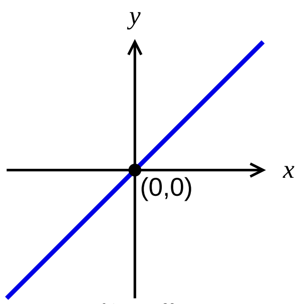
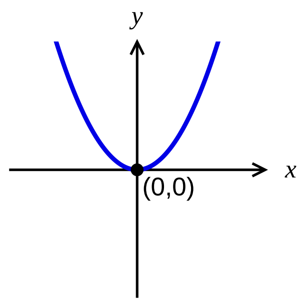
0.0: Graphs from Precalculus

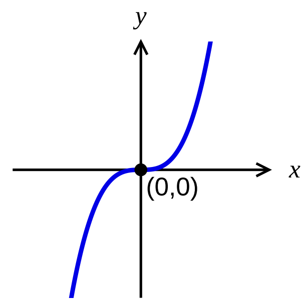
You should be able to sketch the graphs of these function from precalculus, including any intercepts and asymptotes.



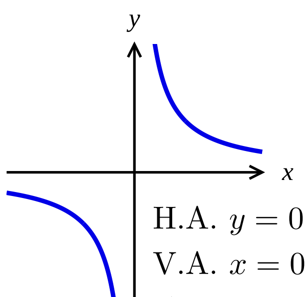
$y = x$



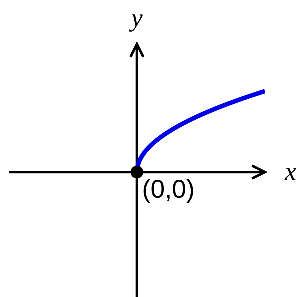
$y = x^2$



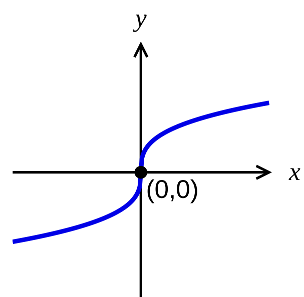
$y = x^3$



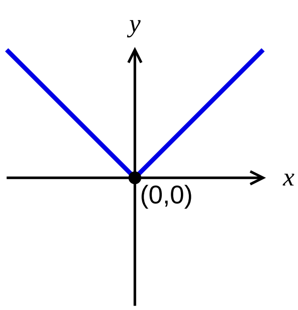
$y = \frac{1}{x}$



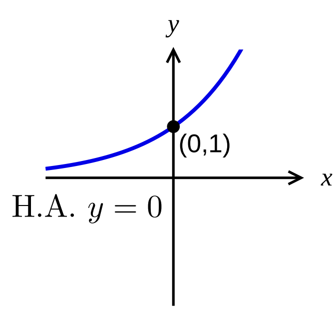
$y = x^{1/2}$



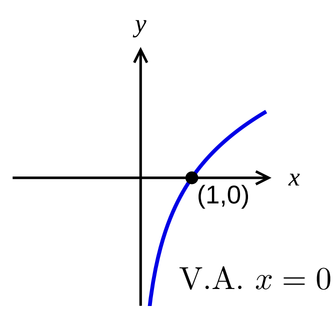
$y = x^{1/3}$



$y = |x|$

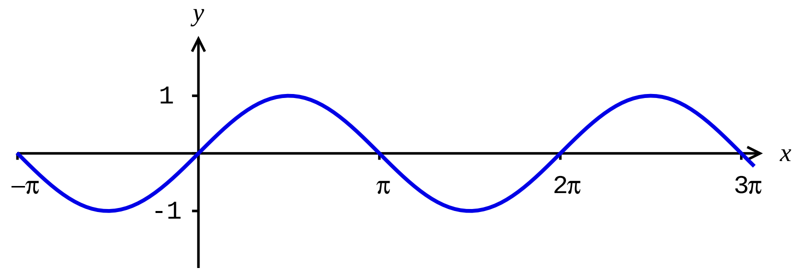


$y = e^x$

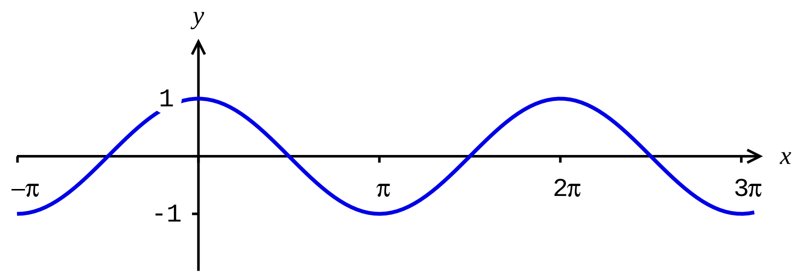


$y = \ln x$

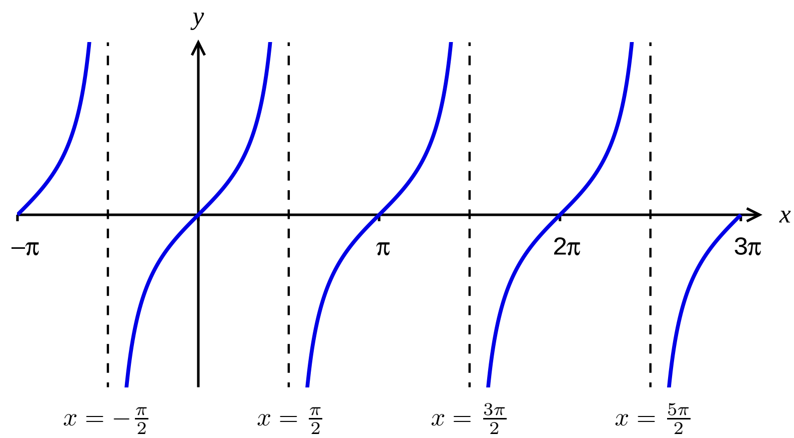
$y = \sin x$



$y = \cos x$

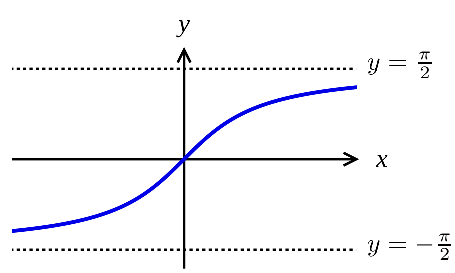


$y = \tan x$



Note that $\tan x = \frac{\sin x}{\cos x} = 0$ where $\sin x = 0$, and $\tan x \to \pm\infty$ where $\cos x = 0$.

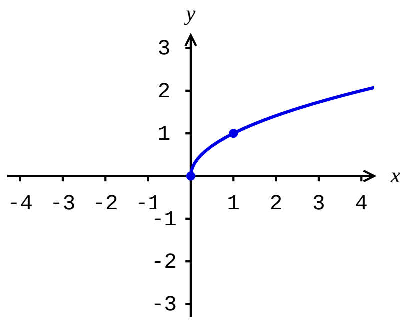
$y = \tan^{-1} x$



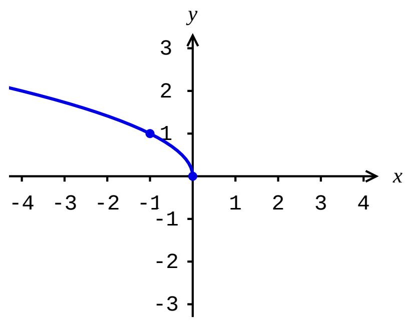
For more on the graphs of the trig functions, including tips for sketching the sine and cosine, see section Ap.D of these notes.

0.1: Transformations on the Graphs of Equations

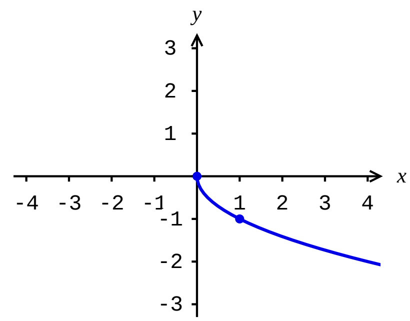
0.1.re1. To understand how changes to the equation change that equation's graph, let's look at some examples using $y = \sqrt{x}$. Check for yourself that the coordinates of the points marked with a dot (•) on each graph satisfy the given equation.



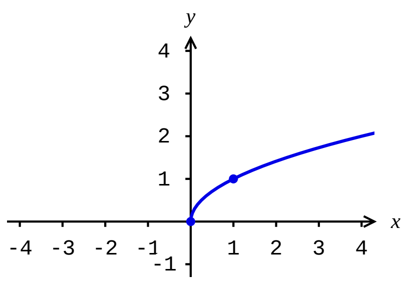
$y = \sqrt{x}$



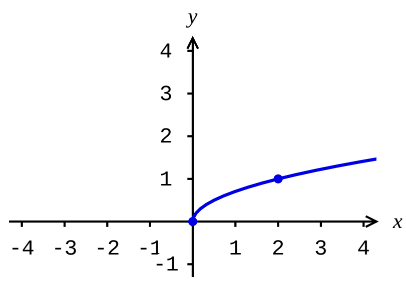
$y = \sqrt{-x}$



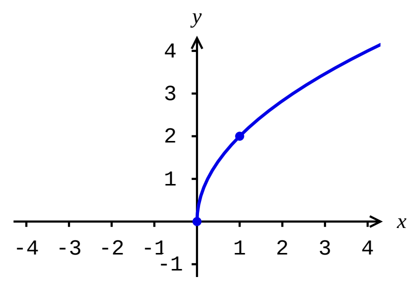
$-y = \sqrt{x}$,



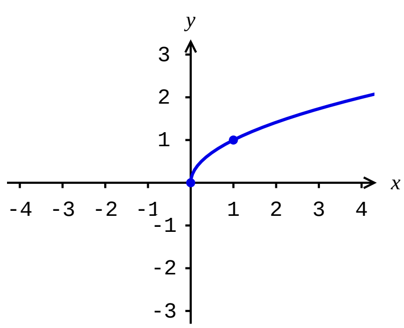
$y = \sqrt{x}$



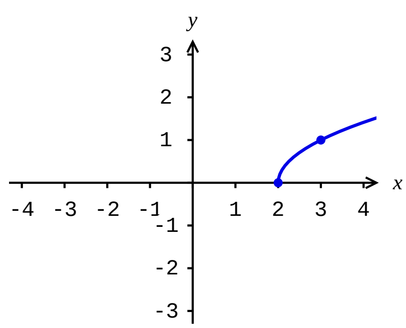
$y = \sqrt{\frac{x}{2}}$



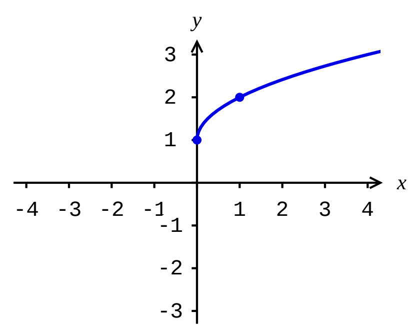
$\frac{y}{2} = \sqrt{x}$,



$y = \sqrt{x}$



$y = \sqrt{x-2}$



$y - 1 = \sqrt{x}$,

end example 0.1.re1

Suppose $E(x, y)$ is an equation in x and y. The following rules are illustrated in 0.1.re1.

Reflections:

1. The graph of $E(-x, y)$ is obtained by reflecting the graph of $E(x, y)$ across $x = 0$.
2. The graph of $E(x, -y)$ is obtained by reflecting the graph of $E(x, y)$ across $y = 0$.

Scaling:

3. The graph of $E(\frac{x}{a}, y)$ is obtained by stretching the graph of $E(x, y)$ horizontally by a factor of a.
4. The graph of $E(x, \frac{y}{b})$ is obtained by stretching the graph of $E(x, y)$ vertically by a factor of b.

Translations:

5. The graph of $E(x - h, y)$ is obtained by shifting the graph of $E(x, y)$ h units right.
6. The graph of $E(x, y - k)$ is obtained by shifting the graph of $E(x, y)$ k units up.

In 5. and 6., "right" and "up" are interpreted according the signs of h and k. For instance, -2 units to the right means 2 units left.

Similar rules apply to the graphs of equations $E(x, y, z, \ldots)$ in more than two variables seen in calculus III.

Transformations on the graphs of functions

If the equation $E(x, y)$ has the special form $y = f(x)$, then

replacing y with $-y$ is equivalent to replacing $f(x)$ with $-f(x)$,
replacing y with $\frac{y}{b}$ is equivalent to replacing $f(x)$ with $bf(x)$, and
replacing y with $y - k$ is equivalent to replacing $f(x)$ with $f(x) + k$.

