h)(3-a)} = \frac{2}{(3-a)^2} \end{aligned}$$

2.7.re.e5. Find the derivative at  $a$  of the given function:

a.  $f(x) = x^3 - 2x^2$       b.  $g(x) = \sqrt{x+1}$       c.  $k(x) = \frac{2x}{x-1}$

2.7.re.e6. A particle moving along a coordinate axis is at position  $f(t) = t^3 - 2t^2$  (meters) at time  $t$  (seconds). Find the particle's velocities at times  $t = 1$  and  $t = 2$ . What are the units in your answer?

2.7.re.e7. Find  $f(x)$  and  $a$  if the following limit is  $f'(a)$ . (There is more than one correct answer.)

a.  $\lim_{h \rightarrow 0} \frac{\sqrt{16+h} - 4}{h}$       b.  $\lim_{x \rightarrow -3} \frac{2x^2 + 4x - 6}{x + 3}$

#### Answers

2.7.re.e1. Here's one such graph: <https://www.desmos.com/calculator/tvcqu53ja8>

2.7.re.e2. The graph of  $k$  is at: <https://www.desmos.com/calculator/seosuy3hs2>    2.7.re.e3.  $p(2) = 5$ .

$p'(2) = -6$ .    2.7.re.e5a.  $f'(a) = 3a^2 - 4a$     2.7.re.e5b.  $g'(a) = \frac{1}{2\sqrt{a+1}}$     2.7.re.e5c.  $k'(a) = \frac{-2}{(a-1)^2}$

2.7.re.e6. Can use derivative found in 3.1.re.e2, part a. Velocity at time 1 is  $f'(1) = -1\text{m/sec}$ . Velocity at time 2 is  $f'(2) = 4\text{m/sec}$ .    2.7.re.e7a.  $f(x) = \sqrt{x}$ ,  $a = 16$ .    2.7.re.e7b.  $f(x) = 2x^2 + 4x$ ,  $a = -3$ .

## 2.8: The Derivative as a Function

**Definition 2.7.re.1**, restated. The derivative of the function  $f(x)$  is **the function**

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{t \rightarrow x} \frac{f(t) - f(x)}{t - x}$$

On exams and quizzes, you may be asked to find the derivative of a function using Definition 2.7.re.1. On such a question, you would not receive credit for using the derivative rules of Chapter 3.

2.8.re.e1. Find  $f'(x)$  if  $f(x)$  is the given function:

a.  $f(x) = x^4 + 5x - 3$

b.  $f(x) = \sqrt{2x - 1}$

c.  $f(x) = \frac{x}{4x - 1}$

d.  $f(x) = \frac{1}{\sqrt{3x + 2}}$

For more practice taking the derivative using the definition, see [http://kunklet.people.cofc.edu/MATH120/derivative\\_prac.pdf](http://kunklet.people.cofc.edu/MATH120/derivative_prac.pdf)

**Definition 2.8.re.1.** The function  $f(x)$  is said to be **differentiable** at  $a$  if  $f'(a)$  exists.

**Fact 2.8.re.2.** If  $f(x)$  is differentiable at  $a$ , then it must be continuous at  $a$ .

Reasons why  $f(x)$  might fail to be differentiable at  $a$ :

1.  $f(x)$  is not continuous at  $a$ . See Section 2.5.
2. The graph of  $f(x)$  has a **corner** at  $a$ , meaning the one-sided limits

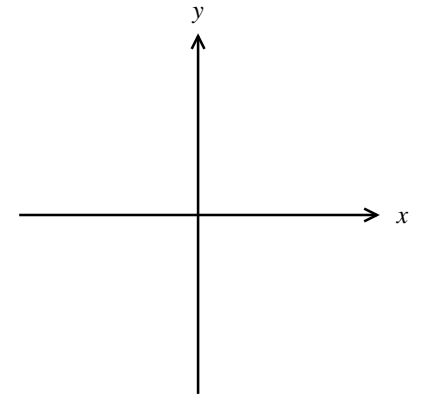
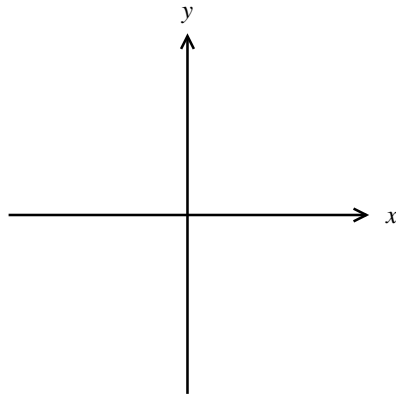
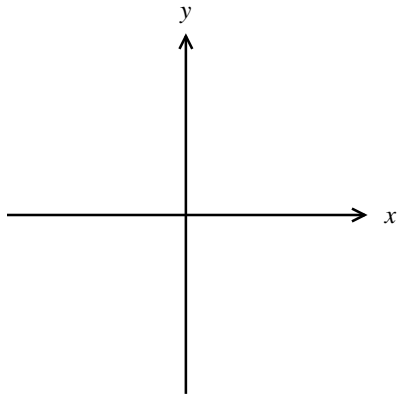
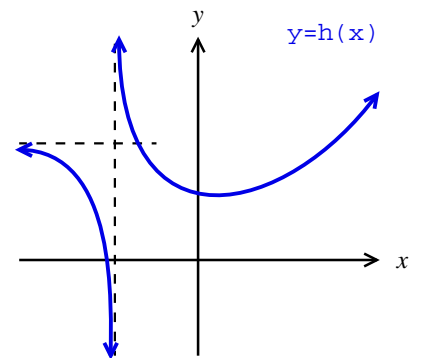
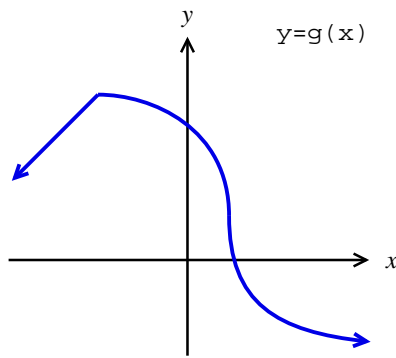
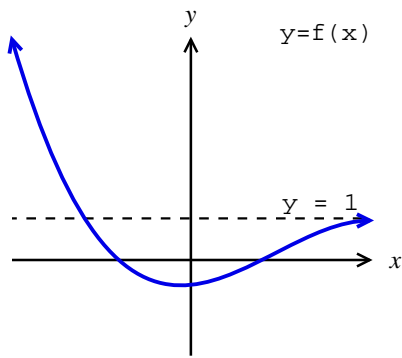
$$\lim_{h \rightarrow 0^-} \frac{f(a+h) - f(a)}{h} \quad \text{and} \quad \lim_{h \rightarrow 0^+} \frac{f(a+h) - f(a)}{h}$$

exist but disagree. This would happen, for instance, if that  $f'(x)$  has a jump discontinuity at  $a$ .

3. The graph of  $f(x)$  has a vertical tangent line at  $x = a$ , meaning that one or both of the one-sided limits above are infinite.
4. None of the above.

Tip: to sketch the graph of  $f'(x)$ , note where the graph of the  $f(x)$  is horizontal, vertical (or approaching vertical), positively or negative sloped, and where  $f'(x)$  fails to exist .

2.8.re.e2. Sketch the derivative of the function graphed.



*Higher order derivatives*

The derivative of a function is also called its **first derivative**.

The **second derivative** of a function is the derivative of its first derivative.

The **third derivative** of a function is the derivative of its second derivative, and so on.

*Various notation for derivatives.*

The symbol  $\frac{d}{dx}$  means ‘the derivative of’, just as  $\sqrt{\quad}$  means ‘the square root of’.

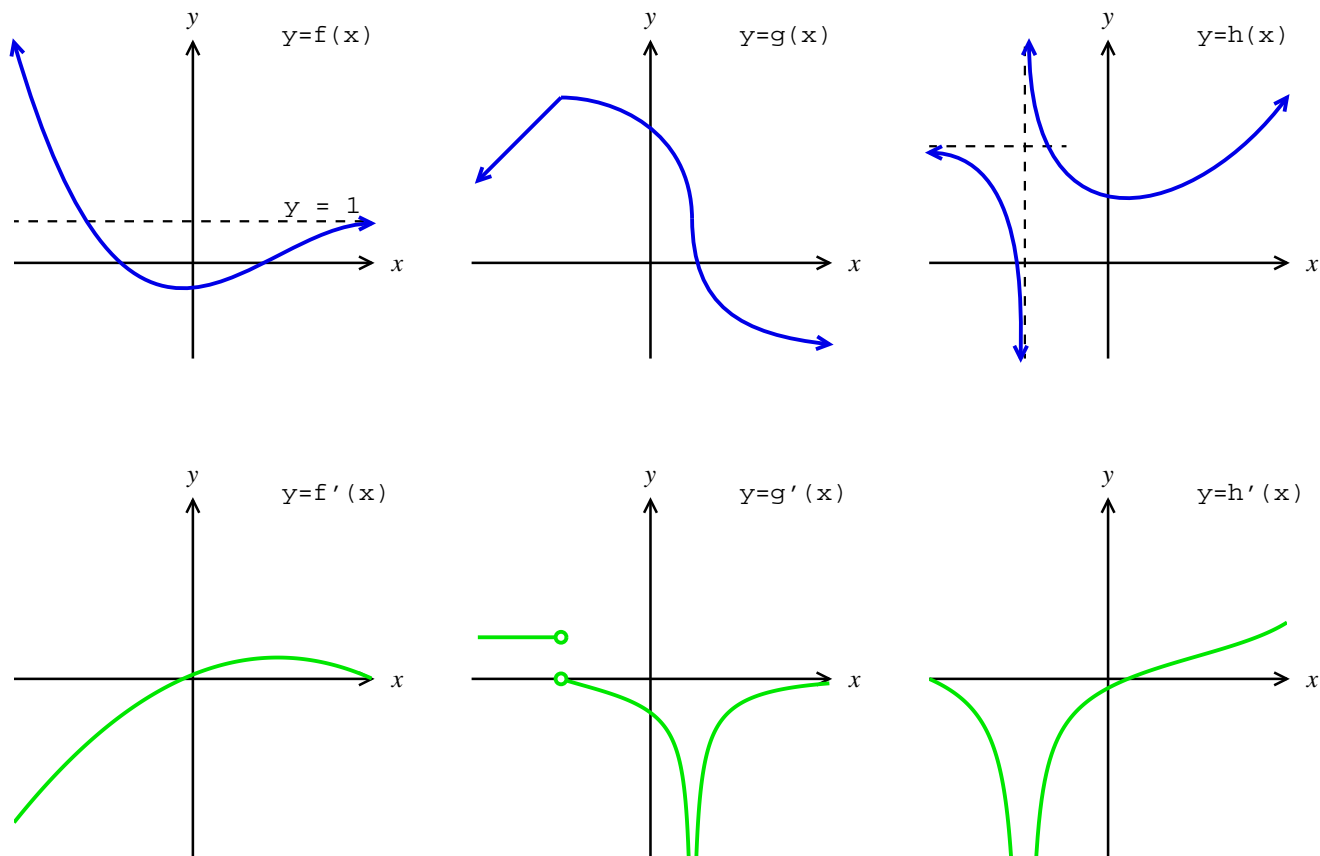
First derivative of $y = f(x)$	$f'(x)$	$\frac{dy}{dx}$	$\frac{d}{dx}y$	$\frac{d}{dx}f(x)$
Second derivative of $y = f(x)$	$f''(x)$	$\frac{d^2y}{dx^2}$	$\frac{d}{dx}\left(\frac{dy}{dx}\right)$	$\left(\frac{d}{dx}\right)^2 f(x)$
Third derivative of $y = f(x)$	$f'''(x)$	$f^{(3)}(x)$	$\frac{d^3y}{dx^3}$	$\left(\frac{d}{dx}\right)^3 f(x)$
Fourth derivative of $y = f(x)$	$f^{(4)}(x)$	$\frac{d^4y}{dx^4}$		

The derivative of a function is simultaneously the slope of its tangent line and its rate of change measured in units-output per units-input.

2.8.re.e3. Suppose the position at time  $t$  of a particle on an axis is  $s(t)$ . If  $s$  is measured in meters (m) and  $t$  in seconds (s), what are the units of its first three derivatives?

Answers

2.8.re.e1a.  $f'(x) = 4x^3 + 5$     2.8.re.e1b.  $f'(x) = \frac{1}{\sqrt{2x-1}}$     2.8.re.e1c.  $f'(x) = \frac{-1}{(4x-1)^2}$   
 2.8.re.e1d.  $f'(x) = \frac{-3}{2(3x+2)^{3/2}}$     2.8.re.e2. See below.



2.8.re.e3. Units of  $s'(t)$  are meters per second, or m/s. Units of  $s''(t)$  are meters per second per second, or  $m/s^2$ . Units of  $s'''(t)$  are meters per second per second per second, or  $m/s^3$ .

### 3.1: Derivatives of Polynomials and Exponentials

In this chapter we learn Differentiation Laws that allow us to find derivatives more easily than by the definition in Section 2.7. They come in two types.

Below,  $f(x)$  and  $g(x)$  can be any differentiable functions, and  $c$  and  $n$  can be any constants, and  $e$  is the mathematical constant mentioned in Section 1.4.

#### I. A catalog of elementary functions and their derivatives.

$(c)' = 0$	$(x^n)' = nx^{n-1}$	$(e^x)' = e^x$
------------	---------------------	----------------

#### II. Combination laws.

$(cf(x))' = cf'(x)$
$(f(x) + g(x))' = f'(x) + g'(x)$
$(f(x) - g(x))' = f'(x) - g'(x)$

Warning: the value of the derivative of a function is different from the derivative of the value of the function. The latter is always 0.

3.1.re.e1. Find the derivative of the given function:

- |   |                                     |                         |
|---|-------------------------------------|-------------------------|
| a. $x^4 + 5x - 3$                         | b. $\sqrt{2x} + \sqrt{\frac{9}{x}}$ | c. $\sqrt[4]{x} - 3x$   |
| d. $\frac{4}{x^3}$                        | e. $\pi r^2$                        | f. $\frac{4}{3}\pi r^3$ |
| g. $4e^x - x^3 + 3^4$                     | h. $4x^{1.2} - 2e^{1.2}$            | i. $(3x - 1)(2x + 1)$   |
| j. $\frac{2\sqrt{x} + 3\sqrt[3]{x^4}}{x}$ |                                     |                         |

3.1.re.e2. Find the second derivative of the given function:

- |                         |                          |   |
|-------------------------|--------------------------|---|
| a. $x^4 + 5x - 3$       | b. $\sqrt[4]{x} - 3x$    | c. $\frac{4}{x^3}$                        |
| d. $\frac{4}{3}\pi r^3$ | e. $4x^{1.2} - 2e^{1.2}$ | f. $\frac{2\sqrt{x} + 3\sqrt[3]{x^4}}{x}$ |

3.1.re.e3. Find numbers  $a$ ,  $b$ , and  $c$  so that  $k(x) = ax^2 + bx + ce^x$  so that

$$k(0) = 2 \quad k'(0) = 1 \quad k''(0) = -3.$$

#### Answers

- 3.1.re.e1a.  $4x^3 + 5$  3.1.re.e1b.  $\frac{1}{\sqrt{2}}x^{-1/2} - \frac{3}{2}x^{-3/2}$  3.1.re.e1c.  $\frac{1}{4}x^{-3/4} - 3$  3.1.re.e1d.  $\frac{-12}{x^4}$  3.1.re.e1e.  $2\pi r$   
 3.1.re.e1f.  $4\pi r^2$  3.1.re.e1g.  $4e^x - 3x^2$  3.1.re.e1h.  $4.8x^{0.2}$  3.1.re.e1i.  $12x + 1$  3.1.re.e1j.  $-x^{-3/2} + x^{-2/3}$   
 3.1.re.e2a.  $12x^2$  3.1.re.e2b.  $-\frac{3}{16}x^{-7/4}$  3.1.re.e2c.  $-48x^{-5}$  3.1.re.e2d.  $8\pi r$  3.1.re.e2e.  $0.96x^{-0.8}$   
 3.1.re.e2f.  $\frac{3}{2}x^{-5/2} - \frac{2}{3}x^{-5/3}$  3.1.re.e3.  $a = -5/2, b = -1, c = 2.$

## 3.2: The Product and Quotient Rules

## II. Combination laws (continued).

**Product Rule:** If  $f(x)$  and  $g(x)$  are differentiable, then so is their product, and

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x).$$

**Quotient Rule:** If  $f(x)$  and  $g(x)$  are differentiable, then so is their quotient, and

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} \quad (\text{provided } g(x) \neq 0).$$

Always simplify at every step to make your next step easier and more likely to be correct. When more than one differentiation rule applies, the last operation determines the first differentiation rule we must use.