For instance, in 0.1.re1, the three equations

$$-y = \sqrt{x} \qquad \frac{y}{2} = \sqrt{x} \qquad y - 1 = \sqrt{x}$$

are equivalent to

$$y = -\sqrt{x} \qquad y = 2\sqrt{x} \qquad y = 1 + \sqrt{x}$$

For equations of the form $y = f(x)$, rules 1–6 become

Reflections:

1′. The graph of $f(-x)$ is obtained by reflecting the graph of $f(x)$ across $x = 0$.
2′. The graph of $-f(x)$ is obtained by reflecting the graph of $f(x)$ across $y = 0$.

Scaling:

3′. The graph of $f\left(\frac{x}{a}\right)$ is obtained by stretching the graph of $f(x)$ horizontally by a factor of a.
4′. The graph of $bf(x)$ is obtained by stretching the graph of $f(x)$ vertically by a factor of b.

Translations:

5′. The graph of $f(x - h)$ is obtained by shifting the graph of $f(x)$ h units right.
6′. The graph of $f(x) + k$ is obtained by shifting the graph of $f(x)$ k units up.

To find how the graph of an equation (or function) is changed by several transformations, you must find exactly which transformations are applied and in which order.

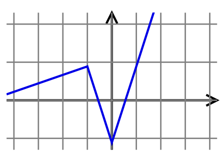
0.1.re2. The function $1 + f(-\frac{1}{2}x - 1)$ can be obtained from $f(x)$ by this sequence:

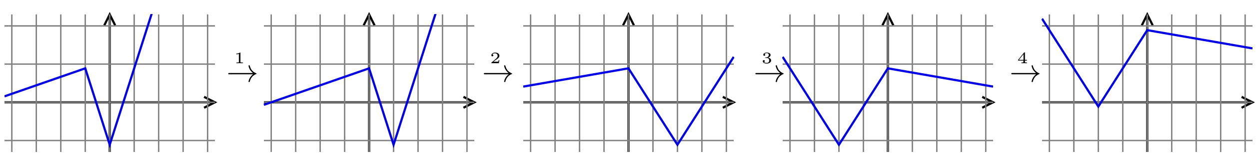
$$f(x) \overset{1}{\to} f(x-1) \overset{2}{\to} f(\tfrac{1}{2}x - 1) \overset{3}{\to} f(-\tfrac{1}{2}x - 1) \overset{4}{\to} 1 + f(-\tfrac{1}{2}x - 1)$$

1: Replace x by $x - 1$.
2: Replace x by $\frac{1}{2}x$.
3: Replace x by $-x$
4: Add 1 to the resulting function.

So the graph of $1 + f(-\frac{1}{2}x - 1)$ can be obtained from the graph of $f(x)$ by performing the corresponding steps in the same order:

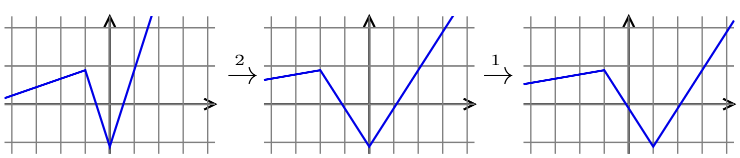
1: Shift the graph 1 unit to the right.
2: Stretch the graph horizontally by the factor 2.
3: Reflect the graph across the line $x = 0$ (the y-axis).
4: Shift the graph one unit up.

For instance, if this:  was the graph of $f(x)$, then 1–4 would look like



$y = f(x)$ $y = f(x-1)$ $y = f(\frac{1}{2}x - 1)$ $y = f(-\frac{1}{2}x - 1)$ $y = 1 + f(-\frac{1}{2}x - 1)$

In general, order matters. If we performed step 2 before step 1, the result would be



$y = f(x)$ $y = f(\frac{1}{2}x)$ $y = f\left(\frac{1}{2}(x-1)\right)$

There may be more than one correct sequence of transformations. For instance,

$$f(x) \overset{1'}{\to} f(-x) \overset{2'}{\to} f(-\tfrac{1}{2}x) \overset{3'}{\to} 1 + f(-\tfrac{1}{2}x) \overset{4'}{\to} 1 + f(-\tfrac{1}{2}(x+2)) = 1 + f(-\tfrac{1}{2}x - 1)$$

1′: Replace x by $-x$.
2′: Replace x by $\frac{1}{2}x$
3′: Add 1 to the resulting function.
4′: Replace x by $x + 2$.

end example 0.1.re2

0.1.re3. Review the basic graphs seen in section 0.0. Then sketch the graphs of the given functions by hand. Describe in words how each is obtained from the one of the basic graphs. Find the equations of all asymptotes and the coordinates of intercepts. Graph on Desmos.com to check your work.

a. $(x-1)^3+1$ b. $\dfrac{-2}{x+1}$ c. $\ln(\frac{1}{3}x)$

d. $\sqrt{1-x}$ e. $6-2|x+1|$ f. $-e^{-x-2}$

Answers

0.1.re3a. Graph of x^3 is shifted 1 unit right and 1 unit up. Intercept at $(0,0)$. 0.1.re3b. Graph of $\frac{1}{x}$ is shifted left 1, reflected about y-axis, the stretched vertically by a factor of 2. VA is $x=-1$; HA is $y=0$. Intercept is $(0,-2)$. 0.1.re3c. Graph of $\ln x$ is stretched horizontally by a factor of 3. VA remains $x=0$. x-intercept is $(3,0)$. 0.1.re3d. Graph of $\sqrt{x}$ is shifted left 1 unit and then reflected about the y-axis. Intercepts are $(1,0)$ and $(0,1)$. 0.1.re3e. Graph of $|x|$ is shifted left 1, reflected about x-axis, stretched by vertically by a factor of 2, and then shifted up 6 units. Intercepts are $(0,4)$, $(-4,0)$, $(2,0)$. 0.1.re3f. Graph of e^x is shifted right 2, then reflected about y-axis and about x-axis. Intercept is $(0,-e^{-2})$. HA remains $y=0$.

0.2: The Binomial Theorem and Pascal's Triangle.

The formulas

$$(x+y)^2 = x^2 + 2xy + y^2, \text{ and}$$
$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3.$$

are special instances of the **Binomial Theorem**, which says that the coefficients $\square$ in the expansion

$$(x+y)^n = \square x^n + \square x^{n-1}y + \square x^{n-2}y^2 + \cdots + \square x^2y^{n-2} + \square xy^{n-1} + \square y^n$$

are found in **Pascal's Triangle**:

```
        1
      1   1
    1   2   1
  1   3   3   1
1   4   6   4   1
```

Pascal's Triangle has infinitely many rows. We refer to the top row as its 0th row. Except for the 1's that begin and end each row, each entry in the triangle is the sum of the two entries directly above.

See http://kunklet.people.cofc.edu/MATH111/pascal.pdf for more details.

0.2.re1. The third row "1 3 3 1" and the fourth row "1 4 6 4 1" tell us that

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$$
$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

0.2.re2. Generate the fifth, sixth, and seventh rows of Pascal's Triangle.

When we expand $(x-y)^n$, the successive terms alternate in sign:

0.2.re3.

$$(x-y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$$
$$(x-y)^4 = x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4$$

0.2.re4. Expand the following.

a. $(x+3)^4$ b. $(u+v)^5$ c. $(u-v)^6$
d. $(x^3+y)^4$ e. $(x-x^{-1})^5$ f. $(\xi-2)^6$

Answers

0.2.re2. 5th row: 1 5 10 10 5 1. 6th row: 1 6 15 20 15 6 1. 7th row: 1 7 21 35 35 21 7 1.

0.2.re4a. $x^4 + 12x^3 + 54x^2 + 108x + 81$ 0.2.re4b. $u^5 + 5u^4v + 10u^3v^2 + 10u^2v^3 + 5uv^4 + v^5$

0.2.re4c. $u^6 - 6u^5v + 15u^4v^2 - 20u^3v^3 + 15u^2v^4 - 6uv^5 + v^6$ 0.2.re4d. $x^{12} + 4x^9y + 6x^6y^2 + 4x^3y^3 + y^4$

0.2.re4e. $x^5 - 5x^3 + 10x - 10x^{-1} + 5x^{-3} + x^{-5}$ 0.2.re4f. $\xi^6 - 12\xi^5 + 60\xi^4 - 160\xi^3 + 240\xi^2 - 192\xi + 64$

0.3: Nonlinear Inequalities

To solve an inequality involving a nonlinear function, try to get zero on one side, factor the other, and make a sign chart.

0.3.re1. Solve the inequality $\dfrac{x+2}{x-1} \le -4$.

Solution:
Add 4 to both sides

$$\frac{x+2}{x-1} + 4 \le 0$$

and add the fractions on the left side.

$$\frac{x+2}{x-1} + 4 = \frac{x+2}{x-1} + \frac{4}{1} \cdot \frac{x-1}{x-1} = \frac{x+2+4(x-1)}{x-1} = \frac{x+2+4x-4}{x-1} = \frac{5x-2}{x-1}$$

The top and bottom can't be simplified any further by factoring. To decide when this is less or equal zero, make a sign chart. Find the x-values where the numerator and denominator are each zero, and their signs on either side of these. From these determine the signs of the fraction. Note the fraction is undefined when its denominator is zero.

$5x-2$:	$------$	0	$++++++++++++$		$++++$
$x-1$:	$------$	$-$	$------------$	0	$++++$
$\dfrac{5x-2}{x-1}$:	$++++++$	0	$--------$	DNE	$++++$
x :		$\frac{2}{5}$		1	

So, $\dfrac{5x-2}{x-1} \le 0$ on the interval $[\frac{2}{5}, 1)$.

end example 0.3.re1

Never multiply both sides of an inequality by an expression of unknown sign. For instance, we do **not** multiply both sides of the inequality in 0.3.re1 by $(x-1)$; doing so may or may not flip the direction of the $\le$, depending on whether $(x-1)$ is negative or positive.

0.3.re2. Solve the inequality $\dfrac{1}{x-3} > \dfrac{2}{x+5}$.

Solution:
Subtract $\dfrac{2}{x+5}$ from both sides:

$$\frac{1}{x-3} - \frac{2}{x+5} > 0$$

Subtract the fractions:

$$\frac{x+5}{x+5}\frac{1}{x-3} - \frac{2}{x+5}\frac{x-3}{x-3} = \frac{x+5-2(x-3)}{(x+5)(x-3)} = \frac{11-x}{(x+5)(x-3)}$$

We want the denominator factored, so don't multiply it out. Make a sign chart to find where this is negative.

$11-x:$	$+\ +\ +\ +\ +\ +\ +\ +\ +\ +\ +\ 0\ -\ -\ -\ -$
$x-3:$	$-\ -\ -\ -\ -\ -\ -\ -\ 0\ +\ +\ +\ +\ +\ +\ +\ +\ +$
$x+5:$	$-\ -\ -\ -\ 0\ +\ +\ +\ +\ +\ +\ +\ +\ +\ +\ +\ +\ +\ +$
$\frac{11-x}{(x+5)(x-3)}:$	$+\ +\ +\ \text{DNE}\ -\ -\ -\ \text{DNE}\ +\ +\ +\ 0\ -\ -\ -\ -$
$x:$	$-5 \quad 3 \quad 11$

The solution set to $\frac{11-x}{(x+5)(x-3)} < 0$ is $(-5,3)\cup(11,\infty)$.

end example 0.3.re2

0.3.re3. Solve and write the solution set in interval notation.

a. $\dfrac{1}{x-2} < \dfrac{1}{x+3}$ b. $\dfrac{x+1}{x-2} \geq 2$ c. $\dfrac{1}{x+4} \leq \dfrac{1}{1-x}$

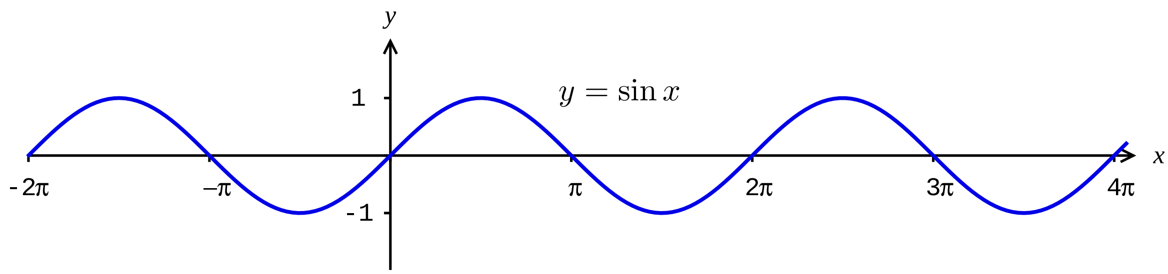
d. $\dfrac{1}{4-x} \geq \dfrac{1}{2+x}$ e. $\dfrac{2}{x+3} \leq 3$ f. $1 \geq \dfrac{9x+19}{(x+3)^2}$

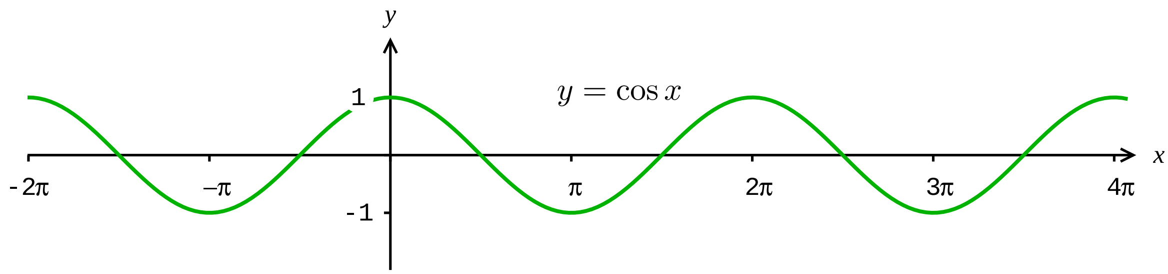
Answers

0.3.re3a. $(-3,2)$ 0.3.re3b. $(2,5]$ 0.3.re3c. $(-\infty,-4)\cup[-3/2,1)$ 0.3.re3d. $(-\infty,-2)\cup[1,4)$
0.3.re3e. $(-\infty,-3)\cup[-7/3,\infty)$ 0.3.re3f. $(-\infty,-3)\cup(-3,-2]\cup[5,\infty)$

Ap.D: Trigonometry

For a more complete review of trignometry, see Appendix D of our text.
The two basic functions in trigonometry are the sine and cosine, graphed here:



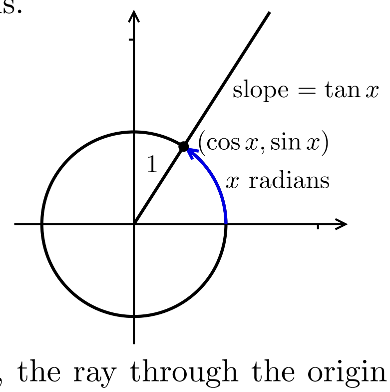


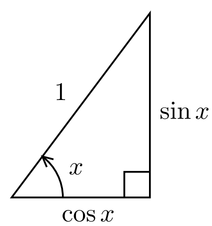
The other four trig functions are defined using sine and cosine:

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x}$$

$$\sec x = \frac{1}{\cos x} \qquad \csc x = \frac{1}{\sin x}$$

$\sin x$ and $\cos x$ are defined for all real numbers x, but $\tan x$ and $\sec x$ are undefined whenever $\cos x = 0$, and $\cot x$ and $\csc x$ are undefined whenever $\sin x = 0$.
By definition, $\cos x$ and $\sin x$ are the coordinates of the point on the **unit circle** (i.e., the circle of radius one centered at the origin) x radians counterclockwise from the positive horizontal axis.





Consequently, the ray through the origin x radians from the positive horizontal axis has slope $\tan x$, and, when x is an acute angle, $\cos x$ and $\sin x$ are the legs of this right triangle with hypotenuse 1 and interior angle x

These basic trigonometric identities follow from the definitions of sine and cosine.

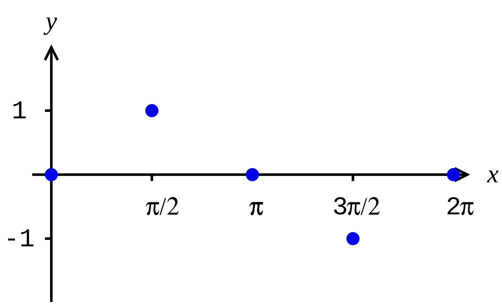
PYTHAGOREAN IDENTITES
$\sin^2 x + \cos^2 x = 1$
$\tan^2 x + 1 = \sec^2 x$
$1 + \cot^2 x = \csc^2 x$
EVEN & ODD IDENTITIES
$\cos(-x) = \cos x$
$\sin(-x) = -\sin x$

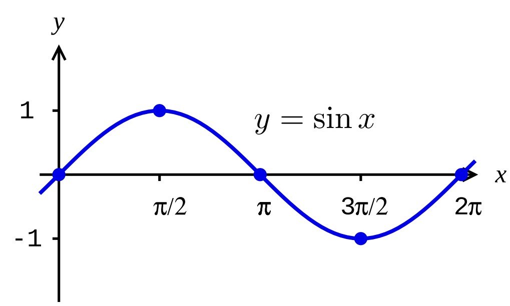
Sketching the sine and cosine

The sine and cosine are periodic functions having period 2π, meaning

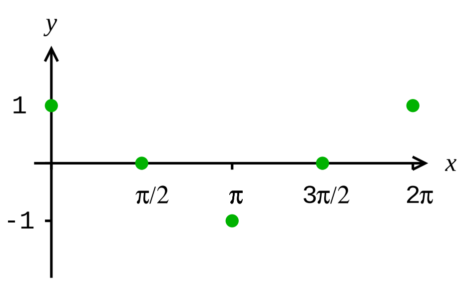
$$\sin(x + 2\pi) = \sin x \qquad \text{and} \qquad \cos(x + 2\pi) = \cos x.$$

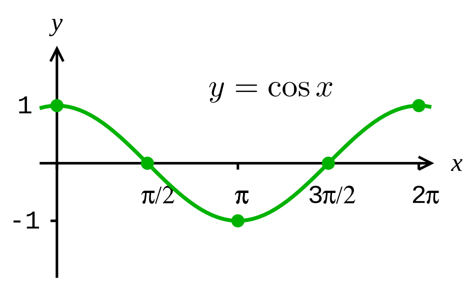
To sketch one cycle of the sine, plot the vertical coordinates on the unit circle at the angles 0, $\pi/2$, π, $3\pi/2$, and 2π. Be careful to make these five points equally spaced horizontally and vertically. Then connect them with a smooth curve.





To sketch the cosine, start by plotting the horizontal coordinates on the unit circle at the same five angles.



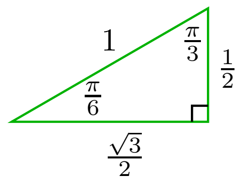


Ap.D.re1. Sketch the graphs of the sine and cosine on the given interval. Label hashmarks so as to clearly indicate all points where $y = -1, 0, 1$ along your curve. (You can check your answers using Desmos.com.)

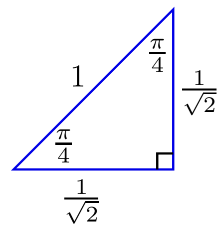
a. $[0, 2\pi]$ b. $[0, 3\pi]$ c. $[-\pi, \pi]$

Known values of sine and cosine

We already know the values of sine and cosine at the four points where the unit circle intersects the x and y axes. By placing these two triangles:

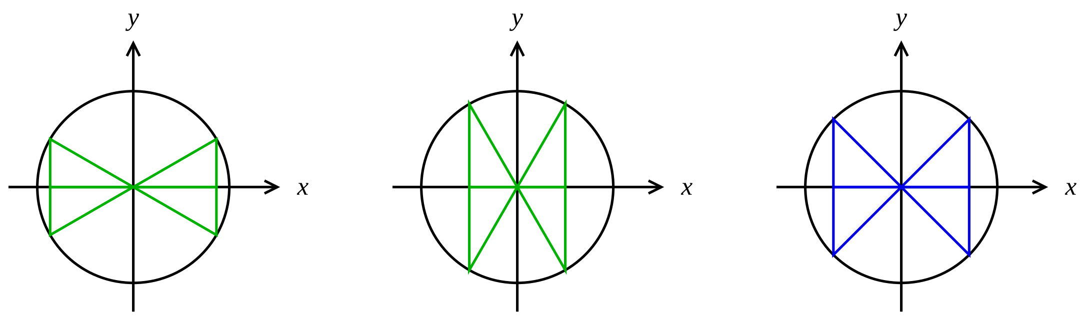


30-60-90



45-45-90

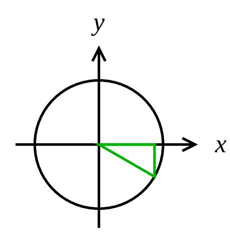
around the unit circle like this:



we find the sines and cosines at 12 more points on the unit circle (and the infinitely many angles that reach those points).

Ap.D.re2. Find the cosine and sine of $\frac{11\pi}{6}$.

Solution:
Rewriting $\frac{11\pi}{6}$ as $2\pi - \frac{\pi}{6}$ allows us to see that the ray $\theta = \frac{11\pi}{6}$ makes an angle $\frac{\pi}{6}$ with the x-axis in quadrant IV as in the figure to the right. Therefore $|\cos\frac{11\pi}{6}| = \frac{\sqrt{3}}{2}$ and $|\sin\frac{11\pi}{6}| = \frac{1}{2}$, since these are the horizontal and vertical legs of the 30-60-90 triangle. In quadrant IV, $\cos\frac{11\pi}{6} > 0$ and $\sin\frac{11\pi}{6} < 0$, and so $\cos\frac{11\pi}{6} = \frac{\sqrt{3}}{2}$ and $\sin\frac{11\pi}{6} = -\frac{1}{2}$.

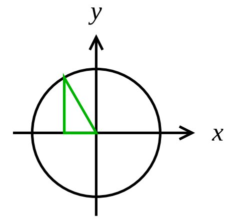


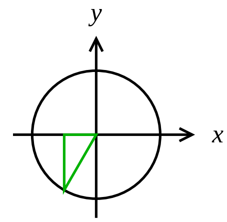
end example Ap.D.re2

Ap.D.re3. Evaluate the six trig functions at the given angle.

a. -14π	b. 111π	c. $\frac{15\pi}{4}$	d. $\frac{-11\pi}{6}$	e. $\frac{13\pi}{3}$
f. $\frac{-19\pi}{3}$	g. $\frac{23\pi}{6}$	h. $\frac{-19\pi}{4}$	i. $\frac{17\pi}{6}$	j. $\frac{-25\pi}{3}$
k. $\frac{43\pi}{4}$	l. $\frac{-36\pi}{3}$	m. $\frac{45\pi}{2}$	n. $\frac{-5\pi}{6}$	o. $\frac{-5\pi}{2}$
p. $\frac{7\pi}{6}$	q. $\frac{-2\pi}{3}$	r. $\frac{9\pi}{4}$	s. $\frac{5\pi}{6}$	t. $\frac{-20\pi}{3}$

Ap.D.re4. Find all angles whose cosine is $-\frac{1}{2}$.





Solution:
We recognize 1/2 as the short side of the 30-60-90 triangle, so for the cosine to be $-1/2$, the angle must be one of the two pictured at right. Find one angle that matches each drawing, for instance,

$$x = \pi - \pi/3 = 2\pi/3 \text{ above, and} \quad x = \pi + \pi/3 = 4\pi/3 \text{ below.}$$

Then add all multiples of 2π to describe *all* angles that fit the drawings:

$$x = 2\pi/3 + 2\pi n \quad \text{and} \quad x = 4\pi/3 + 2\pi n \quad \text{(where } n \text{ is any integer).}$$

end example Ap.D.re4

Ap.D.re5. Find all solutions x to the given equation.

a. $\sin x = -1/\sqrt{2}$ b. $\cos x = 0$ c. $\tan x = -1$
d. $\sec x = -2$ e. $\csc x = 2/\sqrt{3}$ f. $\cot x = \sqrt{3}$

Solving trigonometric equations

When the variable of the equation appears inside a trig function, first solve for the function, and then solve for the variable.

Ap.D.re6. Solve for x in the equation $4\sin^2 x - 8\sin x + 3 = 0$

Solution:
This is a quadratic equation in $\sin x$ which can be solved by factoring:

$$(2\sin x - 3)(2\sin x - 1) = 0$$

which implies either $\sin x = \frac{3}{2}$ or $\sin x = \frac{1}{2}$. The first of these has no real solutions, so the solution set of the original equation is the same as the solution set of $\sin x = \frac{1}{2}$, namely $x = \pi/6 + 2\pi n$ or $x = 5\pi/6 + 2\pi n$ (for any $n \in \mathbb{Z}$).

end example Ap.D.re6

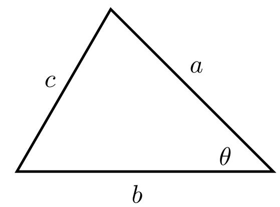
It sometimes helps to use the Pythagorean identities to rewrite the equation entirely in terms of one trig function.

Ap.D.re7. Find all solutions x to the given equation.

a. $1 - \sin x - 2\cos^2 x = 0$ b. $3\cos x - 2\sin^2 x = 0$ c. $\sin^2 x + 3\sin x + 2 - \cos^2 x = 0$
d. $1 - \cos^2 x = 0$ e. $\cos^2 x - 3 = 0$ f. $2\sin^2 x - 2\cos^2 x = 1$

The Law of Cosines

When we label the sides and any one angle of a triangle as shown in the figure, the Law of Cosines states that



$$c^2 = a^2 + b^2 - 2ab\cos\theta.$$

In case θ is a right angle, the Law of Cosines reduces to the Pythagorean identity.