3.2.re.e1. Find the first derivative of the given function.

a.  $x^2 e^x$

b.  $x^2 e^x (2x - 3)$

c.  $\frac{3x^3 + 2}{x^2}$

d.  $\frac{x}{3x^2 + 1}$

e.  $(r - \sqrt{r})(r + \sqrt{r})$

f.  $\frac{ze^z}{z + 1}$

g.  $\frac{x}{x + \frac{2}{x}}$

h.  $\frac{x^2 + 4}{x^3 - 8}$

3.2.re.e2. Find the equation of the line tangent to  $y = \frac{x}{3x^2 + 1}$  at  $x = 1$ .

3.2.re.e3. Find all  $x$ -values at which the slope of  $y = \frac{x - 1}{x + 3}$  equals ...

a. 9

b. -8

c. 0

## Answers

3.2.re.e1a.  $e^x(2x + x^2)$  3.2.re.e1b.  $e^x(2x^3 + 3x^2 - 6x)$  3.2.re.e1c.  $3 - 4x^{-3}$  3.2.re.e1d.  $\frac{1 - 3x^2}{(3x^2 + 1)^2}$   
 3.2.re.e1e.  $2r - 1$  3.2.re.e1f.  $\frac{(z^2 + z + 1)e^z}{(z + 1)^2}$  3.2.re.e1g.  $\frac{4x^{-1}}{(x + 2x^{-1})^2}$  3.2.re.e1h.  $\frac{-x^4 - 12x^2 - 16x}{(x^3 - 8)^2}$  3.2.re.e2.  $y - \frac{1}{4} = -\frac{1}{8}(x - 1)$  3.2.re.e3a.  $x = -3 \pm \frac{2}{3}$ , or  $x = -\frac{11}{3}, -\frac{7}{3}$  3.2.re.e3b. no solutions. The slope of this curve is always positive. 3.2.re.e3c. no solutions.

### 3.3: The Derivatives of the Trig Functions

In this section we add 6 more functions to our derivative catalog.

#### I. Catalog.

$(\sin x)' = \cos x$	$(\cos x)' = -\sin x$
$(\tan x)' = \sec^2 x$	$(\cot x)' = -\csc^2 x$
$(\sec x)' = \sec x \tan x$	$(\csc x)' = -\csc x \cot x$

It is useful to note that the last four follow from the first two and the quotient rule.

3.3.re.e1. Show that  $\frac{d}{dx} \cot x = \csc^2 x$  using the derivatives of  $\sin x$  and of  $\cos x$ .

By definition,  $\cot x = \frac{\cos x}{\sin x}$ , and so its derivative

$$\left(\frac{\cos x}{\sin x}\right)' = \frac{(\cos x)' \sin x - \cos x (\sin x)'}{\sin^2 x} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-1}{\sin^2 x} = -\csc^2 x.$$

(Here, I used the pythagorean identity  $\sin^2 x + \cos^2 x = 1$ , and the definition  $\csc x = \frac{1}{\sin x}$ .)

3.3.re.e2. Find the derivative of the given function.

- |                               |                                  |                                  |
|-------------------------------|----------------------------------|----------------------------------|
| a. $\frac{x}{3 + \cos x}$     | b. $x^2 \cos x \sin x$           | c. $e^x(x \sec x + \cot x)$      |
| d. $\frac{x \cos x}{x^2 + 1}$ | e. $\cos x \sin x \sec x \cot x$ | f. $\frac{(\sin x + 1)}{\tan x}$ |
| g. $3e^x + x^5 \csc x$        | h. $e^x \cos x \sin x$           |                                  |

3.3.re.e3. Find the equation of the line tangent to the given function at the given  $x$ -value.

- |   |                                       |
|---|---------------------------------------|
| a. $2 \sin x - \cos x; \quad x = \pi/2$ | b. $\tan x + \cot x; \quad x = \pi/4$ |
|---|---------------------------------------|

#### Answers

- 3.3.re.e2a.  $\frac{3 + \cos x + x \sin x}{(3 + \cos x)^2}$     3.3.re.e2b.  $2x \cos x \sin x - x^2 \sin^2 x + x^2 \cos^2 x$     3.3.re.e2c.  $e^x(x \sec x \tan x + \sec x + x \sec x \tan x \cot x - \csc^2 x)$     3.3.re.e2d.  $\frac{(\cos x - x \sin x)(x^2 + 1) - 2x^2 \cos x}{(x^2 + 1)^2}$     3.3.re.e2e.  $-\sin x$   
 3.3.re.e2f.  $-\sin x - \csc^2 x$     3.3.re.e2g.  $3e^x + 5x^4 \csc x - x^5 \csc x \cot x$     3.3.re.e2h.  $e^x(\cos x \sin x - \sin^2 x + \cos^2 x)$   
 3.3.re.e3a.  $y - 2 = (x - \pi/2)$     3.3.re.e3b.  $y = 2$



## 3.4: The Chain Rule

Our final Combination law will allow us to differentiate the *composition* of two functions.  
**II. Combination laws** (continued).

**Chain Rule:** If  $y$  is a differentiable function of  $u$ , and  $u$  is a differentiable function of  $x$ , then  $y$  is a differentiable function of  $x$ , and

$$\frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}.$$

If function notation, if  $f$  and  $g$  are differentiable functions, and if the range of  $g$  lies in the domain of  $f$ , then  $f \circ g(x) = f(g(x))$  is differentiable, and

$$[f(g(x))]' = f'(g(x))g'(x)$$

3.4.re.e1. Find the derivative of the given function.

- |                             |                             |                       |
|-----------------------------|-----------------------------|-----------------------|
| a. $\cos x^2$               | b. $\tan^2 x$               | c. $x^2 e^{-\cos x}$  |
| d. $\sqrt{x^2 - 5e^x}$      | e. $(2x - 1)^3(x^3 - 4x)^4$ | f. $e^{x^2 - e^x}$    |
| g. $\sqrt{\frac{x+1}{x-1}}$ | h. $\sin(\tan^3 x)$         | i. $e^{2e^{3e^{4x}}}$ |
| j. $e^{2+e^{3+e^{4+x}}}$    |                             |                       |

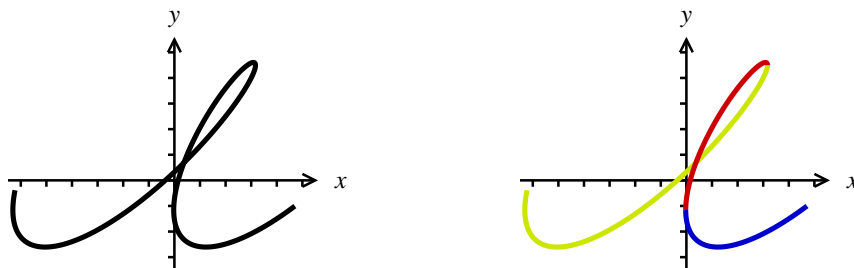
3.4.re.e2. At what  $x$ -values is the graph of  $y = e^{\tan^2 x}$  horizontal?

## Answers

- 3.4.re.e1a.  $-2x \sin(x^2)$  3.4.re.e1b.  $2 \tan x \sec^2 x$  3.4.re.e1c.  $e^{-\cos x}(2x + x^2 \sin x)$  3.4.re.e1d.  $\frac{2x-5e^x}{2\sqrt{x^2-5e^x}}$   
 3.4.re.e1e.  $6(2x-1)^2(x^3-4x)^4 + 4(2x-1)^3(x^3-4x)^3(3x^2-4)$ , or  
 $(2x-1)^2(x^3-4x)^3(6(x^3-4x) + 4(2x-1)(3x^2-4))$  3.4.re.e1f.  $(2x - e^x)e^{x^2 - e^x}$   
 3.4.re.e1g.  $\left(\frac{-1}{(x-1)^2}\right)\left(\frac{x-1}{x+1}\right)^{1/2}$  3.4.re.e1h.  $\cos(\tan^3 x)3 \tan^2 x \sec^2 x$  3.4.re.e1i.  $(e^{2e^{3e^{4x}}})(2e^{3e^{4x}})(3e^{4x})4$   
 3.4.re.e1j.  $(e^{2+e^{3+e^{4+x}}})(e^{3+e^{4+x}})(e^{4+x})$  3.4.re.e2. Solution: set the derivative  $2e^{\tan^2 x} \tan x \sec^2 x = 0$ .  
 Since  $e^{\tan^2 x}$  and  $\sec x$  are never zero, the only solutions are where  $\tan x = 0$ , that is,  $x = n\pi$  for any integer  $n$ .

### 3.5: Implicit Differentiation and the Inverse Trig Functions

Implicit differentiation is a method to find  $\frac{dy}{dx}$  along the graph of an equation. It's based on the fact that, even if the graph of an equation fails to be the graph of a function, it can



still consist of the graphs of *several* functions. So, if we differentiate both sides of an  $x$ - $y$  equation, it's safe to assume that  $y$  is some unspecified function of  $x$ . The result is an equation in  $x$ ,  $y$ , and  $\frac{dy}{dx}$  that, regardless of the original  $x$ - $y$  equation, is **linear** in  $\frac{dy}{dx}$ .

3.5.re.e1. Find  $\frac{dy}{dx}$  along the curve  $x^2 - y^4 = x^4y + 19$ .

Differentiate both sides with respect to  $x$ , remembering that  $y$  is some (unspecified) function of  $x$ . Wherever the product or chain rules say the derivative of  $y$  should appear, write  $\frac{dy}{dx}$ :

$$2x - 4y^3 \frac{dy}{dx} = 4x^3y + x^4 \frac{dy}{dx}$$

Now solve for the unknown  $\frac{dy}{dx}$ :

$$\begin{aligned} 2x - 4x^3y &= 4y^3 \frac{dy}{dx} + x^4 \frac{dy}{dx} \\ &= (4y^3 + x^4) \frac{dy}{dx} \\ \frac{2x - 4x^3y}{4y^3 + x^4} &= \frac{dy}{dx} \end{aligned}$$

3.5.re.e2, *continued*. The point  $(2, -1)$  is on the curve, since  $x = 2$ ,  $y = -1$  is a solution to  $x^2 - y^4 = x^4y + 19$ . Find the equation of the line tangent to the curve at this point.

Evaluate  $\frac{dy}{dx}$  at  $(2, -1)$  to find  $\frac{dy}{dx} = \frac{36}{12} = 3$ , so the line is given by  $y + 1 = 3(x - 2)$ .

3.5.re.e2. Find  $\frac{dy}{dx}$  along the given curve.

a.  $x^3 - 3x^2y - y^3 = 3$

b.  $\frac{x + y}{x^2 + y^2} = \sin y$

c.  $e^{xy} = \cos(x^2 + y^2)$

d.  $\sqrt{x + 2y} = \tan(2x + y)$

e.  $x^2 + e^y = x$

3.5.re.e3. Find  $\frac{d^2y}{dx^2}$  along the curve  $x^2 + e^y = x$ .

3.5.re.e4. Use these values of  $f(x)$  and  $g(x)$  and their derivatives at  $x = 1, 2,$  and  $3$

$x$	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	3	-3	2	6
2	1	-4	3	7
3	2	-5	1	4

to evaluate the **derivatives** of the following functions at  $x = 1$ :

- a.  $g(f(x))$
- b.  $f(g(x))$
- c.  $f(x)g(x)$
- d.  $f(x)(g(x))^2$

*Derivatives of the inverse trig functions*

**I. Catalog.**

$(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}}$	$(\cos^{-1} x)' = -\frac{1}{\sqrt{1-x^2}}$
$(\tan^{-1} x)' = \frac{1}{1+x^2}$	$(\cot^{-1} x)' = -\frac{1}{1+x^2}$
$(\sec^{-1} x)' = \frac{1}{x\sqrt{x^2-1}}$	$(\csc^{-1} x)' = -\frac{1}{x\sqrt{x^2-1}}$

3.5.re.e5. Find the derivative of the given function.

- a.  $\tan^{-1}(\sin^{-1} x)$
- b.  $(\tan^{-1} x)(\sin^{-1} x)$
- c.  $e^x \arccos x^3$
- d.  $\sec(\sec^{-1} x^2)$
- e.  $\cot(\arctan e^x)$

**Answers**

- 3.5.re.e2a.  $\frac{dy}{dx} = \frac{x^2-2xy}{x^2+y^2}$
- 3.5.re.e2b.  $\frac{dy}{dx} = \frac{1-2x \sin y}{(x^2+y^2) \cos y + 2y \sin y - 1}$
- 3.5.re.e2c.  $\frac{dy}{dx} = -\frac{ye^{xy} + 2x \sin(x^2+y^2)}{2y \sin(x^2+y^2) + xe^{xy}}$
- 3.5.re.e2d.  $\frac{dy}{dx} = \frac{1-4 \sec^2(2x+y)\sqrt{x+2y}}{-2+2 \sec^2(2x+y)\sqrt{x+2y}}$
- 3.5.re.e2e.  $\frac{dy}{dx} = \frac{2x}{1-e^y}$
- 3.5.re.e3.  $\frac{d^2y}{dx^2} = \frac{2(1-e^y)^2 + 4x^2 e^y}{(1-e^y)^3}$
- 3.5.re.e4a.  $g'(f(1))f'(1) = g'(3)f'(1) = 4(-3) = -12$
- 3.5.re.e4b.  $f'(g(1))g'(1) = f'(2)g'(1) = -4 \cdot 6 = -24$
- 3.5.re.e4c.  $f'(1)g(1) + f(1)g'(1) = -3 \cdot 2 + 3 \cdot 6 = 12$
- 3.5.re.e4d.  $f'(1)(g(1))^2 + f(1)2g(1)g'(1) = -3 \cdot 4 + 3 \cdot 4 \cdot 6 = 60$
- 3.5.re.e5a.  $(1 + (\sin^{-1} x)^2)^{-1}(1 - x^2)^{-1/2}$
- 3.5.re.e5b.  $\frac{\sin^{-1} x}{1+x^2} + \frac{\tan^{-1} x}{\sqrt{1-x^2}}$
- 3.5.re.e5c.  $e^x(\arccos x^3 - \frac{3x^2}{\sqrt{1-x^6}})$
- 3.5.re.e5d.  $2x$
- 3.5.re.e5e.  $-e^{-x}$

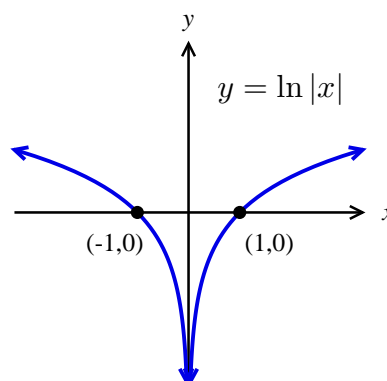
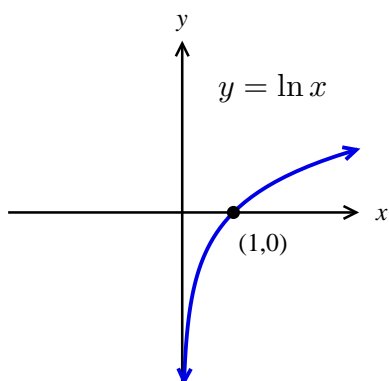
### 3.6: Derivatives of Logarithmic Functions

Two more entries for our catalog of functions and their derivatives.

#### I. Catalog.

$(\ln x)' = \frac{1}{x}$	$(\ln  x )' = \frac{1}{x}$
--------------------------	----------------------------

The derivative of  $\ln |x|$  has exactly the same algebraic formula as the derivative of  $\ln x$ , but these two functions are different.  $\ln x$  and  $\ln |x|$  agree on  $(0, \infty)$ , but, of the two, only  $\ln |x|$  is defined on  $(-\infty, 0)$ .



3.6.re.e1. Find the derivative of the given function.

a.  $\log_3 x$

b.  $(x + 1)^{(x-1)}$

See the properties of logs in section 1.5 of these notes.

a. Use the change of base formula to rewrite  $\log_3 x = \frac{1}{\ln 3} \ln x$ . The derivative of this is  $\frac{1}{\ln 3} \frac{1}{x}$ , or  $\frac{1}{x \ln 3}$ .

b. It is convenient to rewrite expressions in which the variable appears in an exponent so as to have base  $e$ :

$$y = (x + 1)^{(x-1)} = e^{\ln(x+1)^{(x-1)}} = e^{(x-1) \ln(x+1)}$$

Now differentiate using the chain and product rules:

$$\frac{dy}{dx} = e^{(x-1) \ln(x+1)} \left( \ln(x+1) + \frac{x-1}{x+1} \right).$$

3.6.re.e2. Find the derivative of the given function.

a.  $\ln 2$

b.  $\ln(2x)$

c.  $\ln(e^{-x}5x^2 \sin x)$

d.  $\ln |\sec x|$

e.  $\log_{10} x^2$

f.  $e^{\frac{1}{2} \ln x}$

g.  $x^{x/2}$

h.  $(x^2 + 1)^{\sin x}$

#### Answers

3.6.re.e2a. 0   3.6.re.e2b.  $\frac{1}{x}$    3.6.re.e2c.  $-1 + \frac{2}{x} + \cot x$    3.6.re.e2d.  $\tan x$    3.6.re.e2e.  $\frac{2}{x \ln 10}$   
3.6.re.e2f.  $\frac{1}{2}x^{-1/2}$    3.6.re.e2g.  $e^{\frac{1}{2}x \ln x}(\frac{1}{2} \ln x + \frac{1}{2})$    3.6.re.e2h.  $e^{\sin x \ln(x^2+1)}(\cos x \ln(x^2 + 1) + \frac{2x \sin x}{x^2+1})$

3.7: Rates of change in the sciences

$\frac{dy}{dx}$  stands for the derivative, or rate of change, of the function  $y$  with respect to  $x$ . If it were constant,  $\frac{dy}{dx}$  would equal the change in  $y$  corresponding to a 1 unit increase in  $x$ . The units of  $\frac{dy}{dx}$  are the units of  $y$  per units of  $x$ .

3.7.re.e1. Find the units of the derivative, given those of  $x$  and  $y$ .

If the units of $x$ are:	and the units of $y$ are:	then the units of $dy/dx$ are:
sec	m	m/sec
sec	m/sec	
century	g	
in <sup>2</sup>	lb	
fortnight	hectares	

3.7.re.e2. An object moving along a coordinate axis is at position  $p(t) = (t^2 - 4)^2$  meters at time  $t$  seconds. Assume that  $-\infty < t < \infty$ .

- Express the object's velocity and acceleration as functions of  $t$ . Label your answers and include the correct units.
- At what time(s), if any, is velocity zero?
- At what time(s), if any, is acceleration zero?
- Over what interval(s) of time is the object moving in the positive direction?
- Sketch a schematic diagram to illustrate the object's motion between times  $t = -2$  and  $t = 3$ .
- What is the object's *net* distance travelled between  $t = -2$  and  $t = 3$ ?
- Find the total distance the object travels between times  $t = -2$  and  $t = 3$ .

Solution:

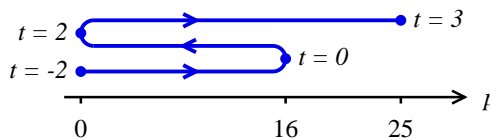
- Position =  $p = (t^2 - 4)^2$  m; velocity =  $v = p' = 2(t^2 - 4)2t = 4t(t^2 - 4) = 4t^3 - 16t$  m/sec; acceleration =  $a = p'' = 12t^2 - 16$  m/sec<sup>2</sup>.
- Factor to solve:  $v = 2(t^2 - 4)2t = 4t(t - 2)(t + 2) = 0$  at  $t = -2, 0,$  and  $2$ .
- $a = 12t^2 - 16 = 4(3t^2 - 4) = 4(\sqrt{3}t - 2)(\sqrt{3}t + 2) = 0$  at  $t = \pm 2/\sqrt{3}$ .
- 0d. The object is moving in the positive direction when position is increasing. This is when the derivative of position, or velocity, is positive. Make a sign chart for  $v$  as in precalculus:

$$\begin{array}{ccccccc}
 4t(t - 2)(t + 2) : & - & - & - & 0 & + & + & + & + & 0 & - & - & - & - & 0 & + & + & + & + \\
 \hline
 t : & & & & -2 & & & & 0 & & & & & & 2 & & & & & & 
 \end{array}$$

So, the object is moving forward for  $t$  in  $(-2, 0)$  and in  $(2, \infty)$ .

e. The object moves forward from  $t = -2$  to  $t = 0$ , then backward between  $t = 0$  and  $t = 2$ , and then forward from  $t = 2$  to  $t = 3$ . Calculate  $p$  at these times and draw the object's motion on a schematic diagram like so:

$t$	-2	0	2	3
$p(t)$	0	16	0	25



f. Net distance traveled, or **displacement**, between  $t = -2$  and  $t = 3$  is the change in position:  $p(3) - p(-2) = 25 - 0 = 25\text{m}$ .

g. Total distance traveled, as the diagram illustrates, between  $t = -2$  and  $t = 3$  is 16 forward +16 backward +25 forward = 57m.

end example 3.7.re.2

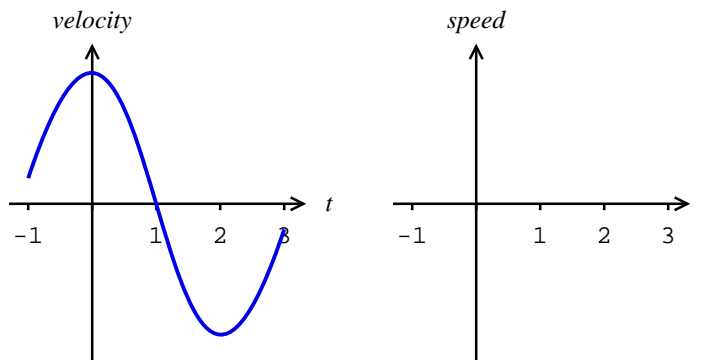
3.7.re.e3. The given function is the position  $s$  at time  $t$  of an object moving along an axis. Assume  $-\infty < t < \infty$ . Find the following.

- (1). The velocity, speed and acceleration of the object at time  $t$ .
- (2). When the object is moving forward and when it is moving backward.
- (3). A schematic diagram of the object's motion on  $-\infty < t < \infty$ .
- (4). The net and total distance traveled by the object between times 0 and 3.

a.  $t^4 - 8t^2 + 16$       b.  $\frac{4}{3}t^3 - 36t$       c.  $\frac{t^2}{t^2 + 4}$       d.  $\frac{t}{t^2 + 1}$

Be careful to distinguish between velocity and speed, the absolute value of velocity.

3.7.re.e4. An object moves along a coordinate axis from time  $t = -1$  to  $t = 3$ . The figure on the left shows the graph of the object's velocity.

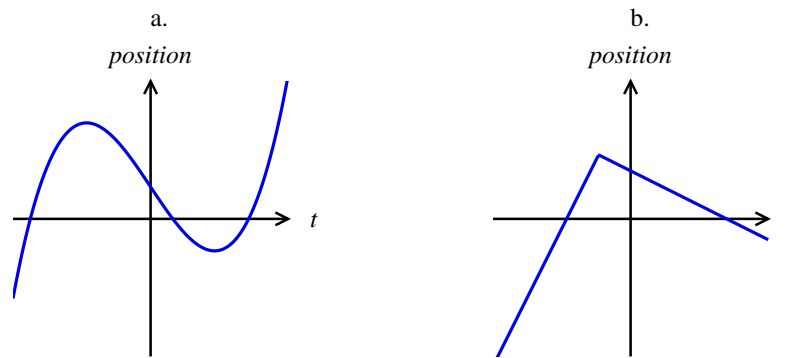


- a. On the axes provided, sketch the graph of the object's speed.
- b. Over what interval(s) of time is the object speeding up?

3.7.re.e5. A ball launched upward is at  $s = 140 + 60t - 5t^2$  meters at time  $t$  seconds after launch. Find the following.

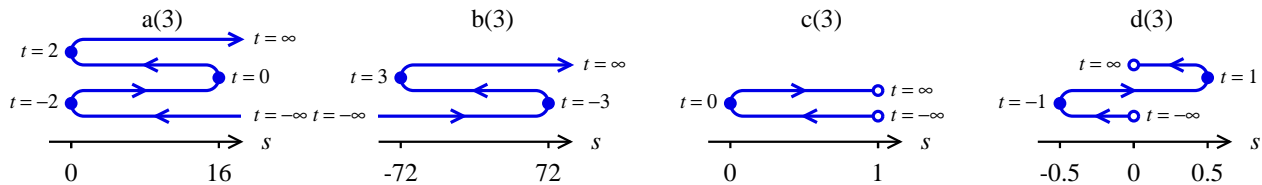
- a. The ball's velocity and acceleration at time  $t$ , and the units of each.
- b. When the ball is traveling upward and when it is traveling downward.
- c. When the ball was launched, and when it hit the ground?
- d. A schematic diagram of the ball's motion
- e. The graph of the ball's velocity and speed.
- f. When the ball's velocity is increasing.
- g. When the ball is speeding up and when it is slowing down.

3.7.re.e6. Graph velocity and speed for the given position function.

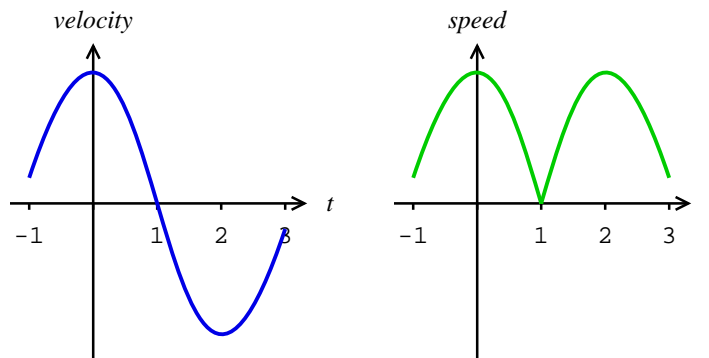


Answers

3.7.re.e1. m/sec<sup>2</sup>, g/century, lb/in<sup>2</sup>, hectares/fortnight 3.7.re.e3a. (1)  $v = 4t^3 - 16t$ .  $|v| = |4t^3 - 16t|$ .  $a = 12t^2 - 16$ . (2) Forward when  $-2 < t < 0$  or  $2 < t$ . Backward when  $-\infty < t < -2$  or  $0 < t < 2$ . (3) below. (4) Net =  $s(3) - s(0) = 9$ . Total = 41. 3.7.re.e3b. (1)  $v = 4(t^2 - 9)$ .  $|v| = 4|t^2 - 9|$ .  $a = 8t$ . (2) Forward when  $t < -3$  or  $3 < t$ . Backward when  $-3 < t < 3$ . (3) below. (4) Net =  $s(3) - s(0) = -72$ . Total = 72. 3.7.re.e3c. (1)  $v = \frac{8t}{(t^2+4)^2}$ .  $|v| = \frac{8|t|}{(t^2+4)^2}$ .  $a = \frac{32-24t^2}{(t^2+4)^3}$ . (2) Forward when  $0 < t$ . Backward when  $t < 0$ . (3) below. (4) Net =  $s(3) - s(0) = \frac{9}{13} =$  Total. 3.7.re.e3d. (1)  $v = \frac{1-t^2}{(t^2+1)^2}$ .  $|v| = \frac{|1-t^2|}{(t^2+1)^2}$ .  $a = \frac{2t^3-6t}{(t^2+1)^3}$ . (2) Forward when  $-1 < t < 1$ . Backward when  $t < -1$  or  $1 < t$ . (3) below. (4) Net =  $s(3) - s(0) = \frac{3}{10}$ . Total =  $\frac{7}{10}$ .



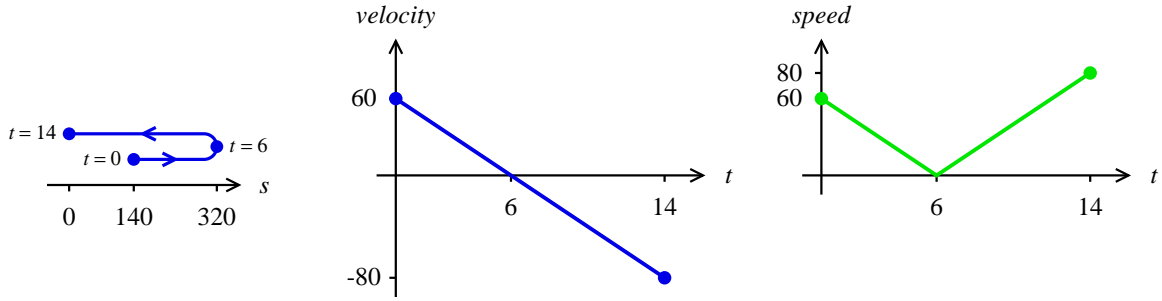
3.7.re.e4. a. speed = |velocity|. See graph:



b. The object is speeding up when speed is increasing. That is, when  $-1 \leq t \leq 0$  and  $1 \leq t \leq 2$ .

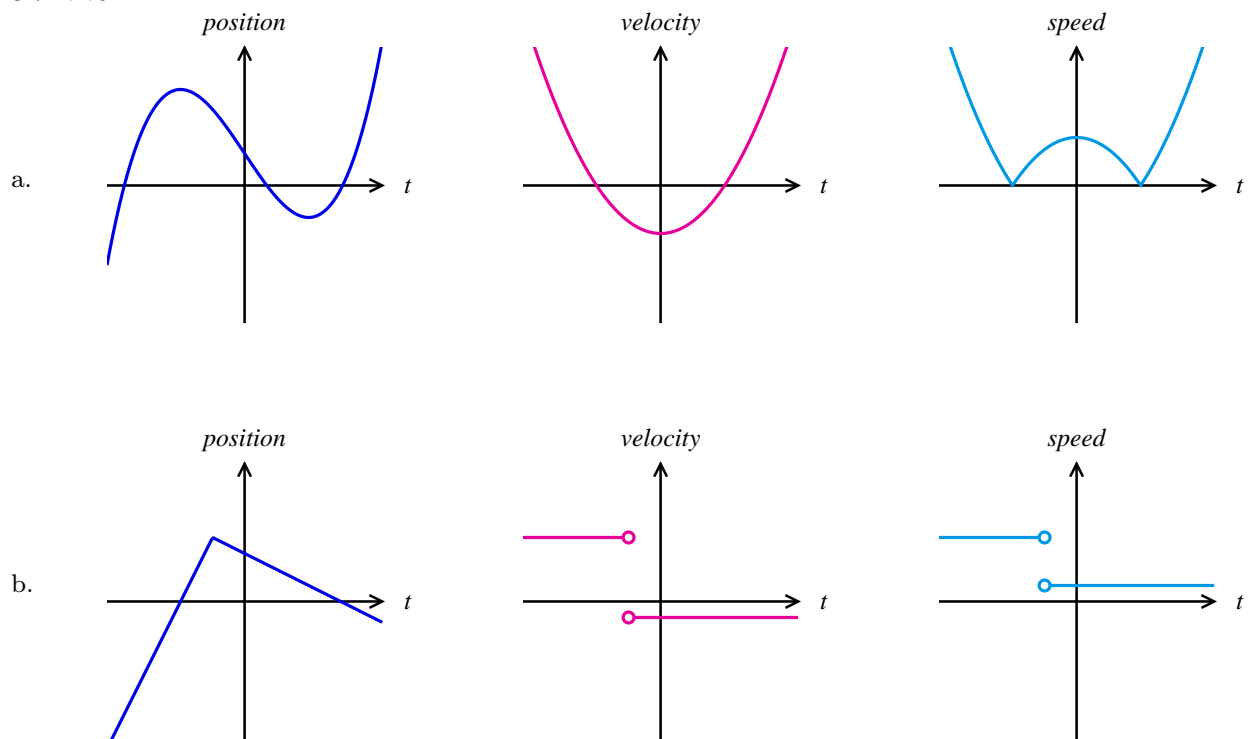
3.7.re.e5a.  $v = 60 - 10t$  m/sec.  $a = -10$  m/sec<sup>2</sup>. 3.7.re.e5b. Up for  $t < 6$ . Down  $t > 6$ . 3.7.re.e5c. Question implies launch is at time  $t = 0$ . Ball reaches ground level when  $0 = 140 + 60t - 5t^2 = -5(t - 14)(t + 2)$ . Since  $t = -2$  is before time of launch, the only relevant solution is  $t = 14$  sec. 3.7.re.e5d. Assuming the ball stops when it strikes the ground, schematic diagram looks like the one below.