The Area of a Triangle

Since the height of the triangle pictured above is $a\sin\theta$, its area is

$$A = \frac{1}{2}ab\sin\theta$$

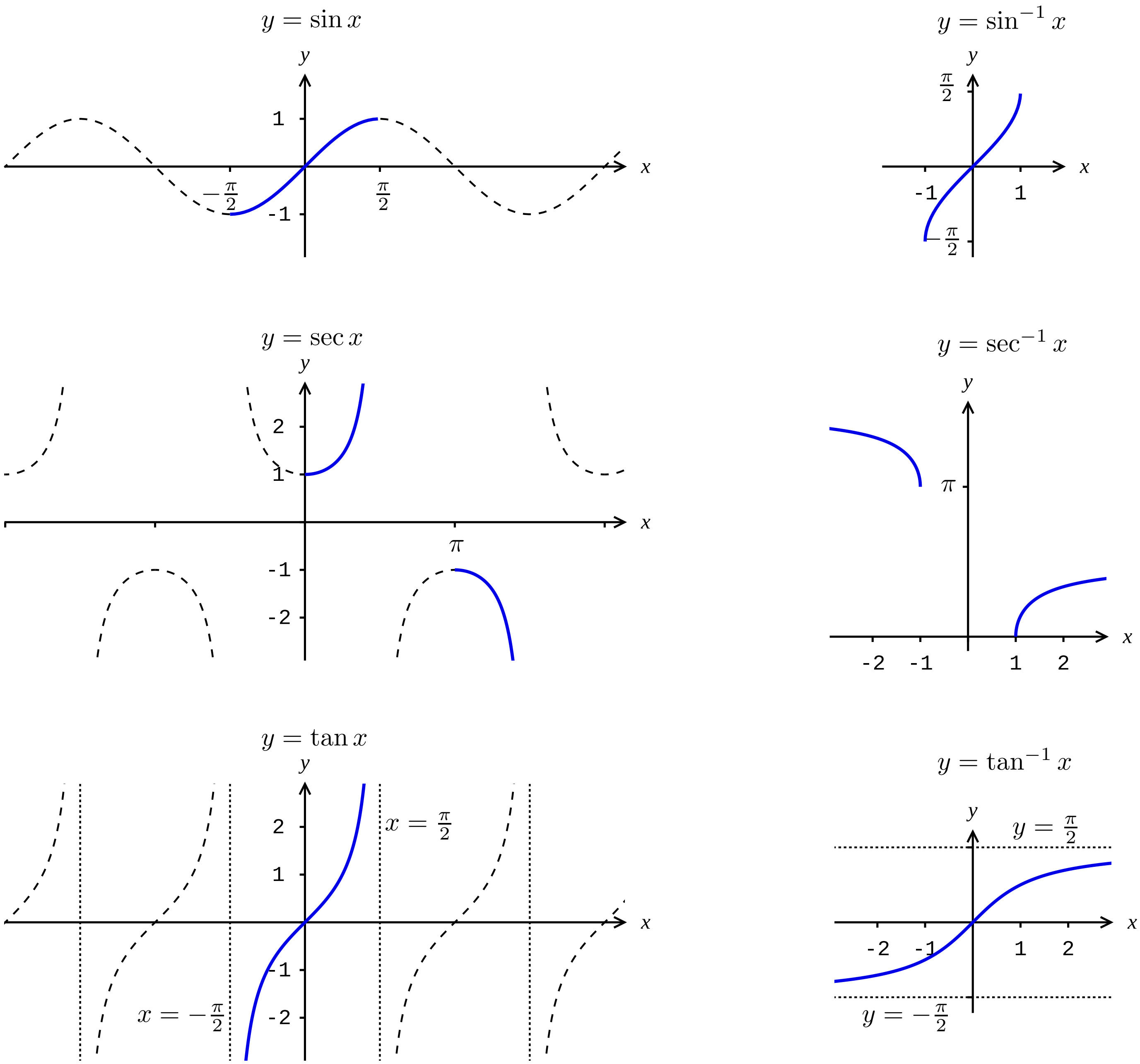
Ap.D.re8. Two ships leave the port of Charleston. One sails due east at a speed of 5 knots (nautical miles per hour) while the other sails in a direction 30° north of due east at 6 knots. Assume for this problem that the small part of the ocean sailed by the two ships is relatively flat.

What is the distance (in nautical miles) between the ships after two hours?

What is the area of the triangle formed by the two ships and the port of Charleston?

The inverse trig functions

To define inverses of the trig functions, we restrict each to a domain on which it takes each value in its range exactly once. The three we'll see most often in calculus are the inverses of the sine, tangent, and secant.



Traditionally, each of the inverse trig functions has two names. The inverse function of $\sin x$ is called $\sin^{-1} x$ or $\arcsin x$, the inverse function of $\sec x$ is called $\sec^{-1} x$ or $\text{arcsec}\, x$, etc..

As indicated in the graphs above,

	Domain	Range
$\sin^{-1}$	$[-1,1]$	$\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]$
$\sec^{-1}$	$(-\infty,-1]\cup[1,\infty)$	$\left[0,\dfrac{\pi}{2}\right)\cup\left[\pi,\dfrac{3\pi}{2}\right)$
$\tan^{-1}$	$(-\infty,\infty)$	$\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right)$

although there are few times in this course when you'll need to know these details.

Tip: if $x>0$, and "arctrig" is any of the inverse trig functions, then $\operatorname{arctrig} x \in [0,\frac{\pi}{2}]$.

It is helpful to remember the definitions of the inverse trig functions in words:

If $-1\le x\le 1$, then $\sin^{-1}x$ is the angle in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ whose sine is x. That is,

$$\sin(\sin^{-1}x)=x \quad \text{for any } x\in[-1,1].$$

If $|x|\ge 1$, then $\sec^{-1}x$ is the angle in $\left[0,\frac{\pi}{2}\right)\cup\left[\pi,\frac{3\pi}{2}\right)$ whose secant is x. That is,

$$\sec(\sec^{-1}x)=x \quad \text{for any } x\in(-\infty,-1]\cup[1,\infty).$$

If $-\infty<x<\infty$, then $\tan^{-1}x$ is the angle in $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ whose tangent is x. That is,

$$\tan(\tan^{-1}x)=x \quad \text{for any real number } x.$$

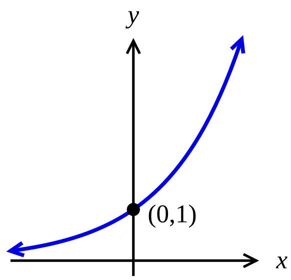
Ap.D.re9. Evaluate: $\sin^{-1}\left(-\frac{1}{2}\right)$

Answers

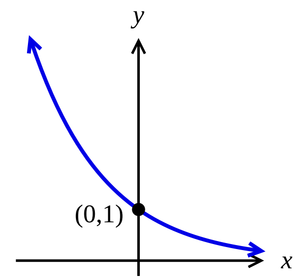
Ap.D.re3a. $(\cos,\sin,\cot,\tan,\sec,\csc)=(1,0,\text{dne},0,1,\text{dne})$ Ap.D.re3b. $(-1,0,\text{dne},0,-1,\text{dne})$
Ap.D.re3c. $(\frac{1}{\sqrt2},-\frac{1}{\sqrt2},-1,-1,\sqrt2,-\sqrt2)$ Ap.D.re3d. $(\frac{\sqrt3}{2},\frac12,\sqrt3,\frac{1}{\sqrt3},\frac{2}{\sqrt3},2)$
Ap.D.re3e. $(\frac12,\frac{\sqrt3}{2},\frac{1}{\sqrt3},\sqrt3,2,\frac{2}{\sqrt3})$ Ap.D.re3f. $(\frac12,-\frac{\sqrt3}{2},-\frac{1}{\sqrt3},-\sqrt3,2,-\frac{2}{\sqrt3})$
Ap.D.re3g. $(\frac{\sqrt3}{2},-\frac12,-\sqrt3,-\frac{1}{\sqrt3},\frac{2}{\sqrt3},-2)$ Ap.D.re3h. $(-\frac{1}{\sqrt2},-\frac{1}{\sqrt2},1,1,-\sqrt2,-\sqrt2)$
Ap.D.re3i. $(-\frac{\sqrt3}{2},\frac12,-\sqrt3,-\frac{1}{\sqrt3},-\frac{2}{\sqrt3},2)$ Ap.D.re3j. $(\frac12,-\frac{\sqrt3}{2},-\frac{1}{\sqrt3},-\sqrt3,2,-\frac{2}{\sqrt3})$
Ap.D.re3k. $(-\frac{1}{\sqrt2},\frac{1}{\sqrt2},-1,-1,-\sqrt2,\sqrt2)$ Ap.D.re3l. $(1,0,\text{dne},0,1,\text{dne})$
Ap.D.re3m. $(0,1,0,\text{dne},\text{dne},1)$ Ap.D.re3n. $(-\frac{\sqrt3}{2},-\frac12,\sqrt3,\frac{1}{\sqrt3},-\frac{2}{\sqrt3},-2)$
Ap.D.re3o. $(0,-1,0,\text{dne},\text{dne},-1)$ Ap.D.re3p. $(-\frac{\sqrt3}{2},-\frac12,\sqrt3,\frac{1}{\sqrt3},-\frac{2}{\sqrt3},-2)$
Ap.D.re3q. $(-\frac12,-\frac{\sqrt3}{2},\frac{1}{\sqrt3},\sqrt3,-2,-\frac{2}{\sqrt3})$ Ap.D.re3r. $(\frac{1}{\sqrt2},\frac{1}{\sqrt2},1,1,\sqrt2,\sqrt2)$
Ap.D.re3s. $(-\frac{\sqrt3}{2},\frac12,-\sqrt3,-\frac{1}{\sqrt3},-\frac{2}{\sqrt3},2)$ Ap.D.re3t. $(-\frac12,\frac{\sqrt3}{2},-\frac{1}{\sqrt3},-\sqrt3,-2,\frac{2}{\sqrt3})$
Ap.D.re5a. $x=-\pi/4+2\pi n$ or $x=-3\pi/4+2\pi n$ Ap.D.re5b. $x=\pi/2+2\pi n$ or $x=-\pi/2+2\pi n$
Ap.D.re5c. $x=3\pi/4+2\pi n$ or $x=11\pi/4+2\pi n$ Ap.D.re5d. (same as in example Ap.D.re4)
Ap.D.re5e. $x=\pi/3+2\pi n$ or $x=2\pi/3+2\pi n$ Ap.D.re5f. $x=\pi/6+2\pi n$ or $x=7\pi/6+2\pi n$
Ap.D.re7a. $\sin x=-1/2$ or 1, $x=\pi/2+2\pi n, 7\pi/6+2\pi n, 11\pi/6+2\pi n$ Ap.D.re7b. $\cos x=-2$ (no sol'ns) or $1/2$, $x=\pi/3+2\pi n, 5\pi/3+2\pi n$ Ap.D.re7c. $\sin x=-1/2$ or -1, $x=3\pi/2+2\pi n$, $7\pi/6+2\pi n, 11\pi/6+2\pi n$ Ap.D.re7d. $\cos x=\pm1$, $x=n\pi$ Ap.D.re7e. $\cos x=\pm\sqrt3$ no real sol'ns Ap.D.re7f. $\sin x=\pm\sqrt3/2$, $x=\pi/3+2\pi n, 2\pi/3+2\pi n, 4\pi/3+2\pi n, 5\pi/3+2\pi n$ Ap.D.re8. The two lines sailed by the ships are 10 and 12 nautical miles, and the angle between them is 30°. The distance between the ships is the third leg of the triangle, or $\sqrt{100+144-2\cdot10\cdot12\cos\pi/6}=\sqrt{244-120\sqrt3}$ nautical miles.
The area of the triangle is $\frac12 10\cdot12\sin\pi/6=30$ square nautical miles. Ap.D.re9. $-\pi/6$

1.4: Exponential Functions

An **exponential function** is one of the form $f(x) = a^x$ for some positive number a.
All exponential functions have the same graph, in the sense that the graph of any one can be obtained from that of any other by rescaling in the x-direction and, possibly, reflecting about the y-axis:



$y = a^x$
$a > 1$



$y = a^x$
$0 < a < 1$

The symbol e represents a mathematical constant, roughly equal to 2.718. For reasons we'll go into in section 3.1, e^x is the most commonly occurring exponential function in calculus.