3.7.re.e5e. Above 3.7.re.e5f. Velocity is never increasing 3.7.re.e5g. Ball is slowing down when its speed is decreasing, or  $0 < t < 6$ . It is speeding up when its speed is increasing, or  $6 < t < 14$ .

3.7.re.e6.



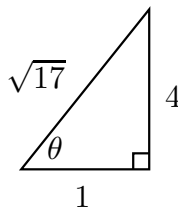
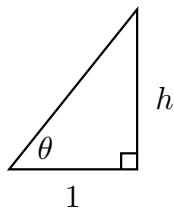
## 3.9: Related Rates

**Tips for solving a related rates problem:**

1. Draw a picture that's true for all time, not just one moment in time. To do this, you must read carefully and distinguish between quantities in your picture that are variable and those that are fixed. Watch out for particular values of variables that are true at only one moment.
2. Identify all rates referred to in the problem as derivatives of variable quantities in your picture (lengths, areas, volumes, angle measurements, etc.) and make those quantities your variables. Find an equation\* that relates just those variables **and no others**.
3. Differentiate implicitly with respect to time. *Then*, plug in any given values of variables at the moment in question and solve for the desired derivative.

3.9.re.e1. A rocket flies vertically away from its launchpad as an observer standing 1 km from the launchpad records the rocket's flight. How fast is the angle of elevation from the observer to the rocket increasing when the rocket is 2 km above the launchpad and traveling 4 km/sec?

Solution: See below, left. Given  $\frac{dh}{dt} = 4$  km/sec, we're asked for  $\frac{d\theta}{dt}$  when  $h = 2$ .



We need an equation relating  $h$  and  $\theta$ , and it comes from the definition of the tangent:

$$\frac{h}{1} = \tan \theta.$$

Now differentiate both sides implicitly with respect to the variable time  $t$ :

$$\frac{dh}{dt} = \sec^2 \theta \frac{d\theta}{dt}$$

Use Pythagoras to find hypotenuse when  $h = 2$  (See above, right). At that moment,  $\sec \theta = \frac{1}{\cos \theta} = \sqrt{17}$ . Plug this and  $\frac{dh}{dt} = 4$  into the equation immediately above and solve for  $\frac{d\theta}{dt}$ :

$$4 = \sqrt{17}^2 \frac{d\theta}{dt} \implies \frac{d\theta}{dt} = \frac{4}{17}.$$

That is, at the moment in question, the angle of elevation from the observer to the rocket is increasing  $\frac{4}{17}$  radians per second.

*end example 3.9.re.1*

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\* The most common sources of equations in these problems is the pythagorean theorem, similar triangles, right-triangle trigonometry, 3-D geometry formulas, and, occasionally, the law of cosines. See the trig review earlier in these notes.

3.9.re.e2. A 10-ft long ladder is leaning against a vertical wall on horizontal ground. The bottom of the ladder is pulled away from the wall at a rate of 1 ft/sec.

- How fast is the ladder sliding down the wall when the bottom is 6 ft from the wall?
- How fast is the area of the triangle formed by the ladder, wall and ground changing at that moment? Is the area increasing or decreasing?

(See an animation of this at <https://www.desmos.com/calculator/akotvavges> Click the play button on line 2. Notice that if the bottom of the ladder moves at a constant rate, the top does not.)

3.9.re.e3. A duck flying at altitude 50 m is moving horizontally 3 m/sec away from a hunter on the ground.

- How fast is the distance between the hunter and her prey increasing when the duck is 130 m away from her?
- At that moment, how fast is the angle of elevation from the hunter to the duck decreasing in radians/sec?

3.9.re.e4. A conical paper cup has height 3 in and radius 1.5 in. Water is leaking from the bottom of the cup  $\frac{1}{18}$  in<sup>3</sup>/sec. Find how fast the radius of the water's surface decreasing when the water is 2 in deep.

3.9.re.e5. Two sides of a triangle have lengths 2 and 3 cm. The angle between the two sides is increasing 2 radian/sec.

- How fast is the length of the third side changing when that angle is 60°? Is the length increasing or decreasing at that time?
- How fast is the area of the triangle changing when that angle is 30°? Is the area increasing or decreasing at that time?

3.9.re.e6. A lighthouse stands on a small island 1 mile from a point  $P$  on a straight shoreline. If the light atop the light house rotates 2 revolutions per minute, how fast is the beam of light moving along the shore when it's shining on a point  $\frac{3}{4}$  mile from  $P$ ? (Animation at <https://www.desmos.com/calculator/d09fdiiwah>)

#### Answers

3.9.re.e2a. The height of the ladder is decreasing  $\frac{3}{4}$  ft/sec 3.9.re.e2b.  $\frac{7}{4}$  ft<sup>2</sup>/sec. Since  $dA/dt > 0$ , the area is increasing. 3.9.re.e3a.  $\frac{36}{13}$  m/sec 3.9.re.e3b.  $3/338$  radians/sec 3.9.re.e4. At the moment in question, the radius decreases  $-1/(36\pi)$  in/sec. 3.9.re.e5a. Increasing  $6\sqrt{3/7}$  cm/sec. 3.9.re.e5b. Increasing  $3\sqrt{3}$  cm<sup>2</sup>/sec. 3.9.re.e6.  $25\pi/4$  mi/min.

## 3.10: Linear Approximation

**Definition 3.10.re.1.** If  $f$  is differentiable at  $a$ , then the **linearization** of  $f$  at  $a$  is the function

$$L(x) = f(a) + f'(a)(x - a).$$

$L(x)$  is the function whose graph is the line tangent to  $y = f(x)$  at  $x = a$ .

3.10.re.e1. Find the linearization of the function at the given point.

- a.  $\frac{1}{1+2x}$ ,  $a = 0$       b.  $\ln x$ ,  $a = 1$       c.  $\sin x$ ,  $a = \pi$   
 d.  $\cos x$ ,  $a = 0$       e.  $\sqrt{x}$ ,  $a = 16$       f.  $e^{2x} + 4e^{3x}$ ,  $a = 0$

Generally, **Linear Approximation** refers to the approximation of the function  $f(x)$  by the function  $L(x)$  for  $x$  near  $a$ , which we vaguely indicate by

$$f(x) \approx L(x) \quad (x \approx a).$$

We can use linear approximation to approximate a number if (1) we recognize that number as  $f(x)$  for some function  $f$  at some  $x$ -value, and (2) that  $x$ -value is near another  $x$ -value  $a$  at which we know the value of  $f$  and  $f'$ .

3.10.re.e2. Use linear approximation to approximate the given number. You are not required to state your answer in decimal form. What function are you approximating, at what  $x$  is the function evaluated, and what  $a$  did you use?

- a.  $\sqrt{25.25}$       b.  $\ln(1.05)$       c.  $\cos(89^\circ)$

**Definition 3.10.re.2.** The **differential** of  $y$  is  $\frac{dy}{dx} dx$ , which we write as  $dy$ .

The differential is of great importance in Chapter 5.

3.10.re.e3. Find the differential of the function.

- a.  $\ln|\sec x|$       b.  $e^{-x} \sin x$       c.  $\ln(\sqrt{x})$       d.  $\frac{x-1}{2x+1}$

## Answers

3.10.re.e1a.  $1 - 2x$     3.10.re.e1b.  $x - 1$     3.10.re.e1c.  $-(x - \pi)$     3.10.re.e1d.  $1$     3.10.re.e1e.  $4 + \frac{1}{8}(x - 16)$   
 3.10.re.e1f.  $5 + 14x$     3.10.re.e2a.  $\sqrt{25.25} \approx 5 + \frac{1}{10}(25.25 - 25) = 5.025$ , using  $f(x) = \sqrt{x}$ ,  $x = 25.25$ , and  $a = 25$ .  
 3.10.re.e2b.  $\ln(1.05) \approx 0.05$ , using  $f(x) = \ln x$ ,  $x = 1.05$ , and  $a = 1$ .    3.10.re.e2c.  $\cos(89^\circ) = \cos(\frac{89\pi}{180} \text{ rad}) \approx \frac{\pi}{180}$ , using  $f(x) = \cos x$ ,  $x = \frac{89\pi}{180}$ , and  $a = \frac{1}{2}\pi$ .    3.10.re.e3a.  $\tan x dx$     3.10.re.e3b.  $e^{-x}(\cos x - \sin x) dx$   
 3.10.re.e3c.  $\frac{dx}{2x}$     3.10.re.e3d.  $3(2x + 1)^{-2} dx$

4.1: Extrema

Vocabulary:

**maxima/minima:** plural of maximum/minimum

**extremum (pl extrema):** a maximum or minimum

We study two types of extrema in calculus: absolute and local.

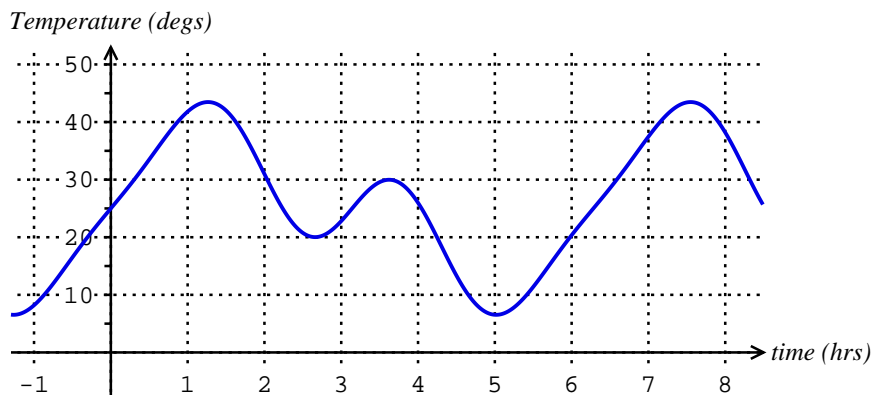
**Definition 4.1.re.1.** Suppose a function  $f$  is defined on some interval  $D$ . The **absolute maximum** of  $f$  on  $D$  is the largest value that  $f$  attains on  $D$ , if such a value exists.

The **absolute minimum** of  $f$  on  $D$  is its smallest value on  $D$ , if it exists.

**Definition 4.1.re.2.** The function  $f$  has a **local maximum** at  $x = c$  if  $f(c)$  is the absolute maximum of  $f$  on some (possibly small) open interval containing  $c$ .

$f$  has a **local maximum** at  $x = c$  if  $f(c)$  is the absolute maximum of  $f$  on some open interval containing  $c$ .

4.1.re.e1. In the graph below, temperature  $T$  is a function of time  $t$ .



Estimate the absolute extrema of  $T$  on the given intervals of  $t$ .

	$[0, 3]$	$[2, 4]$	$[0, 8]$	$[6, 8]$	$(6, 8]$
abs max $T$					
abs min $T$					

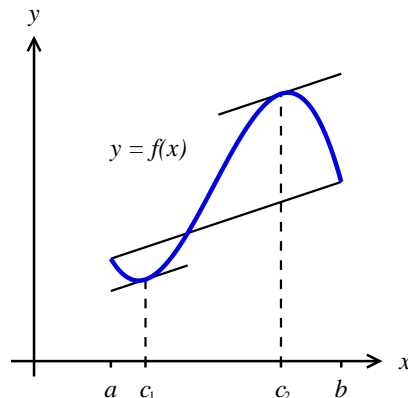
4.1.re.e2. Identify the  $t$ -values at which the function  $T$  in 4.1.re.e1 attains a local max or local min.



## 4.2: The Mean Value Theorem

**Mean Value Theorem (MVT) 4.2.re.1.** If  $f$  is continuous on the closed interval  $[a, b]$  and differentiable on the open interval  $(a, b)$ , then there exists a number  $c$  in  $(a, b)$  for which

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



If, in addition,  $f(a) = f(b)$ , then the MVT guarantees a  $c$  in  $(a, b)$  at which  $f'(c) = 0$ . This is known as **Rolle's Theorem**.

4.2.re.e1. Check whether the function satisfies the hypothesis of the MVT on the given interval. If it does, find the number(s)  $c$  as in the conclusion.

a.  $f(x) = x^3 + x + 2$   $[0, 2]$

b.  $g(x) = 2x^{2/3} - x^{5/3}$   $[0, 2]$

c.  $h(x) = x^{2/3}$   $[-1, 1]$

d.  $\ell(x) = \sqrt{x+1}$   $[1, 3]$

## Answers

4.2.re.e1a.  $c = 2/\sqrt{3}$ . 4.2.re.e1b.  $c = 4/5$ . 4.2.re.e1c. fails to satisfy hypothesis.

4.2.re.e1d.  $c = -1 + (2 - \sqrt{2})^{-2}$ .

## 4.3: A Function's Derivative and its Graph

**Definition 4.3.re.1.**

$f$  **increasing** means:  $a < b \implies f(a) < f(b)$

$f$  **decreasing** means:  $a < b \implies f(a) > f(b)$

**Theorem 4.3.re.2.** Suppose  $f$  is differentiable on an interval  $I$ . Then:

a.  $f' > 0$  on  $I \implies f$  is increasing on  $I \implies f' \geq 0$  on  $I$ .

b.  $f' < 0$  on  $I \implies f$  is decreasing on  $I \implies f' \leq 0$  on  $I$ .

That is, the sign of the first derivative determines a function's **monotonicity** (increasingness and decreasingness). The *second* derivative determines a function's **concavity**:

**Definition 4.3.re.3.** The graph of  $f$  is **concave up** when  $f'$  is increasing (or  $f'' > 0$ ), and **concave down** when  $f'$  is decreasing (or  $f'' < 0$ ).

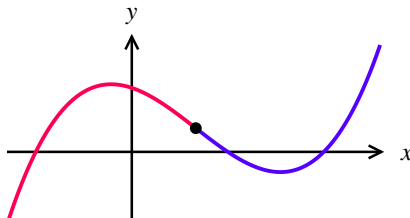
An **inflection point** is a point on the graph of  $f$  where  $f$  changes concavity (and is continuous).

Monotonicity and concavity are independent, in that a function can be any of

increasing and concave up,  
decreasing and concave up, or

increasing and concave down,  
decreasing and concave down.

as this figure demonstrates:



Generally, a curve is concave up where its tangent lines lie below the curve, and concave down where its tangent lines lie above the curve. You can see this on an interactive graph of the function graphed above,  $f(x) = \frac{1}{3}(x+1)(x-1)(x-2)$ , by going to <https://www.desmos.com/calculator/necjpvhc6j> for





4.3.re.e4, continued. b. Sketch the graph of function  $f$  satisfying

$$f'(x) < 0 \text{ if } x < 0$$

$$f'(x) > 0 \text{ if } 0 < x < 4 \text{ or } 4 < x$$

$$f'(x) = 0 \text{ at } x = 0, 4$$

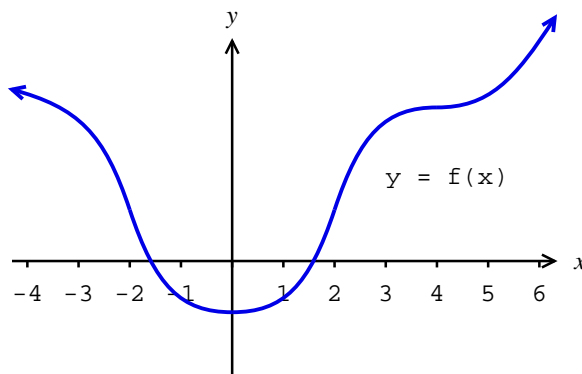
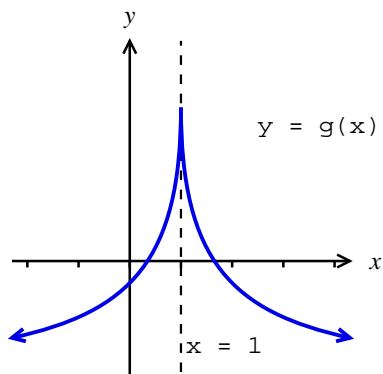
$$f''(x) < 0 \text{ if } x < -2 \text{ or } 2 < x < 4$$

$$f''(x) > 0 \text{ if } -2 < x < 2 \text{ or } 4 < x$$

$$f''(x) = 0 \text{ at } x = -2, 2, 4$$

#### Answers

4.3.re.e1a. increases on  $(-\infty, 1]$  and on  $[3, \infty)$ . decreases on  $[1, 3]$ . local max is 4 (at  $x = 1$ ) and local min is 0 (at  $x = 3$ ). 4.3.re.e1b. increases on  $[0, \pi/6]$  and on  $[5\pi/6, 2\pi]$ . decreases on  $[\pi/6, 5\pi/6]$ . local max is  $\pi/12 + \sqrt{3}/2$  at  $x = \pi/6$ ; local min is  $5\pi/12 - \sqrt{3}/2$  at  $x = 5\pi/6$ . 4.3.re.e1c. Curve has vertical asymptotes at  $x = \pm 2$ .  $\ell$  increases on  $(\infty, -2)$  and on  $(-2, 0]$ . decreases on  $[0, 2)$  and on  $(2, \infty)$ . local max  $= -1/4$  at  $x = 0$ . no local min. 4.3.re.e1d. increases on  $[0, 1/3]$ . decreases on  $[1/3, \infty)$ . local max is  $2/(3^{3/2})$  at  $x = 1/3$ . no local min. 4.3.re.e2a. concave up on  $(2, \infty)$ , concave down on  $(-\infty, 2)$ . inflection point is at  $x = 2$ . 4.3.re.e2b. concave down on  $[0, \pi/2]$  and on  $[3\pi/2, 2\pi]$ . concave up on  $[\pi/2, 3\pi/2]$ . inflection points at  $x = \pi/2$  and  $x = 3\pi/2$ . 4.3.re.e2c. concave up on  $(-\infty, -2)$  and on  $(2, \infty)$ . concave down on  $(-2, 2)$  no inflection points (since  $x = \pm 2$  are not points on the curve). 4.3.re.e2d. concave down on its entire domain,  $[0, \infty)$ . no inflection point. 4.3.re.e4.



## 4.4: l'Hospital's Rule and indeterminate forms.

*Limit values and Indeterminate forms*

4.4.re.e1. The function  $\frac{\cos(1+h)-\cos 1}{h}$  does not exist at  $h = 0$  because  $\frac{0}{0}$  is not a number. However, as seen in 2.2 of these notes,  $\lim_{h \rightarrow 0} \frac{\cos(1+h)-\cos 1}{h}$  exists and is approximately 0.841. We say that this limit has the “form”  $\frac{0}{0}$  because 0 is the limit of both the numerator and denominator. But there's no way for us to deduce value the limit (0.841) from the initial form of the limit ( $\frac{0}{0}$ ).

4.4.re.e2.  $\lim_{x \rightarrow \infty} \frac{2x-1}{x+3}$  initially has the form  $\frac{\infty}{\infty}$ , but, as seen in 2.6.re.e3c, the value of the limit is 2. It's not possible to deduce the value of the limit from this initial form.

$\frac{0}{0}$	$\frac{\infty}{\infty}$	$0 \cdot \infty$	$\infty - \infty$
---------------	-------------------------	------------------	-------------------

are called **indeterminate forms** of limits, since they tell us nothing about the *value* or even the *existence* of such limits.

That's not simply because these forms are not numbers. For example,  $\frac{\text{nonzero}}{0}$  and  $\frac{\text{finite}}{\infty}$  are not numbers, but these forms tell us something about the limit value:

A limit of the form $\frac{\text{nonzero}}{0}$ can only equal $\pm\infty$ ;
A limit of the form $\frac{\text{finite}}{\infty}$ must equal zero.

Many interesting limits in calculus are of an indeterminate form. l'Hôpital's Rule is a tool to help us evaluate some of these.

**l'Hospital's Rule 4.4.re.1.** If  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$  has the form  $\frac{0}{0}$  or  $\frac{\infty}{\infty}$ , and if  $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)} = L$ , then  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$  exists and also equals  $L$ .

*Notes on l'Hôpital's Rule*

1.  $x \rightarrow a$  can be replaced throughout by  $x \rightarrow \infty$ ,  $x \rightarrow -\infty$ ,  $x \rightarrow a^-$ , or  $x \rightarrow a^+$ .
2.  $L$  can be  $\infty$ ,  $-\infty$ , or any real number.
3. l'Hospital's Rule applies only to the limits of the form  $\frac{0}{0}$  or  $\frac{\infty}{\infty}$ . Limits of the form  $0 \cdot \infty$  or  $\infty - \infty$  would have to be rewritten as quotients for l'Hôpital's Rule to be of any use.
4. l'Hospital's Rule never allows us to conclude that  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$  does not exist.

4.4.re.e3. Evaluate the limit.

a.  $\lim_{x \rightarrow 0} \frac{e^{-x} - e^{-3x}}{x}$

b.  $\lim_{x \rightarrow 4} \frac{2 - \sqrt{x}}{16 - x^2}$

c.  $\lim_{x \rightarrow 0} \frac{e^x - e^{-x} - 2x}{x - \sin x}$

d.  $\lim_{x \rightarrow 1} \frac{x^{1/2} - x^{1/3}}{x^{1/3} - x^{1/4}}$

e.  $\lim_{x \rightarrow 8} \frac{\sqrt{1+2x} - x + 5}{\sqrt{x+17} - x + 3}$

f.  $\lim_{x \rightarrow 0} \frac{x}{x - \sin x}$

g.  $\lim_{x \rightarrow 0^-} \frac{x}{\cos x - 1}$

h.  $\lim_{x \rightarrow 0} \frac{x}{\cos x - 1}$

i.  $\lim_{x \rightarrow 2^+} \left( \frac{x}{x-2} - \frac{x^2}{x^2-4} \right)$

j.  $\lim_{x \rightarrow \infty} (3x - \sqrt{9x^2 - 2x})$

k.  $\lim_{x \rightarrow \infty} (\ln(6x+1) - \ln(3x-1))$

l.  $\lim_{x \rightarrow \infty} (\ln x^3 - \ln x^2)$

m.  $\lim_{x \rightarrow \infty} \left( \frac{\ln x^3}{\ln x^2} \right)$

n.  $\lim_{x \rightarrow \infty} x^2 e^{-x}$

o.  $\lim_{x \rightarrow 0} x^3 \ln x$

When  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$  is indeterminate but  $\lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$  is not an easier problem, perhaps l'Hôpital's Rule is not the way to go. Instead, try factoring out (what you believe to be) the dominant terms in the top and bottom.

p.  $\lim_{x \rightarrow \infty} \frac{\sqrt{x^2 + x}}{1 + 2x}$

q.  $\lim_{x \rightarrow -\infty} \frac{\sqrt{x^2 + x}}{1 + 2x}$

r.  $\lim_{x \rightarrow \infty} \frac{2e^x - e^{-x}}{e^x + 3e^{-x}}$

s.  $\lim_{x \rightarrow -\infty} \frac{2e^x - e^{-x}}{e^x + 3e^{-x}}$

#### Answers

4.4.re.e3a. 2. 4.4.re.e3b.  $1/32$ . 4.4.re.e3c. 2. 4.4.re.e3d. 2. 4.4.re.e3e.  $20/27$ . 4.4.re.e3f.  $\infty$ .  
 4.4.re.e3g.  $\infty$ . 4.4.re.e3h. does not exist. 4.4.re.e3i.  $\infty$ . 4.4.re.e3j.  $1/3$ . 4.4.re.e3k.  $\ln 2$ .  
 4.4.re.e3l.  $\infty$ . 4.4.re.e3m.  $3/2$ . 4.4.re.e3n. 0. 4.4.re.e3o. 0. 4.4.re.e3p.  $1/2$ . 4.4.re.e3q.  $-1/2$ .  
 4.4.re.e3r. 2. 4.4.re.e3s.  $-1/3$ .

### 4.5: Summary of Curve Sketching

In this section, we'll graph functions without calculating more than a few points, using ideas of calculus and precalculus:

- Domain (the set of all  $x$ -values at which  $f(x)$  exists)
- Intercepts
  - $y$ -axis:  $f(0) = ?$
  - $x$ -axis:  $f(?) = 0$
- Symmetry of even and odd functions
  - $f(x)$  is even if  $f(-x) = f(x)$ .  
The graph of of an even function is symmetric across the  $y$ -axis.
  - $f(x)$  is odd if  $f(-x) = -f(x)$ .  
The graph of of an odd function is symmetric through the origin.
- Asymptotes
  - $x = c$  is a Vertical Asymptote if  $|y| \rightarrow \infty$  as  $x \rightarrow c$ .
  - $y = c$  is a Horizontal Asymptote if  $y \rightarrow c$  as either  $x \rightarrow \infty$  or  $x \rightarrow -\infty$ .
- Monotonicity
  - $f(x)$  is increasing when  $f'(x)$  is positive.
  - $f(x)$  is decreasing when  $f'(x)$  is negative.
  - The local extrema of  $f(x)$  are where its graph changes monotonicity.
- Concavity
  - The graph of  $f(x)$  is concave up when  $f''(x)$  is positive .
  - The graph of  $f(x)$  is concave down when  $f''(x)$  is negative.
  - The infection points of  $f(x)$  are where its graph changes concavity.

*Tips from precalculus for finding a function's domain*

If a function includes the expression	then it requires
$\frac{\text{anything}}{A}$	$A \neq 0$
$\sqrt{A}$	$A \geq 0$
$\ln A$	$A > 0$
$\tan A$ or $\sec A$	$\cos A \neq 0$
$\cot A$ or $\csc A$	$\sin A \neq 0$
$\sin^{-1} A$ or $\cos^{-1} A$	$ A  \leq 1$
$\sec^{-1} A$ or $\csc^{-1} A$	$ A  \geq 1$

4.5.re.e1. Sketch the graph of the function  $\rho(x) = \frac{x^2 + 4x + 4}{x}$ .

Domain:  $\rho(x)$  is defined for all  $x$  other than  $x = 0$ . (That is, its domain =  $(-\infty, 0) \cup (0, \infty)$ .)

Symmetry:  $\rho(-x) = \frac{x^2 - 4x + 4}{-x}$ , which equals neither  $\rho(x)$  nor  $-\rho(x)$ , so  $\rho$  is neither even nor odd.

Intercepts:  $x \neq 0 \implies y$  does not exist. No  $y$ -intercept.

$y = 0 \implies 0 = x^2 - 4x + 4 = (x + 2)^2 \implies x = -2$ . That is,  $(-2, 0)$  is the only  $x$ -intercept.

Asymptotes: Because  $\frac{\text{nonzero}}{0}$  indicates an infinite limit,  $\lim_{x \rightarrow 0^\pm} \rho(x)$  is infinite, so the graph has a vertical asymptote at  $x = 0$ .

The graph has no horizontal asymptote because the limits

$$\lim_{x \rightarrow \pm\infty} \frac{x^2 + 4x + 4}{x} = \frac{\infty}{\infty} \stackrel{HR}{\hookrightarrow} \lim_{x \rightarrow \pm\infty} \frac{2x + 4}{1} = \pm\infty$$

are not finite. What's more,

$$\begin{aligned} y &\rightarrow \infty \text{ as } x \rightarrow \infty, \text{ and} \\ y &\rightarrow -\infty \text{ as } x \rightarrow -\infty \end{aligned}$$

mean that, when we draw the graph of  $\rho$  on a finite screen, the graph should leave the screen in the upper right and lower left corners.

Monotonicity: Rewrite  $\rho(x) = x + 4 + 4x^{-1}$ , so  $\rho'(x) = 1 - 4x^{-2} = x^{-2}(x^2 - 4) = \frac{(x + 2)(x - 2)}{x^2}$ . Make a sign chart for  $\rho'(x)$ :

$x - 2$ :	- - - - - 0 + + + +
$x^2$ :	+ + + + + 0 + + + + +
$x + 2$ :	- - - - 0 + + + + + + + + + +
$\frac{(x-2)(x+2)}{x^2}$ :	+ + + + 0 - - - - DNE - - - - 0 + + + +
$x$ :	-2                      0                      2

So,  $\rho$  is increasing on the intervals  $(-\infty, -2)$  and  $(2, \infty)$  and decreasing on the intervals  $(-2, 0)$  and  $(0, 2)$ .  $\rho$  has a local maximum at  $x = -2$  and a local minimum at  $x = 2$ .

Concavity:  $\rho''(x) = 8x^{-3} = \frac{8}{x^3}$ . Sign chart:

$x^3$ :	- - - - - 0 + + + + +
$\frac{8}{x^3}$ :	- - - - DNE + + + + +
$x$ :	0

The graph of  $\rho$  is concave up on  $(0, \infty)$  and concave down on  $(-\infty, 0)$ . (Since  $\rho(x)$  is undefined at  $x = 0$ , it has no inflection point at  $x = 0$ .)

Try to sketch the graph of  $\rho$ , illustrating all of the above. Then compare your drawing to the one found here: <https://www.desmos.com/calculator/k3mcjc7fo0>

4.5.re.e2. For each of the following functions, determine the domain, intercepts, symmetry, asymptotes, monotonicity, concavity, and the  $x$ -values where the function has a local extremum or point of inflection. Sketch the function and compare your graph with the one produced by Desmos.