1.4.re1. Make a rough sketch by hand of the graphs of of the following. Label all intercepts and asymptotes. How do the four graphs compare?

a. $y = 2^x$ b. $y = e^x$ c. $y = (\frac{1}{2})^x$ d. e^{-x} e. $y = e^{2x}$

In the sciences, exponential functions are used for modeling population growth, asset appreciation/depreciation, radioactive decay, and other quantities that grow or decay **exponentially** over time, that is, at a constant percent per unit of time. Radioactive isotopes exhibit exponential decay. For example, a sample of Fe-59 (iron 59) will lose about 1.55% of its mass per day.
A quantity y that grows/decays exponentially can be expressed as

$$y = C \cdot a^t$$

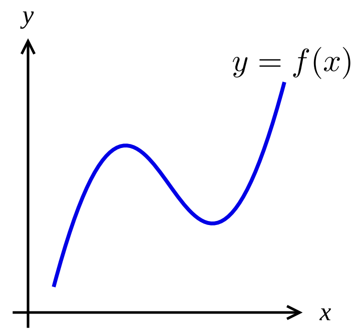
for some constants C and a.

Answers

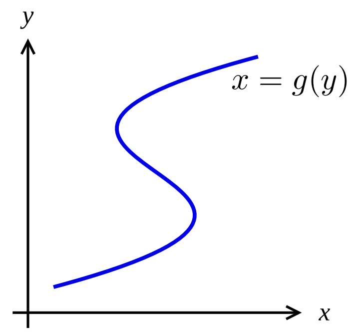
1.4.re1. All five curves have the same y-intercept $(0, 1)$ and no x-intercept. All five lie above the x-axis. $y = 0$ is a horizontal asymptote of 2^x, e^x, and e^{2x} as $x \to -\infty$ and of e^{-x} and $(\frac{1}{2})^x$ as $x \to +\infty$. $y = e^x$ rises more steeply than $y = 2^x$; $y = e^{2x}$ rises more steeply than $y = e^x$. $y = e^{-x}$ is the reflection of $y = e^x$ across the y-axis; $y = (\frac{1}{2})^x = 2^{-x}$ is the reflection of $y = 2^x$ across the y-axis.

1.5: Inverse Functions, including Logarithms and Inverse Trig Functions

A curve that passes the **vertical line test** has an equation the form $y = f(x)$ for some function f.

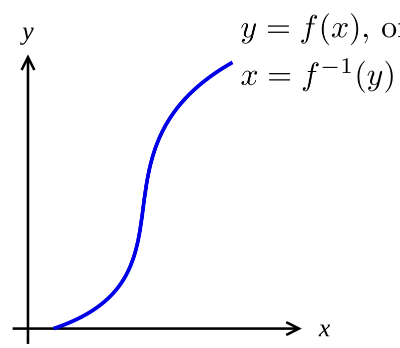


A curve that passes the **horizontal line test** has an equation the form $x = g(y)$ for some function g.



And a curve that passes both has both types of equations. In this case, we say that f is **invertible**, and we name the other function f^{-1} (read "f-inverse") due to the relations

$$\begin{aligned} x &= f^{-1}\big(f(x)\big) \quad \text{if} \quad x \in \text{domain}\, f = \text{range}\, f^{-1} \\ y &= f\big(f^{-1}(y)\big) \quad \text{if} \quad y \in \text{domain}\, f^{-1} = \text{range}\, f \end{aligned}$$



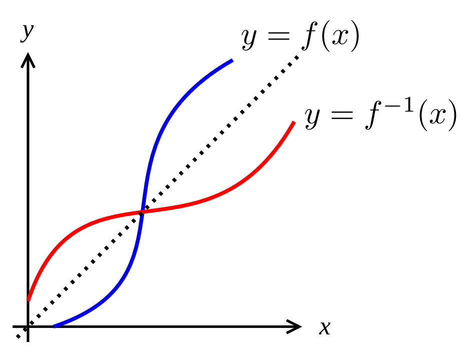
Since the equation

$$y = f^{-1}(x)$$

is obtained from

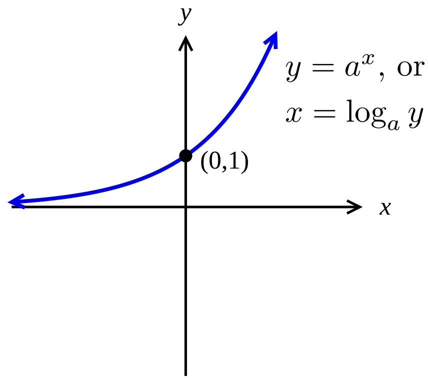
$$x = f^{-1}(y)$$

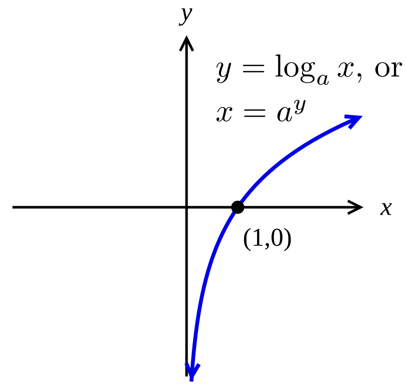
by interchanging x and y, the graph of f^{-1} is obtained by reflecting the graph of f across the line $y = x$.



Logarithms

The inverse of the exponential function a^x is called $\log_a x$ (read "log base a of x").





By convention, "log" stands for $\log_{10}$, the **common logarithm**, and "ln" stands for $\log_e$, the **natural logarithm**. In calculus, we work almost exclusively with the natural log.

1.5.re1. Sketch graphs of the following functions. Find all intercepts and asymptotes. (You can check your answers using Desmos.com.)

a. $\ln x$ b. $\ln(x+2)$ c. $-\ln x$ d. $-2\ln x$

Some important properties of the natural logarithm:

1. $\text{domain} \ln x = (0, \infty) = \text{range}\, e^x$
2. $\text{range} \ln x = (-\infty, \infty) = \text{domain}\, e^x$
3. $x = e^{\ln x}$ for all $x > 0$.
4. $x = \ln(e^x)$ for all real numbers x.
5. $\ln(AB) = \ln A + \ln B$ for all A and $B > 0$.
6. $\ln\left(\dfrac{A}{B}\right) = \ln A - \ln B$ for all A and $B > 0$.
7. $\ln(A^t) = t \ln A$ for all $A > 0$ and for any real t.

The **change of base** formula allows us to rewrite logs of any base in terms of ln:

8. $\log_b x = \dfrac{\ln x}{\ln b}$ for all b and $x > 0$.

Sometimes it is useful to use properties 5-7 to rewrite logarithmic expressions.

1.5.re2. Write

$$\sigma(x) = 3\ln x + 2\ln(x+1) - 4\ln(2x+3)$$

as a single logarithm.

Solution:

First use property 7 to write $\sigma(x) = \ln x^3 + \ln(x+1)^2 - \ln(2x+3)^4$. Now use properties 5 and 6:

$$\begin{aligned}\sigma(x) &= \ln\big(x^3(x+1)^2\big) - \ln(2x+3)^4 \\ &= \ln\left(\frac{x^3(x+1)^2}{(2x+3)^4}\right).\end{aligned}$$

end example 1.5.re2

1.5.re3. Do the opposite of what you did in example 1.5.re2; break

$$\nu(x) = \log\left(\frac{\sqrt{9x^2+1}}{9x^2+6x+1}\right)$$

into multiple logs with simple arguments.

Solution:

Be careful. $\sqrt{9x^2+1}$ is **not** $3x+1$.
The denominator $9x^2+6x+1 = (3x+1)^2$. Therefore

$$\begin{aligned}\nu(x) &= \log(9x^2+1)^{1/2} - \log(3x+1)^2 \\ &= \frac{1}{2}\log(9x^2+1) - 2\log(3x+1).\end{aligned}$$

The quadratic $9x^2+1$ is irreducible, meaning that it cannot be factored into linear factors using real coefficients, so $\nu(x)$ can't be broken down any further.

end example 1.5.re3

1.5.re4. Combine into a single log.

a. $\ln(2x-1) + \ln(x+1)$

b. $\ln(x+3) - \ln(x^2-4x-21)$

c. $2\log x - \frac{1}{2}\log(x+3) + 3\log(x-1)$

d. $\log(x^2-2x) - \log(3x-6)$

1.5.re5. Rewrite in terms of multiple logs with simple arguments.

a. $\ln(12x^3 - 40x^2 + 12x)$

b. $\ln\big((3x^2-13x+4)^4\big)$

c. $\ln\left(\dfrac{x^2-4}{x^3-8}\right)$

Solving equations with exponentials and logs

We can sometimes use the properties of logs, especially 3 and 4, to solve equations involving exponentials and logs.

1.5.re6. Solve for x.

a. $\ln(3x) - \ln(x+1) = 2$ b. $5^{x-1} = e^{2x}$ c. $e^{2x-1} = -3$

Solution:
a. Combine logs: $\ln\left(\frac{3x}{x+1}\right) = 2$. Since $e^{\ln x} = x$ for all $x > 0$, raise e to both sides to obtain

$$\frac{3x}{x+1} = e^2 \quad\Longrightarrow\quad 3x = e^2x + e^2 \quad\Longrightarrow\quad (3-e^2)x = e^2 \quad\Longrightarrow\quad x = \frac{e^2}{3-e^2}$$

b. Take ln of both sides and use property 7 to obtain $(x-1)\ln 5 = 2x$, which implies

$$x\ln 5 - \ln 5 = 2x \quad\Longrightarrow\quad x(-2+\ln 5) = \ln 5 \quad\Longrightarrow\quad x = \frac{\ln 5}{-2+\ln 5}$$

c. No solutions, since e^x is never negative.

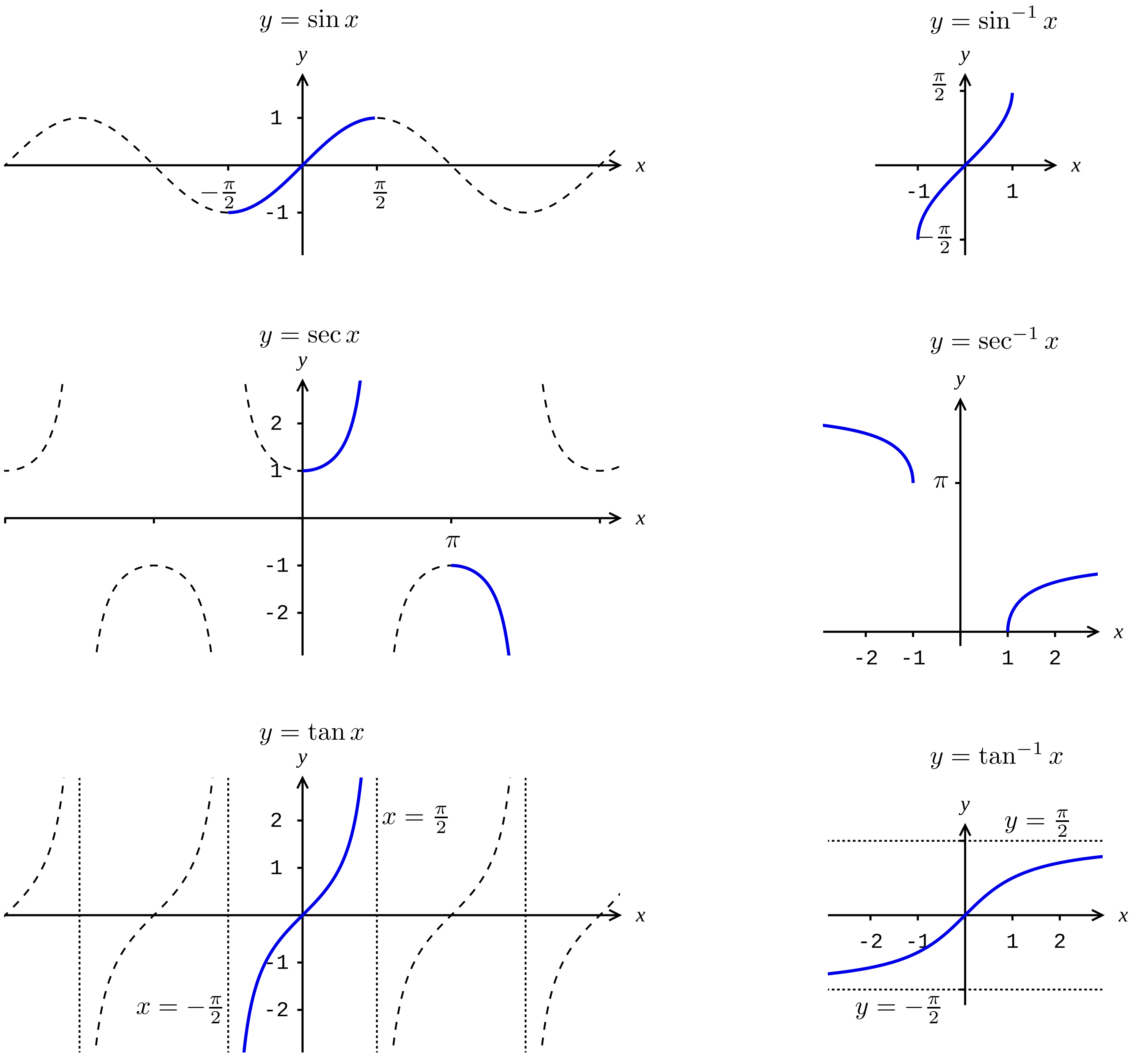
end example 1.5.re6

1.5.re7. Solve for x.

a. $e^{x-4} = 2$ b. $\ln(3x-4) = -1$ c. $3^{2x+1} = 2$
d. $4e^x - 3e^{-x} = 0$ e. $\ln(x+1) + \ln(x-4) = \ln 6$ f. $4e^x + 3e^{-x} = 0$

The inverse trig functions

To define inverses of the trig functions, we restrict each to a domain on which it takes each value in its range exactly once. The three we'll see most often in calculus are the inverses of the sine, tangent, and secant.