a.  $x^4 - 9x^2$

b.  $x^3 - x^2 - 6x$

c.  $\frac{x}{x^2 - 9}$

d.  $\frac{x^2}{x^2 + 4}$

e.  $\frac{x - 2}{x + 3}$

f.  $x^{-1}e^x$

g.  $4e^x - e^{2x}$

h.  $e^{-x^2}$

i.  $x - 2 \sin x$  on  $[0, 2\pi]$

j.  $\sec x - \cos x$  on  $[0, 2\pi]$

## Answers

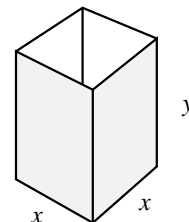
4.5.re.e2a. **domn**  $(-\infty, \infty)$ . **intrept**  $(0, 0)$ ,  $(3, 0)$ ,  $(-3, 0)$ . **symm** across the  $y$ -axis. **asympt** none. **mono** increases on  $(-3/\sqrt{2}, 0)$  and  $(3/\sqrt{2}, \infty)$ . decreases elsewhere. **concav** up on  $(-\infty, -\sqrt{3/2})$  and  $(\sqrt{3/2}, \infty)$ . down elsewhere. **loc extrm at**  $x = 0$  (min) and  $x = \pm 3/\sqrt{2}$  (max). **infl pts at**  $x = \pm \sqrt{3/2}$ . 4.5.re.e2b. **domn**  $(-\infty, \infty)$ . **intrept**  $(0, 0)$ ,  $(3, 0)$ ,  $(-2, 0)$ . **symm** none. **asympt** none. **mono** increases on  $(-\infty, \frac{1}{3}(1 - \sqrt{19}))$  and  $(\frac{1}{3}(1 + \sqrt{19}), \infty)$ . decreases elsewhere. **concav** up on  $(\frac{1}{3}, \infty)$ . down elsewhere. **loc extrm at**  $x = \frac{1}{3}(1 - \sqrt{19})$  (max) and  $x = \frac{1}{3}(1 + \sqrt{19})$  (min). **infl pts at**  $x = 1/3$ . 4.5.re.e2c. **domn**  $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$ . **intrept**  $(0, 0)$ . **symm** through the origin. **asympt**  $x = -3$ ,  $x = 3$ ,  $y = 0$ . **mono** decreases on  $(-\infty, -3)$  and on  $(-3, 3)$  and on  $(3, \infty)$ . increases nowhere. **concav** up on  $(-3, 0)$  and  $(3, \infty)$ . down elsewhere. **loc extrm at** none. **infl pts at**  $x = 0$ . 4.5.re.e2d. **domn**  $(-\infty, \infty)$ . **intrept**  $(0, 0)$ . **symm** across the  $y$ -axis. **asympt**  $y = 1$ . **mono** increases on  $(0, \infty)$ . decreases elsewhere. **concav** up on  $(-2/\sqrt{3}, 2/\sqrt{3})$ . down elsewhere. **loc extrm at**  $x = 0$  (min). **infl pts at**  $x = \pm 2/\sqrt{3}$ . 4.5.re.e2e. **domn**  $(-\infty, -3) \cup (-3, \infty)$ . **intrept** none. **symm** none. **asympt**  $y = 0$  (as  $x \rightarrow -\infty$ ). **mono** increases on  $(-\infty, -3)$  and on  $(-3, \infty)$ . decreases nowhere. **concav** up on  $(-\infty, -3)$ . down elsewhere. **loc extrm at** none. **infl pts at** none. 4.5.re.e2f. **domn**  $(-\infty, 0) \cup (0, \infty)$ . **intrept**  $(0, 0)$ . **symm** through the origin. **asympt**  $x = -3$ ,  $x = 3$ ,  $y = 0$ . **mono** decreases on  $(-\infty, 0)$  and on  $(0, 1)$ . increases on  $(1, \infty)$ . **concav** up on  $(0, \infty)$ . down elsewhere. **loc extrm at**  $x = 1$  (min). **infl pts at** none. 4.5.re.e2g. **domn**  $(-\infty, \infty)$ . **intrept**  $(0, 3)$ ,  $(\frac{1}{3} \ln 4, 0)$ . **symm** none. **asympt**  $y = 0$  (as  $x \rightarrow -\infty$ ). **mono** increases on  $(-\infty, \ln 2)$ . decreases elsewhere. **concav** up on  $(-\infty, 0)$ . down elsewhere. **loc extrm at**  $x = \ln 2$  (max). **infl pts at**  $x = 0$ . 4.5.re.e2h. **domn**  $(-\infty, \infty)$ . **intrept**  $(0, 1)$ . **symm** about  $y$ -axis. **asympt**  $y = 0$  (as  $x \rightarrow \pm\infty$ ). **mono** increases on  $(-\infty, 0)$ . decreases nowhere. **concav** up on  $(-\infty, -1/\sqrt{2})$  and on  $(1/\sqrt{2}, \infty)$ . down elsewhere. **loc extrm at**  $x = 0$  (max). **infl pts at**  $x = \pm 1/\sqrt{2}$ . 4.5.re.e2i. **domn**  $[0, 2\pi]$ . **intrept**  $(0, 0)$ . Finding the other intercept at about  $(1.895, 0)$  requires a calculator and could not appear on an exam. **symm** through the origin, but none on the given interval. **asympt** none. **mono** increases on  $(\frac{1}{3}\pi, \frac{5}{3}\pi)$ . decreases elsewhere. **concav** up on  $(0, \pi)$ . down elsewhere. **loc extrm at**  $x = \frac{1}{3}\pi$  (min),  $x = \frac{5}{3}\pi$  (max). **infl pts at**  $x = \pi$ . 4.5.re.e2j. **domn**  $[0\pi/2) \cup (\pi/2, 3\pi/2) \cup (3\pi/2, 2\pi]$ . **intrept**  $(0, 0)$ ,  $(\pi, 0)$ ,  $(2\pi, 0)$ . **symm** across the  $y$ -axis, but none on the given interval. **asympt**  $x = \pi/2$ ,  $x = 3\pi/2$ . **mono** increases on  $(0, \pi/2)$  and on  $(\pi/2, \pi)$ . decreases on  $(\pi, 3\pi/2)$  and on  $(3\pi/2, 2\pi)$ . **concav** up  $(0, \pi/2)$  and on  $(3\pi/2, 2\pi)$  down elsewhere. **loc extrm at**  $x = 0$  and  $x = 2\pi$  (min),  $x = \pi$  (max). **infl pts at** none.

## 4.7: Optimization Problems

4.7.re.e1. A rectangular box with a square base and an open top is constructed from two different grades of cardboard. The material for the sides costs 1 ¢/in<sup>2</sup> and the material for the base costs 5 ¢/in<sup>2</sup>. If the cost of materials of the box is limited to \$7.35, find the dimensions that will maximize the volume of the box.

Draw a picture. Since the base is a square, we can label the side-lengths  $x$ ,  $x$ , and  $y$ . Identify your **objective function**: the quantity to be maximized or minimized, in this case volume:

$$V = x^2y$$



As often happens, our objective  $V$  is a function of two variables. To rewrite  $V$  in terms of one variable, find the **constraint equation** that ties the variables together. In this case,  $x$  and  $y$  can't be just anything, because the cost of the box—that's the cost of the  $x \times x$  base plus the cost of the 4  $x \times y$  sides—must be \$7.35:

$$5x^2 + 4xy = 735$$

Solve for  $y$  in the constraint and substitute this expression into  $V$ :

$$\begin{aligned} y &= \frac{735 - 5x^2}{4x} \\ \implies V &= x^2 \left( \frac{735 - 5x^2}{4x} \right) = \frac{1}{4}x(735 - 5x^2) \\ &= \frac{1}{4}(735x - 5x^3) \end{aligned}$$

The maximum of  $V$  will occur as its critical point—the  $x$ -value where  $V'(x) = 0$ :

$$0 = \frac{1}{4}(735 - 15x^2) \implies x^2 = \frac{735}{15} = 49 \implies x = 7.$$

(We can ignore the solution  $x = -7$  since  $x$  is a distance.) Make sure you answer the question. Did it ask for the maximum volume, or the dimensions that will produce it? At  $x = 7$ ,  $y = (735 - 5 \cdot 7^2) \div (4 \cdot 7) = \frac{35}{2}$  and so the desired dimensions are  $7 \times 7 \times \frac{35}{2}$ .

(done)

Be careful not to confuse the objective with the constraint (if there is one).

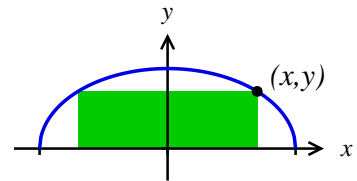
Questions in 4.7 usually ask for either an absolute max or an absolute min, but not both, because one of these is trivial or does not exist. If a question asks only for a max (or a min), it is safe to assume that one exists, and if you find only one critical point, it's safe to assume that you've located the max (or min) in question.



4.7.re.e2. A right circular cylinder is to be inscribed in a sphere of radius  $\sqrt{3}$ . Find the radius and height of the cylinder of maximum volume. Hint: the volume of a cylinder with radius  $r$  and height  $h$  is  $\pi r^2 h$ .

4.7.re.e3. A closed, rectangular box with a square base is have volume 16 cubic inches. If the material for the base costs \$3 per square foot and the material for the sides and top costs \$1 per square foot, what are the dimensions of the least expensive box?

4.7.re.e4. A rectangle is inscribed in the region bounded beneath the ellipse  $2x^2 + 8y^2 = 1$  and above the  $x$ -axis. See figure. Find the dimensions of the rectangle of largest possible area.

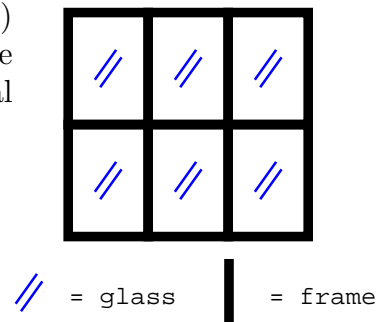


4.7.re.e5. A right circular cone is inscribed in a sphere of radius  $\sqrt{5}$ . Find the maximum possible volume of the cone. Hint: the volume of a cone of height  $h$  and radius  $r$  (measured at its base) is  $\frac{1}{3}\pi r^2 h$ .

4.7.re.e6. A rectangular plot of land will be bounded on one side by a straight river and on the other three sides by a wire fence. With 144 meters of fence at your disposal, what are the dimensions of the largest area you can enclose?

4.7.re.e7. Find the dimensions of the cylinder of largest possible volume that can be generated by rotating a rectangle of perimeter 24 about one of its sides.

4.7.re.e8. A window consists of six identical panes (rectangles) of glass separated by a frame. If the area of each pane must be  $2 \text{ ft}^2$ , find the dimensions of the panes that minimize the total length of frame in the window.



#### Answers

4.7.re.e2.  $r = \sqrt{2}$ ,  $h = 2$  4.7.re.e3.  $2 \times 2 \times 4$  4.7.re.e4. Base = 1, height =  $\frac{1}{4}$ . 4.7.re.e5. Maximum volume  $8\pi/3$  occurs at  $r = h = 2$ . 4.7.re.e6.  $36 \times 72$ , with the longer side parallel the river. 4.7.re.e7.  $r = 8$ ,  $h = 4$ . 4.7.re.e8. Pane has base  $4/3$  and height  $3/2$ .

### 4.9: Antiderivatives

An **antiderivative** for  $f(x)$  is a function whose derivative is  $f(x)$ .

If  $F(x)$  is an antiderivative of  $f(x)$  on an interval, then *every* antiderivative of  $f$  on that interval equals  $F(x) + C$  for some constant  $C$ .

We say that the  $F(x) + C$  is the **General antiderivative** of  $f$  on that interval. The general antiderivative of  $f(x)$  is denoted  $\int f(x) dx$ .

4.9.re.e1.  $x^2$  is an antiderivative of  $2x$  because  $(x^2)' = 2x$ . The general antiderivative of  $2x$  is  $x^2 + C$ .

#### I. A catalog of elementary functions and their antiderivatives.

A table of antiderivatives is practically the same as a table of derivatives read backwards:

$f(x)$	derivative of $f$	$f(x)$	antiderivative of $f$
$f(x)$	$f'(x)$	$f(x)$	$\int f(x) dx$
$x^n$	$nx^{n-1}$	$x^n$	$\begin{cases} \frac{1}{n+1}x^{n+1} + C & \text{if } n \neq -1 \\ \ln x  + C & \text{if } n = -1 \end{cases}$
$\ln x $	$x^{-1}$	$e^x$	$e^x + C$
$e^x$	$e^x$	$\cos x$	$\sin x + C$
$\sin x$	$\cos x$	$\sin x$	$-\cos x + C$
$\cos x$	$-\sin x$	$\sec^2 x$	$\tan x + C$
$\tan x$	$\sec^2 x$	$\csc^2 x$	$-\cot x + C$
$\cot x$	$-\csc^2 x$	$\sec x \tan x$	$\sec x + C$
$\sec x$	$\sec x \tan x$	$\csc x \cot x$	$-\csc x + C$
$\csc x$	$-\csc x \cot x$	$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x + C$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$	$\frac{1}{1+x^2}$	$\tan^{-1} + C$
$\tan^{-1} x$	$\frac{1}{1+x^2}$	$\frac{1}{x\sqrt{x^2-1}}$	$\sec^{-1} x + C$
$\sec^{-1} x$	$\frac{1}{x\sqrt{x^2-1}}$		

#### II. Combination laws.

If  $F$  and  $G$  are antiderivatives for  $f$  and  $g$ , and if  $c$  is a constant, then

1.  $F + G$  is an antiderivative for  $f + g$ .
2.  $F - G$  is an antiderivative for  $f - g$ .
3.  $cF$  is an antiderivative for  $cf$ .

It is important to note that  $FG$  is **not** an antiderivative for  $fg$ , and  $F/G$  is not an antiderivative for  $f/g$ .

4.9.re.e2.  $x^2$  is an antiderivative for  $2x$ , and  $e^x$  is an antiderivative for itself, but  $x^2e^x$  is **not** an antiderivative for  $2xe^x$ , because

$$(x^2e^x)' = 2xe^x + x^2e^x, \text{ not } 2xe^x.$$

4.9.re.e3.  $\sin x$  is an antiderivative of  $\cos x$  and  $x$  is an antiderivative of 1, but  $\frac{\sin x}{x}$  is not an antiderivative for  $\frac{\cos x}{1}$ , because

$$\left(\frac{\sin x}{x}\right)' = \frac{x \cos x - \sin x}{x^2}, \text{ not } \frac{\cos x}{1}$$

4.9.re.e4. Find the general antiderivative of the function:

a.  $5x^3 - 2\sqrt{x} + 6$

b.  $6x^{2/3} + 3x^{5/3} - e^x$

c.  $\sqrt{x^3} - 5\sqrt[3]{x^2}$

d.  $\frac{9}{4} - \frac{2}{x} + \frac{3}{x^2}$

e.  $\frac{x^2 - 2x + 2}{x}$

f.  $(x-2)(x^{1/2} - x^{-1/2})$

g.  $(\sec x)(2 \sec x - 3 \tan x)$

h.  $3 \sin x + 4 \cos x$

Given the derivative of  $f$ , we can only hope to find  $f$  up to a constant.

4.9.re.e5. If  $f'(x) = 2x + e^x$ , then

$$(1) \quad f(x) = x^2 + e^x + C$$

for some constant  $C$ .

To find this constant, we'd also have to know the value of  $f(x)$  at one  $x$ :

4.9.re.e5, continued. If, in addition,  $f(0) = -2$ , then evaluate (1) at  $x = 0$  to find  $C$ :

$$f(x) = x^2 + e^x + C$$

$$f(0) = 0^2 + e^0 + C$$

$$-2 = 1 + C$$

so  $C = -3$ , and  $f(x) = x^2 + e^x - 3$ .

4.9.re.e6. Use the given information to find  $f(x)$ .

a.  $f'(x) = 4x^3 - 2e^x + \sec x \tan x$ ;  $f(0) = 9$

b.  $f'(x) = x(3 - 2x^2)$ ;  $f(2) = -3$

c.  $f'(x) = \frac{5x^2 - 3x + 1}{\sqrt{x}}$ ;  $f(1) = -6$

d.  $f''(x) = -6$ ;  $f'(1) = 2$ ;  $f(1) = 0$

e.  $f''(x) = \sin x$ ;  $f(0) = 1$ ;  $f(\pi/2) = -2$

4.9.re.e7. At time  $t = 0$ , a marble is tossed from a 100 ft tower with an upward velocity of 30 ft/sec. Assuming its acceleration due to gravity is  $-32$  ft/sec<sup>2</sup>, find the marble's altitude at time  $t$ .

#### Answers

4.9.re.e4a.  $\frac{5}{4}x^4 - \frac{4}{3}x^{3/2} + 6x + C$  4.9.re.e4b.  $\frac{18}{5}x^{5/3} + \frac{9}{8}x^{8/3} - e^x + C$  4.9.re.e4c.  $\frac{2}{5}x^{5/2} - 3x^{5/3} + C$   
 4.9.re.e4d.  $\frac{9}{4}x - 2 \ln|x| - 3x^{-1} + C$  4.9.re.e4e.  $\frac{1}{2}x^2 - 2x + 2 \ln|x| + C$  4.9.re.e4f.  $\frac{2}{5}x^{5/2} - 2x^{3/2} + 4x^{1/2} + C$   
 4.9.re.e4g.  $2 \tan x - 3 \sec x + C$  4.9.re.e4h.  $-3 \cos x + 4 \sin x + C$  4.9.re.e6a.  $f(x) = x^4 - 2e^x + \sec x + 10$   
 4.9.re.e6b.  $f(x) = \frac{3}{2}x^2 - \frac{1}{2}x^4 - 1$  4.9.re.e6c.  $f(x) = 2x^{5/2} - 2x^{3/2} + 2x^{1/2} - 8$  4.9.re.e6d.  $f(x) = -3x^2 + 8x - 5$   
 4.9.re.e6e.  $f(x) = -\sin x - \frac{4}{\pi}x + 1$  4.9.re.e7. Solve  $s''(t) = -32$ ,  $s'(0) = 30$ ,  $s(0) = 100$  to find altitude  
 $s = -16t^2 + 30t + 100$

### 5.1: The Area and Distance Problems

The **Area Problem**: how can we find the area between the the graph of a function  $f(x)$  and the  $x$ -axis, between  $x = a$  and  $x = b$ ?

The **Distance Problem**: how can we use the velocity  $v(t)$  of an object to determine its the distance traveled between times  $t = a$  and  $t = b$ ?

Solutions to the area and distance problems can both be estimated by a Riemann sum:

**Definition 5.1.re.1.** Divide the interval  $[a, b]$  into  $n$  subintervals of equal length  $\Delta x$ , and select a number  $x_i^*$  in each ( $1 \leq i \leq n$ ). If the function  $f(x)$  is defined on  $[a, b]$ , then

$$(5.1.re.2) \quad f(x_1^*)\Delta x + f(x_2^*)\Delta x + f(x_3^*)\Delta x + \cdots + f(x_n^*)\Delta x = \sum_{i=1}^n f(x_i^*)\Delta x$$

is called a **Riemann Sum** for  $f(x)$  on  $[a, b]$ .

The symbol  $\sum_{i=1}^n$  means “the sum of values of the following, for  $i$  equal  $1, 2, \dots, n$ .”

$$5.1.re.e1. \quad \sum_{i=1}^5 (i^2 + 1) = (1^2 + 1) + (2^2 + 1) + (3^2 + 1) + (4^2 + 1) + (5^2 + 1) = 60.$$

5.1.re.e2. Calculate the sum.

a.  $\sum_{i=0}^3 (i + 3)$

b.  $\sum_{j=1}^5 2^{j-1}$

c.  $\sum_{k=3}^7 (-1)^k$

When evaluating the Riemann sum (5.1.re.2), it's handy to take out the common factor:

$$\Delta x (f(x_1^*) + f(x_2^*) + f(x_3^*) + \cdots + f(x_n^*)) = \Delta x \sum_{i=1}^n f(x_i^*)$$

5.1.re.e3. Approximate the net signed area under the given curve, on the given interval, with a Riemann sum as described.

a.  $y = x^3 - 2$ ,  $[0, 1]$ ,  $n = 4$  subintervals and their midpoints.

b.  $y = x^3 - 2$ ,  $[0, 6]$ ,  $n = 6$  subintervals and their left endpoints.

c.  $y = e^x$ ,  $[-1, 1]$ ,  $n = 5$  subintervals and their right endpoints.

d.  $y = \frac{1}{1+x}$ ,  $[6, 12]$ ,  $n = 3$  subintervals and their midpoints.

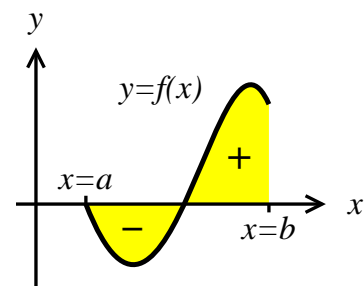
5.1.re.e4. Use the prescribed Riemann sum to estimate the distance traveled between times  $t = 60$  and  $t = 140$  by the object whose velocity is recorded in the following table.

$t(\text{sec})$	60	70	80	90	100	110	120	130	140
$v(\text{m/sec})$	1	3	7	12	14	16	12	7	2

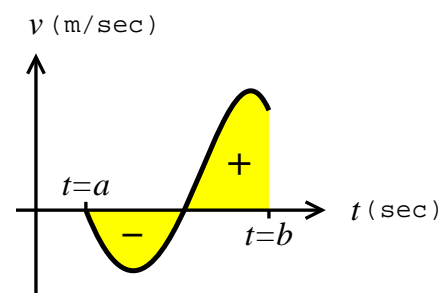
- $n = 8$  subintervals and their left endpoints.
- $n = 4$  subintervals and their right endpoints.
- $n = 4$  subintervals and their midpoints.
- $n = 2$  subintervals and their midpoints.

### Solutions to the area and distance problems

The limit as  $n \rightarrow \infty$  of the Riemann sums for  $f(x)$  on  $a \leq x \leq b$  is the **net signed area** between the graph of  $f$  and the  $x$ -axis between  $x = a$  and  $x = b$ . Areas above the  $x$ -axis are counted as positive and areas below the  $x$ -axis are counted as negative.



The limit as  $n \rightarrow \infty$  of the Riemann sums for  $v(t)$  on  $a \leq t \leq b$  is the **net distance traveled** by the object whose velocity at time  $t$  is  $v(t)$ .



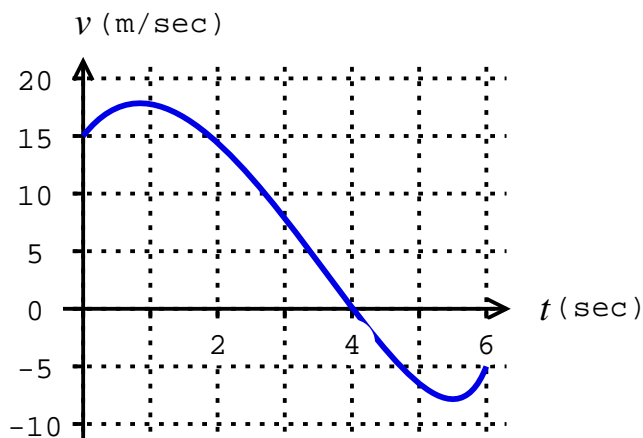
That means that the net distance traveled by the object from time  $a$  to time  $b$  is the same as the net signed area between its velocity curve  $y = v(t)$  and the  $t$ -axis over  $[a, b]$  (since both are limits of the same Riemann sums). Areas above the  $t$ -axis, where velocity is positive, measure motion in the positive direction and areas below the  $t$ -axis, when velocity is negative, measure motion in the negative direction.

5.1.re.e5. The velocity of a car, recorded for six seconds, is graphed in the below.

a. When (between times 0 and 6 sec) is the car moving forward and when is it moving backward?

b. Approximate (not necessary with a Riemann sum) the forward distance traveled by the car between time  $t = 0$  and time  $t = 6$  sec.

c. Approximate the backward distance traveled by the car between time  $t = 0$  and time  $t = 6$  sec.



d. Using these approximations, what do you estimate to be the car's total and net distances traveled between time 0 and 6?

#### Answers

5.1.re.e2a.  $(0+3)+(1+3)+(2+3)+(3+3) = 18$ . 5.1.re.e2b.  $1+2+4+8+16 = 31$ . 5.1.re.e2c.  $-1+1-1+1-1 = -1$ . 5.1.re.e3a.  $\frac{1}{4} \left( \left(\frac{1}{8}\right)^3 - 2 + \left(\frac{3}{8}\right)^3 - 2 + \left(\frac{5}{8}\right)^3 - 2 + \left(\frac{7}{8}\right)^3 - 2 \right)$ . 5.1.re.e3b.  $0^3 - 2 + 1^3 - 2 + 2^3 - 2 + 3^3 - 2 + 4^3 - 2 + 5^3 - 2$  (or 213). 5.1.re.e3c.  $0.4(e^{-1} + e^{-0.6} + e^{-0.2} + e^{0.2} + e^{0.6})$ . 5.1.re.e3d.  $2\left(\frac{1}{8} + \frac{1}{10} + \frac{1}{12}\right)$ . 5.1.re.e4a. 720 m 5.1.re.e4b. 700 m 5.1.re.e4c. 760 m 5.1.re.e4d. 760 m 5.1.re.e5a. Forward when  $0 < t < 4$  and backward when  $4 < t < 6$ . 5.1.re.e5b. Area of each square is  $(5\text{m/sec}) \times (1\text{ sec}) = 5\text{m}$ . Area above the  $t$  axis appears to be approximately 53 m. 5.1.re.e5c. Area below the  $t$  axis appears to be approximately 12m. 5.1.re.e5d. Using the above, the estimated total distance is  $53 + 12 = 65\text{m}$ , and the estimated net distance is  $53 - 12 = 41\text{ m}$ . That is, at  $t = 6$ , the car is about 41 m ahead of its position at time 0, but is traveled 65 m total to get there.

## 5.2: The Definite Integral

**Definition 5.2.re.1.** If  $f(x)$  is defined on the interval  $[a, b]$ , then the limit of its Riemann sums on  $[a, b]$  as  $n \rightarrow \infty$ , if it exists, is called the **definite integral** (or **Riemann integral**) **from  $a$  to  $b$  of  $f(x) dx$** , written  $\int_a^b f(x) dx$ . That is,

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x,$$

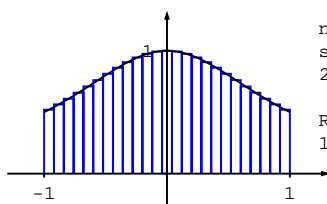
When this limit exists we say that  $f(x)$  is **integrable** on  $[a, b]$ .

Note that  $\int_a^b f(x) dx$ , when it exists, is a number.

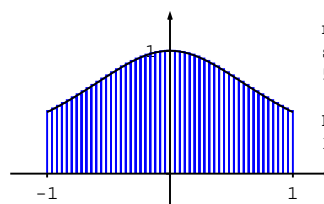
The definite integral  $\int_a^b f(x) dx$  can be interpreted as the:

- 1) net signed area between the graph of  $f(x)$  and the  $x$ -axis between  $x = a$  and  $x = b$ , and
- 2) net distance traveled from time  $a$  to  $b$  by an object whose velocity is  $f(x)$  at time  $x$ .

5.2.re.e1. Here are two Riemann sums for the function  $\frac{1}{x^2 + 1}$  on the interval  $[-1, 1]$ :



number of  
subintervals =  
25  
Riemann sum =  
1.57106



number of  
subintervals =  
50  
Riemann sum =  
1.57086

Based on these two Riemann sums,  $\int_{-1}^1 \frac{1}{x^2 + 1} dx$  appears to equal about 1.5708.

Not every function is integrable, but a great many are:

**Theorem 5.2.re.2.** If  $f(x)$  is continuous on  $[a, b]$  (with the possible exception of finitely many jump or removable discontinuities) then  $f$  is integrable on  $[a, b]$ .



Some useful algebraic properties of the definite integral:

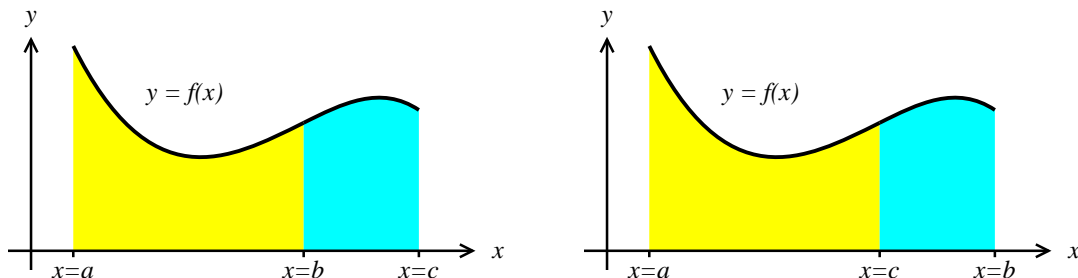
If  $f(x)$  and  $g(x)$  are integrable on  $[a, b]$  and  $c$  is a constant, then

1.  $\int_a^b c \, dx = c(b - a)$
2.  $\int_a^b cf(x) \, dx = c \int_a^b f(x) \, dx$
3.  $\int_a^b (f(x) + g(x)) \, dx = \int_a^b f(x) \, dx + \int_a^b g(x) \, dx$
4.  $\int_a^b f(x) \, dx + \int_b^c f(x) \, dx = \int_a^c f(x) \, dx$

A convenient definition:

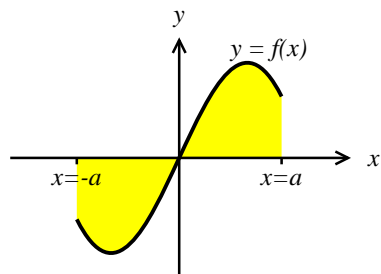
5.  $\int_b^a f(x) \, dx \stackrel{\text{def}}{=} - \int_a^b f(x) \, dx$
6.  $\int_a^a f(x) \, dx \stackrel{\text{def}}{=} 0$

As a consequence of 5, properties 1-4 are valid even when the upper endpoint of an integral is less than its lower endpoint. For example, think about what 4 says in these two pictures:

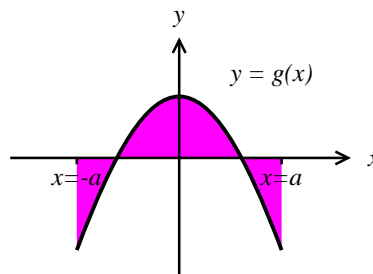


Along similar lines, here's [7](#) from section 5.4:

7. If  $f(x)$  is odd and  $g(x)$  is even, and if  $a > 0$ , then



$$\int_{-a}^a f(x) \, dx = 0$$



$$\int_{-a}^a g(x) \, dx = 2 \int_0^a g(x) \, dx$$

5.2.re.e2. If  $\int_1^3 f(x) dx = 7$ ,  $\int_3^5 f(x) dx = -3$ , and  $\int_1^5 g(x) dx = 2$ , and  $\int_5^3 g(x) dx = 6$ , find the following:

a.  $\int_1^3 g(x) dx$

b.  $\int_3^5 f(x) dx$

c.  $\int_1^5 2f(x) dx$

d.  $\int_1^5 (f(x) - 2g(x)) dx$

e.  $\int_1^5 (6 + g(x)) dx$

f.  $\int_3^1 (3g(x) - 5) dx$

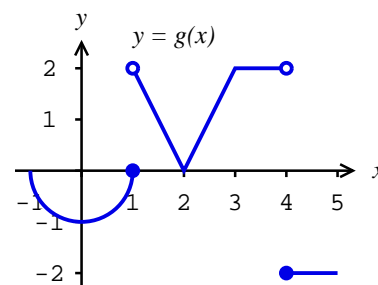
5.2.re.e3. Let  $g(x)$  be the function graphed here (its first piece is a semicircle) and find the following.

a.  $\int_1^{-1} g(x) dx$

b.  $\int_0^3 g(x) dx$

c.  $\int_1^4 g(x) dx$

d.  $\int_1^5 g(x) dx$



5.2.re.e4. Suppose that  $g(x)$  from 5.2.re.e3 is the velocity at time  $x$  of an object moving along a number line. Find the following.

- When the object is traveling forward, and when it's traveling backward.
- The net distance traveled (displacement) by the object between times 0 and 4.
- The total distance traveled by the object between times 0 and 4.
- The net and total distances traveled of the object between time 1 and time 3.

5.2.re.e5. Find the following. (A sketch will help.)

a.  $\int_{-2}^4 |x| dx$

b.  $\int_{-2}^2 1 + \sqrt{4 - x^2} dx$

c.  $\int_{-1}^3 2 - \sqrt{4 - (x - 1)^2} dx$

### Answers

5.2.re.e2a. 8 5.2.re.e2b. 3 5.2.re.e2c. 8 5.2.re.e2d. 0 5.2.re.e2e. 26 5.2.re.e2f. -14 5.2.re.e3a.  $\pi/2$   
 5.2.re.e3b.  $2 - \frac{1}{4}\pi$  5.2.re.e3c. 4 5.2.re.e3d. 2 5.2.re.e4a. Backward when  $-1 < t < 1$  and when  $4 < t < 5$ . Forward when  $1 < t < 4$ . 5.2.re.e4b.  $4 - \frac{1}{4}\pi$  5.2.re.e4c.  $4 + \frac{1}{4}\pi$  5.2.re.e4d. Because the object does not change direction between these times, both equal 2 5.2.re.e5a. 10 5.2.re.e5b.  $4 + 2\pi$  5.2.re.e5c.  $8 - 2\pi$

## 5.3: The Fundamental Theorem of Calculus

**Fundamental Theorem of Calculus I (FTC.I) 5.3.re.1.** If the function  $f$  is integrable on the interval  $[a, b]$ , and if  $F(x)$  is defined as

$$F(x) \stackrel{\text{def}}{=} \int_a^x f(t) dt,$$

then, wherever  $f(x)$  is continuous,  $F(x)$  is differentiable and  $F'(x) = f(x)$ .