Traditionally, each of the inverse trig functions has two names. The inverse function of $\sin x$ is called $\sin^{-1} x$ or $\arcsin x$, the inverse function of $\sec x$ is called $\sec^{-1} x$ or $\operatorname{arcsec} x$, etc..

As indicated in the graphs above,

	Domain	Range
$\sin^{-1}$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\sec^{-1}$	$(-\infty, -1] \cup [1, \infty)$	$\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$
$\tan^{-1}$	$(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

although there are few times in this course when you'll need to know these details.
Tip: if $x > 0$, and "arctrig" is any of the inverse trig functions, then $\operatorname{arctrig} x \in [0, \frac{\pi}{2}]$.

It is helpful to remember the definitions of the inverse trig functions in words:

If $-1 \le x \le 1$, then $\sin^{-1} x$ is the angle in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is x. That is,

$$\sin(\sin^{-1} x) = x \quad \text{for any } x \in [-1, 1].$$

If $|x| \ge 1$, then $\sec^{-1} x$ is the angle in $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$ whose secant is x. That is,

$$\sec(\sec^{-1} x) = x \quad \text{for any } x \in (-\infty, -1] \cup [1, \infty).$$

If $-\infty < x < \infty$, then $\tan^{-1} x$ is the angle in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ whose tangent is x. That is,

$$\tan(\tan^{-1} x) = x \quad \text{for any real number } x.$$

1.5.re8. Use its definition to find $\sin^{-1}\left(-\frac{1}{2}\right)$.

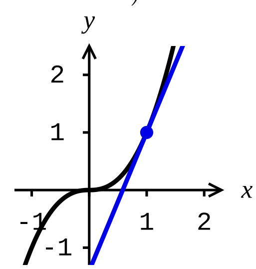
Answers

1.5.re4a. $\ln(2x-1)(x+1)$, or $\ln(2x^2+x-1)$ 1.5.re4b. $\ln\left(\frac{x+3}{x^2-4x-21}\right) = \ln\left(\frac{1}{x-7}\right)$
1.5.re4c. $\log\left(\frac{x^2(x-1)^3}{\sqrt{x+3}}\right)$ 1.5.re4d. $\log\frac{x}{3}$ 1.5.re5a. $\ln(3x-1)+\ln(x-3)+\ln x+2\ln 2$.
1.5.re5b. $4\ln(3x-1)+4\ln(x-4)$. 1.5.re5c. Hint: x^2-4 and x^3-8 have a common factor. Answer $= \ln(x+2)-\ln(x^2+2x+4)$. 1.5.re7a. $x = 4+\ln 2$. 1.5.re7b. $x = \frac{1}{3}(4+e^{-1})$.
1.5.re7c. $x = \frac{1}{2}(\frac{\ln 2}{\ln 3}-1)$. 1.5.re7d. $x = \frac{1}{2}\ln\frac{3}{4}$ 1.5.re7e. $x = 5$. (Omit $x = -2$ since both $x+1$ and $x-4$ must be positive.) 1.5.re7f. $\Longrightarrow 4e^x = -3e^{-x} \Longrightarrow e^{2x} = -\frac{3}{4} < 0$. no solutions. 1.5.re8. $-\pi/6$

2.1: The Tangent Line and Velocity Problems

This section is a motivation for what follows in Chapter 2. Its calculator-based problems are good exercises but would have to be carefully recrafted before they could appear on a no-calculator exam.

Two problems gave rise to differential calculus (that is, the study of derivatives).



The **Tangent Line Problem**: what is the slope of the line tangent to the graph of a function $y = f(x)$ at a given point?

In calculus when we refer to the slope of a curve at a point, we mean the slope of the line tangent to the curve at that point.

The **Velocity Problem**: what is the (instantaneous) velocity of an object at a given time if its position at time x is $f(x)$?

In calculus, unless we specify otherwise, "velocity" always means "instantaneous velocity," not "average velocity."

The solution to both problems uses the **difference quotient**

$$\frac{f(a+h)-f(a)}{h},$$

which can be interpreted both as

1. the slope of the secant line passing through $y = f(x)$ at $x = a$ and $x = a + h$, and
2. the average velocity between times $x = a$ and $x = a+h$ of an object whose position at time x is $f(x)$.

When h is very close to zero, we expect the difference quotient to approximate both

1*. the slope of the line tangent to $y = f(x)$ at $x = a$, and

2*. the velocity at time $x = a$ of an object whose position at time x is $f(x)$.

Consequently, the velocity of an object at time a is the same as the slope of the line tangent to the graph of its position function at that time.

2.1.re1. Approximate the slope of the line tangent to $y = \cos x$ at $x = 1$ with slopes of secant lines. What appears to be the true slope of the tangent line?

Solution:
Using a spreadsheet, I calculated some difference quotients for $\cos x$ at $a = 1$:

h	$\dfrac{\cos(1+h) - \cos 1}{h}$
0.1	−0.8671
0.08	−0.8622
0.06	−0.8572
0.04	−0.8521
0.02	−0.8468
0.01	−0.8442
0.005	−0.8428

h	$\dfrac{\cos(1+h) - \cos 1}{h}$
−0.1	−0.8131
−0.08	−0.8190
−0.06	−0.8248
−0.04	−0.8304
−0.02	−0.8360
−0.01	−0.8388
−0.005	−0.8401

Based on these calculations, the slope of the line tangent to $y = \cos x$ at $x = 1$ appears to be roughly -0.841.

end example 2.1.re1

This interactive graph will allow you to see the secant lines and their slopes along $y = \cos x$ or any other function you input: https://www.desmos.com/calculator/qrvbhvz3wa

2.1.re2. Use the interactive graph above to approximate the slope of the line tangent to the graph of $f(x)$ at the given point. The actual slope appears in the answers.

a. $f(x) = x^4 - 2x^2$ at $a = 0.5$ b. $f(x) = \sqrt{x+1}$ at $a = 2$ c. $f(x) = 2\sin^2 x$ at $a = 1$

Estimating velocities from tabular data

2.1.re3. The table below gives the position s at several times of an object moving along an axis (e.g., the real number line). Estimate the object's velocity at time $t = 4$ seconds.

t (sec)	3.5	3.75	4.0	4.25	4.75
s (m)	4.1	4.5	4.9	5.2	5.4

Solution:
The object's average velocity from time 3.75 to time 4.0 is (4.9-4.5)/0.25 =1.6 m/sec, and its average velocity from time 4.0 to time 4.25 is (5.2-4.9)/0.25 = 1.2 m/sec. We approximate the velocity at time 4 by the average of these, or 1.4 m/sec (which is the same as the average velocity from time 3.75 to time 4.25.) A better approximation would require finer data.

end example 2.1.re3

2.1.re4. Use the positions s recorded below to estimate the object's velocity the given times t. What are the units in your answers?

t (sec)	0	0.25	0.5	0.75	1.0	1.25	1.5	1.75	2.0
s (in)	0	10	21	33	46	60	58	54	48

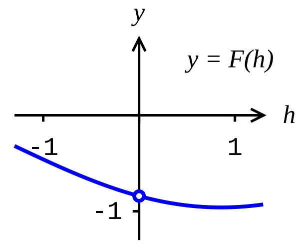
a. 0.25 b. 0.75 c. 1.25 d. 1.75

Answers

2.1.re2a. -1.5 2.1.re2b. 0.28868 2.1.re2c. 1.81859 2.1.re4a. 42 in/sec 2.1.re4b. 50 in/sec 2.1.re4c. 24 in/sec 2.1.re4d. -20 in/sec

2.2: Limits

In Example 2.1.re1, we saw that the slope of the secant line to $y = \cos x$ at the points 1 and $1 + h$ is



$$F(h) = \frac{\cos(1 + h) - \cos 1}{h}$$

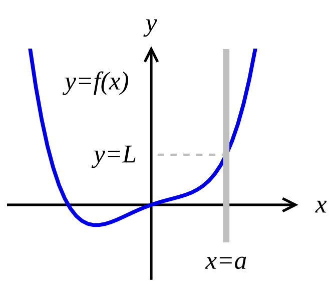
The graph of $F(h)$ appears to the right. We concluded that the slope of the tangent line is roughly -0.841, not because $F(0)$ equals that number (it doesn't), but because -0.841 seems to be the natural altitude of the graph of F at $h = 0$, based on its altitudes at h's *near* 0. The accepted way to indicate this (assuming our guess is correct) is to write

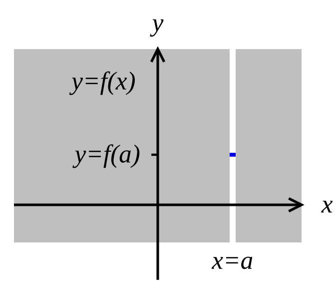
$$\lim_{h\to 0} \frac{\cos(1 + h) - \cos 1}{h} = -0.841,$$

which is read, "the limit as h goes to 0 of $\frac{\cos(1+h)-\cos 1}{h}$ equals -0.841."

Limit at a vs. function value at a

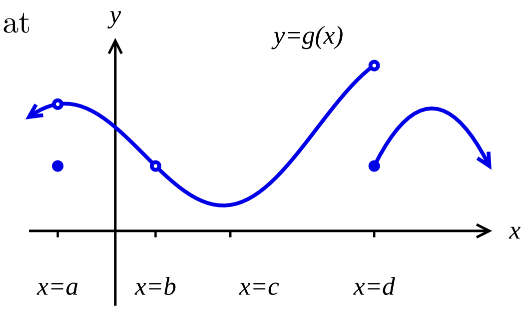
The $\lim_{x\to a} f(x)$ is based on the values of f at x's near a, ignoring what the function does at a itself. The function value $f(a)$ is the opposite. It's based solely on what f does at a, and ignores all other values:



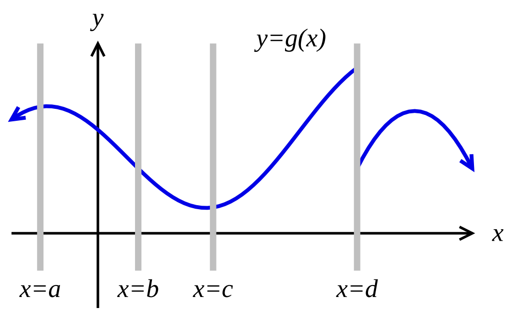


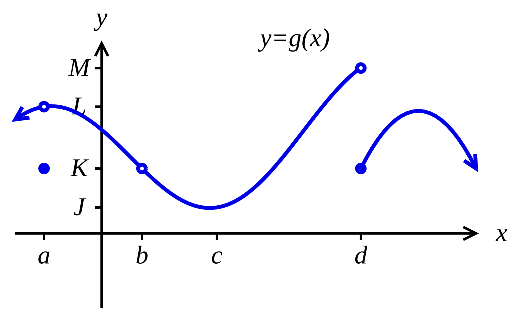
Limit values and function values are y-values. For most functions in calc I, the two are the same thing.

2.2.re1. Compare function values and limits values at a, b, c, and d for the function $g(x)$ graphed at right.



Solution:
When we look at y-values on the graph of $g(x)$ at x near but not equal a, b, c, and d as in below left, we see that the limit exists at a, b, and c, but not at d. In contrast, when we focus on altitudes of the curve at the four points, we see that $g(x)$ exists at $x = a$, c, and d, but not at b.





After drawing and labeling hash marks at the relevant y-values as in above right,

$$g(a) = K, \text{ but } \lim_{x\to a} g(x) = L.$$
$$g(b) \text{ does not exist, but } \lim_{x\to b} g(x) = K.$$
$$g(c) \text{ and } \lim_{x\to c} g(x) \text{ exist and are equal to } J.$$
$$g(d) = K, \text{ but } \lim_{x\to d} g(x) \text{ does not exist.}$$

end example 2.2.re1

In Example 2.2.re1, we would write

$$\lim_{x\to d^-} g(x) = M$$
$$\lim_{x\to d^+} g(x) = K$$

These are read

"the limit as x approaches d from the left of $g(x)$ equals M,"

and

"the limit as x approaches d from the right of $g(x)$ equals K."

As a rule, the (two-sided) limit

$$\lim_{x\to a} f(x)$$

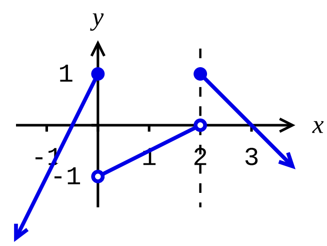
exists if and only if both one-sided limits

$$\lim_{x\to a^-} f(x) \text{ and } \lim_{x\to a^+} f(x)$$

exist and are equal. In that case, all three limits are the same.

2.2.re2. The graph of the piecewise-linear function

$$k(x) = \begin{cases} 2x+1 & \text{if } x \le 0, \\ \frac{1}{2}x - 1 & \text{if } 0 < x \le 2, \text{ and} \\ 3 - x & \text{if } 2 < x. \end{cases}$$



appears at right. As the graph indicates, the one-sided limits

$$\lim_{x\to 0^-} k(x) = 1 \neq \lim_{x\to 0^+} k(x) = -1,$$

so the two-sided limit $\lim_{x\to 0} k(x)$ fails to exist. Likewise, $\lim_{x\to 2} k(x)$ fails to exist because

$$\lim_{x\to 2^-} k(x) = 0 \neq \lim_{x\to 2^+} k(x) = 1.$$

end example 2.2.re2

2.2.re3. Let $f(x) = \begin{cases} -x-1 & \text{if } x < -1, \\ \frac{1}{2}(x+1) & \text{if } -1 \le x \le 2, \text{ and} \\ 4 - x & \text{if } x > 2. \end{cases}$

Evaluate the limit, or explain why it does not exist.

a. $\lim_{x\to -1} f(x)$

b. $\lim_{x\to 2^-} f(x)$

c. $\lim_{x\to 2^+} f(x)$

d. $\lim_{x\to 2} f(x)$

Infinite limits

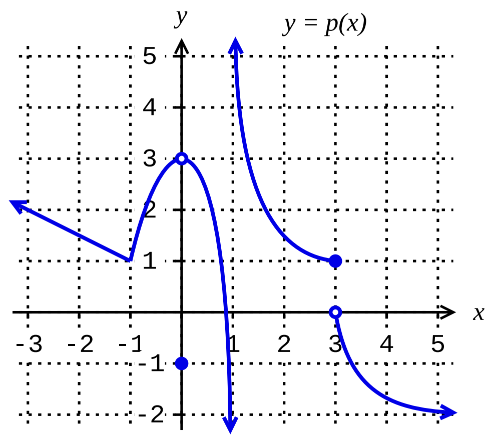
The symbol ∞ always means positive infinity. Negative infinity must be written $-\infty$. You should able to answer problems like the next one without a graphing device.

2.2.re4. Evaluate the limit, or explain why it does not exist. Hint: see section 0.0.

a. $\lim_{x\to 0^+} \ln x$

b. $\lim_{x\to \pi/2^-} \tan x$

2.2.re5. Use the graph of $p(x)$ to find the following. Your response to each part should be a number, infinity, -infinity, or DNE.



a. $\lim_{x\to 0} p(x)$

b. $\lim_{x\to 3} p(x)$

c. $\lim_{x\to 3^+} p(x)$

d. $p(0)$

e. $p(3)$

f. $\lim_{x\to -1} p(x)$

g. $\lim_{x\to 1^-} p(x)$

h. $\lim_{x\to 1} p(x)$

i. $p(-1)$

j. $\lim_{x\to 2} p(x)$

Limits of the form $\dfrac{\textit{nonzero}}{0}$

$\frac{\text{nonzero}}{0}$ indicates a quantity that's blowing up to either $\pm\infty$. To decide between these two, determine the sign of the quotient.

2.2.re6. Evaluate the limit, or explain why it does not exist:

a. $\displaystyle\lim_{x\to1}\frac{x-2}{x-1}$ b. $\displaystyle\lim_{x\to1}\frac{x-2}{(x-1)^2}$

Solution:

a. As $x\to1$, $\frac{x-2}{x-1}\to$ "$\frac{-1}{0}$", which indicates that $\frac{x-2}{x-1}$ is blowing up to one of $\pm\infty$. To decide, determine its sign to the left and right of 1.

$x-1$:	$-----$	0	$+++++$	$++++$	$++$
$x-2$:	$-----$	$---$	$---$	0	$++++$
$\dfrac{x-2}{x-1}$:	$++++$	DNE	$----$	0	$++++$
x:		1		2	

Therefore, $\lim_{x\to1^-}\frac{x-2}{x-1}$ must be ∞, and $\lim_{x\to1^+}\frac{x-2}{x-1}$ must be $-\infty$. Since the one-sided limits disagree, $\lim_{x\to1}\frac{x-2}{x-1}$ does not exist.

b. $\frac{x-2}{(x-1)^2}\to$ "$\frac{-1}{0}$" as $x\to1$, which tells us that $\frac{x-2}{(x-1)^2}$ is blowing up to one of $\pm\infty$. This time the sign chart is

$(x-1)^2$:	$+++++$	0	$+++++$	$++++$	$++$
$x-2$:	$-----$	$---$	$---$	0	$++++$
$\dfrac{x-2}{(x-1)^2}$:	$----$	DNE	$----$	0	$++++$
x:		1		2	

Since $\frac{x-2}{(x-1)^2}$ is negative on both sides of $x=1$, it must be blowing up to $-\infty$. That is, $\lim_{x\to1}\frac{x-2}{(x-1)^2}=-\infty$.

end example 2.2.re6

2.2.re7. Evaluate the limit, if it exists.

a. $\displaystyle\lim_{x\to0}\frac{1}{x^2}$ b. $\displaystyle\lim_{x\to3}\frac{x-4}{(x-3)^4}$ c. $\displaystyle\lim_{x\to0}\frac{1}{x^3}$ d. $\displaystyle\lim_{x\to-1^+}\frac{x-2}{x^2-x-2}$

Answers

2.2.re3a. 0 2.2.re3b. 3/2 2.2.re3c. 2 2.2.re3d. the one-sided limits disagree at $x=2$, so the two-sided limit does not exist. 2.2.re4a. $-\infty$ 2.2.re4b. ∞ 2.2.re5a. 3 2.2.re5b. DNE 2.2.re5c. 0 2.2.re5d. -1 2.2.re5e. 1 2.2.re5f. 1 2.2.re5g. $-\infty$ 2.2.re5h. DNE 2.2.re5i. 1 2.2.re5j. 1.5 2.2.re7a. ∞. 2.2.re7b. $-\infty$. 2.2.re7c. The limit from the right is ∞, but the limit from the left is $-\infty$, so the two-sided limit DNE. 2.2.re7d. ∞.

2.3: Limit Laws

The Limit Laws are rules that allow us to calculate limit more easily. There are two types:

I. Examples of some elementary continuous functions.

1. $\lim\limits_{x\to a} x = a$ 2. $\lim\limits_{x\to a} c = c$ 3. $\lim\limits_{x\to a} e^x = e^a$

II. Combination laws. If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist and are finite, then:

4. $\lim\limits_{x\to a}\big(f(x)+g(x)\big) = \lim\limits_{x\to a} f(x) + \lim\limits_{x\to a} g(x)$

5. $\lim\limits_{x\to a}\big(f(x)-g(x)\big) = \lim\limits_{x\to a} f(x) - \lim\limits_{x\to a} g(x)$

6. $\lim\limits_{x\to a}\big(f(x)\times g(x)\big) = \lim\limits_{x\to a} f(x) \times \lim\limits_{x\to a} g(x)$

7. $\lim\limits_{x\to a}\big(f(x)\div g(x)\big) = \lim\limits_{x\to a} f(x) \div \lim\limits_{x\to a} g(x)$ (except in the case of division by zero).

If c is a constant, then

8. $\lim\limits_{x\to a} cf(x) = c\lim\limits_{x\to a} f(x)$

If r is a rational number, then, except in case of division by zero or roots that do not exist,

9. $\lim\limits_{x\to a}\big(f(x)\big)^r = \big(\lim\limits_{x\to a} f(x)\big)^r$

As a consequence of rules 1. and 9.,

10. $\lim\limits_{x\to a} x^r = a^r$, (except in the case of division by zero or roots that do not exist)

11. If $p(x)$ is any polynomial and a is any number, then $\lim\limits_{x\to a} p(x) = p(a)$.

12. If $r(x)$ is any rational function (i.e., the ratio of two polynomials) and a is any number, then $\lim\limits_{x\to a} r(x) = r(a)$, provided $r(a)$ exists.

2.3.re1. The rational function $\dfrac{x-2}{(x-1)^2}$ is defined for all $x \neq 1$. Its value at $x = 3$ is $\dfrac{3-2}{(3-1)^2} = \dfrac{1}{4}$, so by 12,

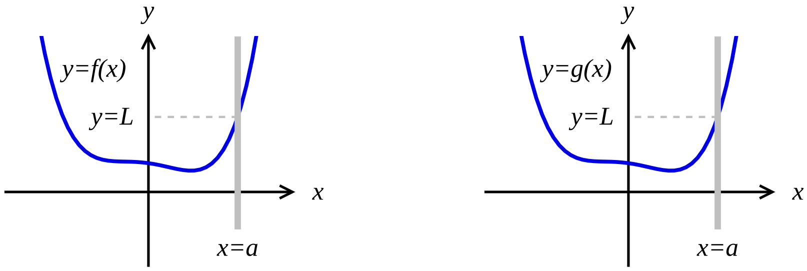
$$\lim_{x\to 3} \frac{x-2}{(x-1)^2} = \frac{1}{4}.$$

end example 2.3.re1

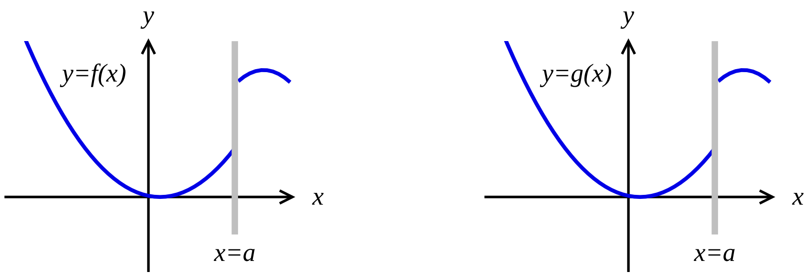
The most commonly used limit law of all:

13. If $f(x) = g(x)$ for all $x \neq a$, then $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ are the same, whether both exist (as in 2.3.re2) or both fail to exist (2.3.re3).

2.3.re2.



2.3.re3.



2.3.re4. Evaluate the limit, if it exists: $\lim_{x\to 2} \dfrac{x-2}{x^2-x-2}$

Solution:

Factor and cancel the common factor.

$$\frac{x-2}{x^2-x-2} = \frac{x-2}{(x-1)(x-2)} =^* \frac{1}{x-1}$$

(* except at $x = 2$, where $\frac{1}{x-1}$ exists but $\frac{x-2}{(x-1)(x-2)}$ does not.) By 13,

$$\lim_{x\to 2} \frac{x-2}{x^2-x-2} = \lim_{x\to 2} \frac{1}{x-1}.$$

Since the rational function $\dfrac{1}{x-1}$ exists at $x = 2$, its limit equals its function value there:

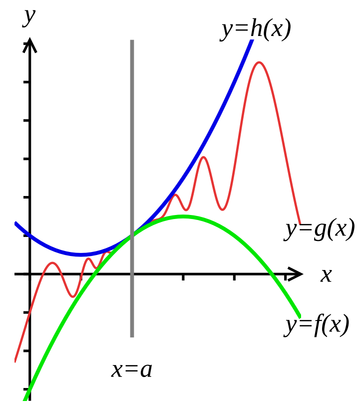
$$\lim_{x\to 2} \frac{1}{x-1} = \frac{1}{2-1} = 1.$$

end example 2.3.re4

2.3.re5. Evaluate the limits:

a. $\lim_{x\to 2} \dfrac{x^2+x-6}{3x^2-6x}$ b. $\lim_{x\to 3} \dfrac{x^2-9}{x^3-27}$ c. $\lim_{x\to 1} \dfrac{\frac{1}{x+3}-\frac{1}{4}}{x-1}$ d. $\lim_{x\to -3} \dfrac{2-\sqrt{x+7}}{x+3}$

The Squeeze Theorem 2.3.re6. *If*



$$f(x) \le g(x) \le h(x)$$

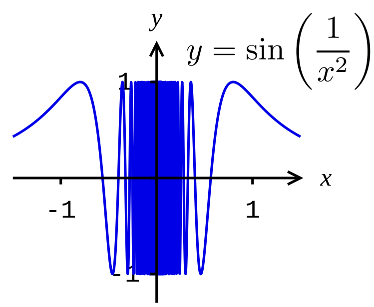
for all $x \ne a$*, and if*

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L,$$

then $\lim_{x\to a} g(x)$ *also exists and equals* L*.*

Examples of the Squeeze Theorem often involve a term that fails to have a limit.