5.3.re.e1. Find the derivative of each of integral-defined functions..

a.  $\int_0^x e^{t^2} dt$

b.  $\int_1^x e^{t^2} dt$

c.  $\int_x^1 e^{t^2} dt$

d.  $\int_x^4 \sin(\cos(\theta)) d\theta$

e.  $\int_0^{2x} \sqrt{1+t^{-1}} dt$

f.  $\int_{3x}^{2x} \frac{1}{1+t^4} dt$

g.  $\int_{\ln x}^{2x} \frac{1}{t} dt$

h.  $\int_1^{x^2} \frac{1+t}{1+t^3} dt$

i.  $\int_{e^x}^{\cos x} \ln(x+5) dt$

j.  $\int_1^{\tan^{-1} x} \tan \theta d\theta$

k.  $\int_0^{\ln x} e^{2u} du$

**Fundamental Theorem of Calculus II (FTC.II) 5.3.re.2.** If  $F(x)$  is any antiderivative of  $f(x)$  and if  $f$  is continuous on  $[a, b]$ , then

$$\int_a^b f(t) dt = F(b) - F(a) \stackrel{\text{def}}{=} F(x) \Big|_a^b$$

5.3.re.e2. Evaluate the definite integral.

a.  $\int_0^2 e^t dt$

b.  $\int_{\pi/4}^{\pi} (\sin \theta + \cos \theta) d\theta$

c.  $\int_{\pi/4}^{\pi} (\sin^2 \theta + \cos^2 \theta) d\theta$

d.  $\int_{-1}^1 (3x-2)(x+1) dx$

e.  $\int_{-3}^{-1} \frac{xe^x + 1}{x} dx$

f.  $\int_1^4 \frac{2x^2 + 1}{\sqrt{x}} dx$

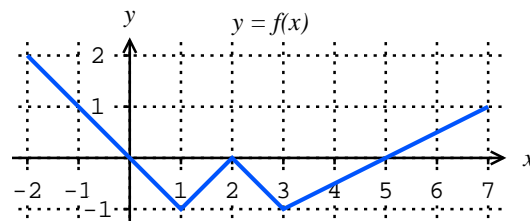
g.  $\int_1^3 \left( \frac{2-t}{t} \right)^2 dt$

h.  $\int_{-\pi/4}^{\pi/4} (\sec^2 x - \cos x) dx$

5.3.re.e3. Find the area of the “triangular” region between the curves  $y = e^x - 1$ ,  $y = 0$ , and  $x = 2$ .

5.3.re.e4. The velocity of a particle moving along a number line is  $v(t) = \cos t - \sin t$  at time  $t$ . Find the displacement of the object between times  $t = -\pi/2$  and  $t = \pi/2$ .

5.3.re.e5. Let  $f(x)$  be the function graphed here and define  $g(x) = \int_{-2}^x f(t) dt$ .



- a. On what intervals is  $g(x)$  increasing? On what intervals is  $g$  decreasing?
- b. Find the critical numbers of  $g$ . Where does  $g$  have a local max or min?
- c. Find the ( $x$ -coordinates of) inflection points of  $g(x)$ . Where is  $g(x)$  concave up and concave down?
- d. Find the absolute extrema of  $g$  on the interval  $[-2, 7]$ .

Answers

- 5.3.re.e1a.  $e^{x^2}$  5.3.re.e1b.  $e^{x^2}$  5.3.re.e1c.  $-e^{x^2}$  5.3.re.e1d.  $-\sin(\cos(x))$  5.3.re.e1e.  $2\sqrt{1+(2x)^{-1}}$
- 5.3.re.e1f.  $\frac{2}{1+16x^4} - \frac{3}{1+81x^4}$  5.3.re.e1g.  $\frac{1}{x} - \frac{1}{x \ln x}$  5.3.re.e1h.  $\frac{(1+x^2)2x}{1+x^6}$  5.3.re.e1i.  $-\sin x \ln(5 + \cos x)$
- $-e^x \ln(e^x + 5)$  5.3.re.e1j.  $\frac{x}{1+x^2}$  5.3.re.e1k.  $x$  5.3.re.e2a.  $e^2 - 1$  5.3.re.e2b. 1 5.3.re.e2c.  $3\pi/4$
- 5.3.re.e2d.  $-2$  5.3.re.e2e.  $e^{-1} - e^{-3} - \ln 3$  5.3.re.e2f.  $\frac{134}{5}$  5.3.re.e2g.  $\frac{14}{3} - 4 \ln 3$  5.3.re.e2h.  $2 - \sqrt{2}$
- 5.3.re.e3.  $e^2 - 3$  5.3.re.e4. 2 5.3.re.e5a.  $g$  is increasing on  $[-2, 0]$  and  $[5, 7]$  and decreasing on  $[0, 5]$ .
- 5.3.re.e5b. Critical numbers of  $g$  are  $x = 0, 2,$  and  $5$ .  $g$  has a local max at  $x = 0$  and local min at  $x = 5$ .
- 5.3.re.e5c. The graph of  $g$  is concave up where  $g''(x) = f'(x) > 0$ , on  $[1, 2]$  and  $[3, 7]$ . The graph of  $g$  is concave down on  $[-2, 1]$  and on  $[2, 3]$ .  $g''(x)$  is never zero, but  $g''(x)$  changes sign at  $x = 1, 2,$  and  $3$ , so the inflection points of  $g$  are at these  $x$ -values. 5.3.re.e5d. Compute  $g$  at critical points and endpoints:

$x$	$-2$	$0$	$2$	$5$	$7$
$g(x)$	$0$	$2$	$1$	$-1/2$	$1/2$

Conclusion: the absolute maximum of  $g(x)$  on  $[-2, 7]$  is 2, and the absolute minimum is  $-1/2$ .

## 5.4: Indefinite integrals and net change

The word “integral” and the integral sign  $\int$  are used two different ways:

**Definition 5.4.re.1.** The **definite integral** from  $a$  to  $b$  of  $f(x) dx$ , written

$$\int_a^b f(x) dx,$$

is the limit of the Riemann sums of  $f$  on the interval  $[a, b]$  as the number of subintervals goes to infinity.

The **indefinite integral** of  $f(x) dx$ , written

$$\int f(x) dx,$$

is the collection of all antiderivatives of  $f(x)$ .

$$\begin{aligned} \int_a^b f(x) dx &= \text{a number.} \\ &= \text{the net signed area trapped between the graph of } f(x) \\ &\quad \text{and the } x\text{-axis from } x = a \text{ to } x = b. \\ &= \text{the net distance traveled from time } x = a \text{ to time } x = b \\ &\quad \text{by an object whose velocity at time } x \text{ is } f(x). \end{aligned}$$

and

$$\begin{aligned} \int f(x) dx &= \text{a collection of functions.} \\ &= \text{the set of all antiderivatives of } f(x). \\ &= F(x) + C, \text{ where } F'(x) = f(x) \end{aligned}$$

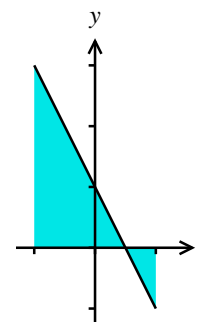
(The **value** of an integral, whether definite or indefinite, refers to the right side above.)

5.4.re.e1. Evaluate the indefinite integral  $\int(1 - 2x) dx$ .

$$\int(1 - 2x) dx = x - x^2 + C, \text{ because } \frac{d}{dx}(x - x^2) = 1 - 2x.$$

5.4.re.e2. Evaluate the definite integral  $\int_{-1}^1(1 - 2x) dx$ .

$\int_{-1}^1(1 - 2x) dx = (x - x^2)|_{-1}^1 = (1 - 1) - (-1 - (-1)^2) = 2$ . The definite integral is positive because, on the interval  $[-1, 1]$ , the line  $y = 1 - 2x$  traps more area above the  $x$ -axis than below.



To restate the results from section 4.9, page 58, here's a list of the indefinite integrals we know so far, and the combination rules for integration:

$f(x)$	$\int f(x) dx$
$x^n$	$\begin{cases} \frac{1}{n+1}x^{n+1} + C & \text{if } n \neq -1, \text{ and} \\ \ln x  + C & \text{if } n = -1. \end{cases}$
$e^x$	$e^x + C$
$\cos x$	$\sin x + C$
$\sin x$	$-\cos x + C$
$\sec^2 x$	$\tan x + C$
$\csc^2 x$	$-\cot x + C$
$\sec x \tan x$	$\sec x + C$
$\csc x \cot x$	$-\csc x + C$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x + C$
$\frac{1}{1+x^2}$	$\tan^{-1} x + C$
$\frac{1}{x\sqrt{x^2-1}}$	$\sec^{-1} x + C$

If  $f$  and  $g$  are functions and  $c$  is a constant, then

$$\int (f(x) + g(x)) dx = \int f(x) dx + \int g(x) dx$$

$$\int cf(x) dx = c \int f(x) dx$$

5.4.re.e3. Evaluate the definite or indefinite integrals:

a.  $\int_1^3 \frac{x^3 + 8}{2} dx$

b.  $\int_0^{\pi/4} (\sin t - 2 \cos t) dt$

c.  $\int \frac{r^3 - 7r^2 + 1}{2r} dr$

d.  $\int (x^2 - 1)^3 dx$

e.  $\int (\sec t \tan t - \csc t \cot t) dt$

f.  $\int_0^{1/2} \frac{du}{\sqrt{1-u^2}}$

g.  $\int (2-t)(3-2t) dt$

h.  $\int_{-1}^1 \sqrt[3]{s^2} ds$

i.  $\int_0^2 (e^x - e) dx$

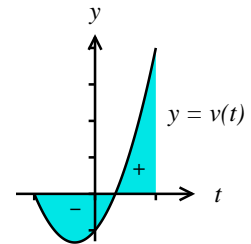
j.  $\int x(5\sqrt{x} - 7\sqrt[3]{x}) dx$

k.  $\int \frac{v^4 - 1}{v^2 - 1} dv$

l.  $\int_{-2}^2 |x| dx$

5.4.re.e4. The velocity of an object moving along a number line is  $v(t) = 3t^2 + 2t - 1$ .

- When is the object moving backward and when it is moving forward?
- Find the *net* distance the object traveled between times  $-1$  and  $1$ .
- Find the *total* distance it traveled between times  $-1$  and  $1$ .
- Suppose the object is at position  $0$  at  $t = 1$ . Find the object's position  $s(t)$  at time  $t$ .
- Draw a schematic diagram of the object's motion for  $-\infty < t < \infty$ .



5.4.re.e5. Suppose an object's acceleration at time  $t$  is  $-2$  m/sec<sup>2</sup> and that its velocity at time  $1$  sec is  $4$  m/sec.

- Find the net and total distances traveled by the object between times  $0$  and  $4$  sec.
- If the object is at position  $12$  m at time  $3$  sec, find its position at time  $4$  sec.

*The definite integral of a rate of change*

The Fundamental Theorem, part II (5.3.re.2), says if  $F(t)$  is any quantity that depends on time  $t$ , and if  $F'(t)$  is continuous, then

$$\int_a^b F'(t) dt = F(b) - F(a).$$

That is

$$\int_a^b (\text{rate of change of } F) dt = \text{change in } F \text{ from time } a \text{ to time } b.$$

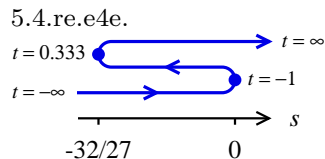
5.4.re.e6. If  $g'(t)$  is the rate at which money in my bank account is changing with respect to time  $t$ , measured in days, what does  $\int_0^{30} g'(t) dt$  represent?

5.4.re.e7. If  $r(t)$  is the altitude of an elevator in the Willis Tower in Chicago and time  $t$  is measured in hours, what do the integrals  $\int_0^{24} r'(t) dt$  and  $\int_0^{24} |r'(t)| dt$  represent?

5.4.re.e8. The number of catfish in a farm pond increases at the rate of  $\frac{300}{t^2}$  fish per month. Here,  $t$  is time measured in months, with  $t = 1$  representing January 1. By how many fish will the population have increased by June 1?

Answers

5.4.re.e3a. 18    5.4.re.e3b.  $-\frac{3}{\sqrt{2}} + 1$     5.4.re.e3c.  $\frac{1}{6}r^3 - \frac{7}{4}x^2 + \frac{1}{2} \ln |r| + C$     5.4.re.e3d.  $\frac{1}{7}x^7 - \frac{3}{5}x^5 + x^3 - x + C$   
 5.4.re.e3e.  $\sec t + \csc t + C$     5.4.re.e3f.  $\pi/6$     5.4.re.e3g.  $6t - \frac{7}{2}t^2 + \frac{2}{3}t^3 + C$     5.4.re.e3h.  $6/5$     5.4.re.e3i.  $e^2 - 2e - 1$   
 5.4.re.e3j.  $2x^{5/2} - 3x^{7/3} + C$     5.4.re.e3k.  $\frac{1}{3}v^3 + v + C$     5.4.re.e3l. 4    5.4.re.e4a. Factor and make a sign chart for  $v$ . Forward  $t < -1$  and  $t > 1/3$ . Backward:  $-1 < t < 1/3$ .  
 5.4.re.e4b. net distance traveled =  $\int_{-1}^1 v(t) dt = 0$ .    5.4.re.e4c. Total distance =  $\int_{-1}^1 |v(t)| dt = -\int_{-1}^{1/3} v(t) dt + \int_{1/3}^1 v(t) dt = \frac{64}{27}$ .  
 5.4.re.e4d.  $s(t) = t^3 + t^2 - t - 1$ .



5.4.re.e5a. 8 m net; 10 m total.    5.4.re.e5b. 11 m.    5.4.re.e6. The difference between my bank balance in 30 days and my bank balance today.    5.4.re.e7. The first is the net distance traveled by the elevator in 24 hours—its stopping position minus its starting position. The second is the total distance traveled by the elevator in the same time period.    5.4.re.e8. 250 fish.



## 5.5: Substitution

The **Substitution Rule** is a mechanism using the chain rule that helps us to simplify an integral. It says that an integral of the form  $\int f(u(x))u'(x) dx$  (for some function  $u$  of  $x$ ) can be rewritten

$$\int f(u) \frac{du}{dx} dx = \int f(u) du$$

and thenceforth be treated as if  $u$  were the variable of integration. For this reason, substitution is also called a **change of variable**.

5.5.re.e1. Evaluate the integral:

a. $\int (x^2 - 1) \cos(x^3 - 3x) dx$	b. $\int x^3 \sqrt{x^4 - 1} dx$	c. $\int \frac{e^x}{e^x + 1} dx$
d. $\int \frac{\sec(\ln x) \tan(\ln x)}{x} dx$	e. $\int x^2 (x^3 + 1)^9 dx$	f. $\int (3x + 4)^{10} dx$
g. $\int \frac{1}{\sqrt{x}(1 + \sqrt{x})^2} dx$	h. $\int \frac{\sin x}{\cos x} dx$	i. $\int e^{2x-1} dx$

## Substitutions in definite integrals

5.5.re.e2. Evaluate the integral:

a. $\int_0^1 \sin\left(\frac{\pi}{2}x\right) dx$	b. $\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$	c. $\int_{-1}^1 x^2 \sin(x^3) dx$
d. $\int_{-2}^2 x^2 \cos(x^3) dx$	e. $\int_0^\pi (\sin x) e^{\cos x} dx$	f. $\int_{-1}^1 (2x + 3)^5 dx$

## Answers

5.5.re.e1a.  $\frac{1}{3} \sin(x^3 - 3x) + C$    5.5.re.e1b.  $\frac{1}{6} (x^4 - 1)^{3/2} + C$    5.5.re.e1c.  $\ln(e^x + 1) + C$    5.5.re.e1d.  $\sec(\ln x) + C$   
 5.5.re.e1e.  $\frac{1}{30} (x^3 + 1)^{10} + C$    5.5.re.e1f.  $\frac{1}{33} (3x + 4)^{11} + C$    5.5.re.e1g.  $-2(1 + \sqrt{x})^{-1} + C$   
 5.5.re.e1h.  $-\ln|\cos x| + C$ , or  $\ln|\sec x| + C$ .   5.5.re.e1i.  $\frac{1}{2} e^{2x-1} + C$    5.5.re.e2a.  $\frac{2}{\pi}$    5.5.re.e2b.  $2e(e - 1)$   
 5.5.re.e2c. 0   5.5.re.e2d.  $\frac{2}{3} \sin 8$    5.5.re.e2e.  $e - e^{-1}$    5.5.re.e2f.  $\frac{1}{12} (5^6 - 1)$