2.3.re7. Evaluate the limit $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x^2}\right)$



Solution:
We try to apply limit law 6,

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x^2}\right) = \lim_{x\to 0} x^2 \cdot \lim_{x\to 0} \sin\left(\frac{1}{x^2}\right),$$

but $\lim_{x\to 0} \sin\left(\frac{1}{x^2}\right)$ does not exist, since, as $\frac{1}{x^2}$ goes to ∞, $\sin\left(\frac{1}{x^2}\right)$ oscillates infinitely many times between -1 and 1.
Instead, begin with

$$-1 \le \sin\left(\frac{1}{x^2}\right) \le 1.$$

Multiply by x^2 to obtain

$$-x^2 \le x^2 \sin\left(\frac{1}{x^2}\right) \le x^2. \qquad (2.3.re8)$$

Since both $\pm x^2 \to 0$ as $x \to 0$, the limit of $x^2 \sin\left(\frac{1}{x^2}\right)$ is also zero, by the Squeeze Theorem. See https://www.desmos.com/calculator/ysugf9i1gp for an illustration of (2.3.re8).

end example 2.3.re7

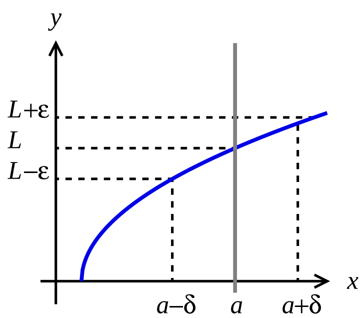
Answers

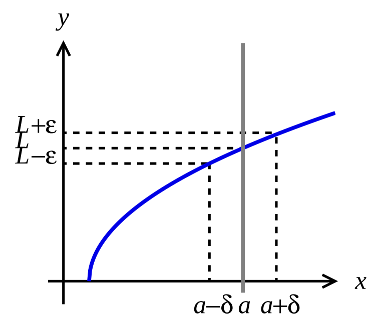
2.3.re5a. 5/6 2.3.re5b. 2/9 2.3.re5c. -1/16 2.3.re5d. $-1/4$

2.4: The Precise Definition of Limit

Definition 2.4.re1. *We say $\lim_{x\to a} f(x) = L$ if, for every positive number ε, there is a corresponding positive number δ for which*

$$|f(x) - L| < \varepsilon \text{ whenever } 0 < |x - a| < \delta.$$



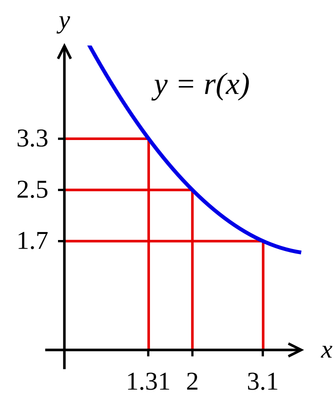


There's an interactive graph illustrating this definition at
https://www.desmos.com/calculator/c1i7yhxjep

2.4.re2. The limit $\lim_{x\to 2} r(x) = 2.5$. Use the graph of $r(x)$ given here to find a positive number δ for which

$$|r(x) - 2.5| < 0.8 \text{ whenever } 0 < |x - 2| < \delta.$$

If δ is a correct answer, so is any smaller number. What's the largest correct δ you can find?

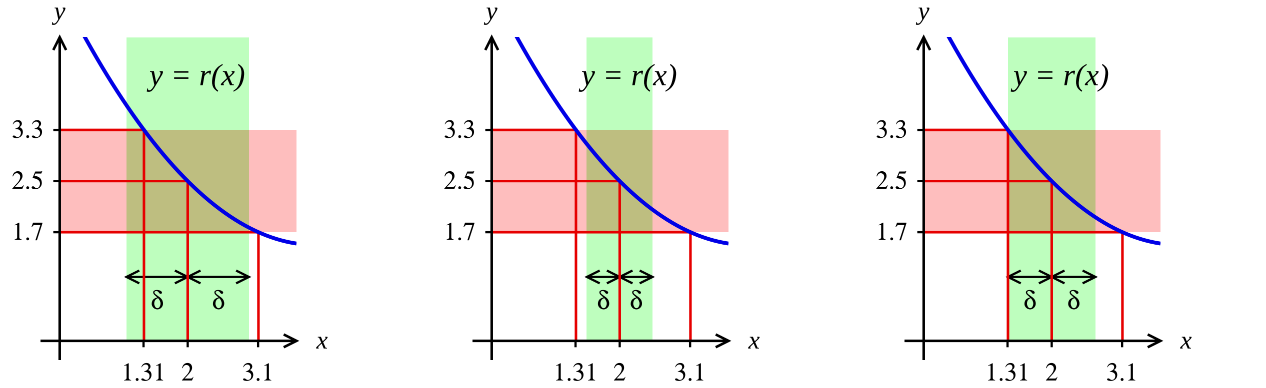


Solution:

Remember that $|x - 2|$ is the distance from x to 2 on the number line, and $|r(x) - 2.5|$ is the distance from $r(x)$ to 2.5. The largest δ for which

$$0 < |x - 2| < \delta \text{ will imply } |r(x) - 2.5| < 0.8$$

is $2 - 1.31 = 0.69$. To illustrate, in the three graphs below, δ is ...



too big

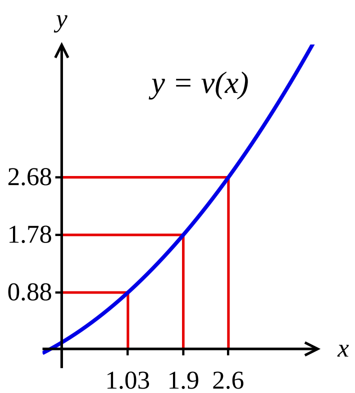
correct but could be bigger

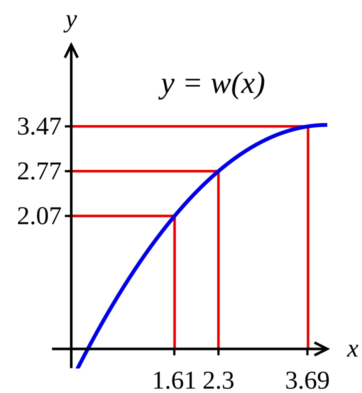
the largest possible correct δ

end example 2.4.re2

2.4.re3. Use the graph of $v(x)$ below to find the largest positive number δ for which

$$|v(x) - 1.78| < 0.9 \text{ whenever } 0 < |x - 1.9| < \delta.$$





2.4.re4. Use the graph of $w(x)$ above to find the largest positive number δ for which

$$|w(x) - 2.3| < 0.7 \text{ whenever } 0 < |x - 2.3| < \delta.$$

2.4.re5. Limit Law 10 promises that $\lim_{x\to 2} x^2 = 4$. Use the table of values to find the positive number δ corresponding to the given ε for this limit.

a. $\varepsilon = 0.1$

b. $\varepsilon = 0.03$

x^2	3.9	3.97	4	4.03	4.1
x	1.9748	1.9925	2	2.0074	2.0248

Writing a limit proof

To write an ε-δ proof of a simple limit requires us to figure out how δ depends on the given ε and explain to the reader why our choice of δ works. The good news is that proofs of limits of linear functions all look the same.

2.4.re6. Write an ε-δ proof of the fact that $\lim_{x\to 4}(3-5x) = -17$.

Solution:
First some analysis. Given $\varepsilon > 0$, we're looking for a number δ so that $0 < |x-4| < \delta$ will imply that $|(3-5x)-(-17)| < \varepsilon$. Start by simplifying:

$$|(3-5x)-(-17)| = |3-5x+17| = |20-5x| = |-5(x-4)| = |-5||x-4| = 5|x-4|,$$

so to make $5|x-4| < \varepsilon$, just make sure that $|x-4| < \frac{1}{5}\varepsilon$.
Now we're ready to write a proof.

Proof: Suppose that $\varepsilon > 0$. Choose $\delta = \frac{1}{5}\varepsilon$. Then, whenever

$$0 < |x-4| < \delta = \frac{1}{5}\varepsilon,$$

$$|(3-5x)-(-17)| = |3-5x+17| = |20-5x| = |-5(x-4)| = |-5||x-4| = 5|x-4| < 5\cdot\frac{1}{5}\varepsilon = \varepsilon,$$

as desired.

end example 2.4.re6

2.4.re7. Write an $\varepsilon - \delta$ proof of the following limits:

a. $\lim\limits_{x\to 3}(2x+1) = 7$ b. $\lim\limits_{x\to 2}(7-4x) = -1$ c. $\lim\limits_{x\to -1}\dfrac{3x^2+x-2}{x+1} = -5$

Answers

2.4.re3. Largest possible $\delta = 0.7$. 2.4.re4. Largest possible $\delta = 0.69$. 2.4.re5a. $\delta = 0.0248$ 2.4.re5b. $\delta = 0.0074$ 2.4.re7a. Suppose $\varepsilon > 0$. Choose $\delta = \frac{1}{2}\varepsilon$. Then, whenever $0 < |x-3| < \delta$,
$|(2x+1)-7| = |2x-6| = |2||x-3| = 2|x-2| < 2\delta = \varepsilon$, as desired.
2.4.re7b. Suppose $\varepsilon > 0$. Choose $\delta = \frac{1}{4}\varepsilon$. Then, whenever $0 < |x-2| < \delta$,
$|(7-4x)-(-1)| = |8-4x| = |-4||x-2| = 4|x-2| < 4\delta = \varepsilon$, as desired.
2.4.re7c. Suppose $\varepsilon > 0$. Choose $\delta = \frac{1}{3}\varepsilon$. Then, whenever $0 < |x-(-1)| < \delta$,
$|\frac{3x^2+x-2}{x+1} - (-5)| = |\frac{(3x-2)(x+1)}{x+1} + 5| = |(3x-2)+5| = |3x+3| = |3||x+1| = 3|x+1| < 3\delta = \varepsilon$, as desired.

2.5: Continuity

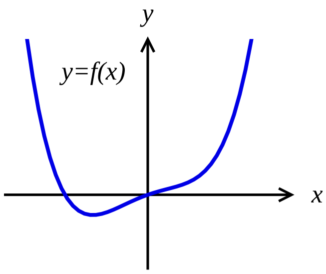
Definition 2.5.re1. *The function $f(x)$ is said to be* **continuous** *at the number a if*

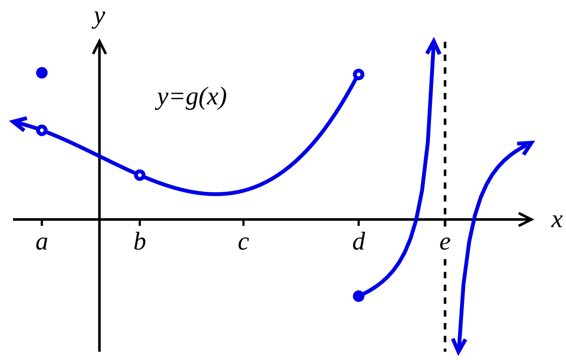
$$\lim_{x\to a} f(x) = f(a).$$

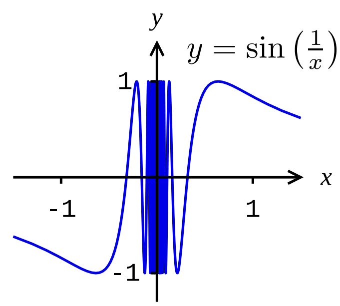
For f to be continuous at a means:

1. the limit $\lim_{x\to a} f(x)$ exists,
2. the function value $f(a)$ exists, and
3. the two are equal.

Informally, if you were drawing the graph of a function from left to right, the function is discontinuous at those x-values where you have to lift your pencil off the paper.







2.5.re2. Where are the above functions continuous? Discontinuous?

a. $f(x)$ b. $g(x)$ c. $\sin(\frac{1}{x})$

Solution:

a. f appears to be continuous everywhere.

b. g is discontinuous at these x-values:

a: limit value and function value exist but are unequal,
b: limit exists but function does not,
d: function exists but limit does not, and
e: neither limit nor function exist.

g is continuous everywhere else.

c. $\sin(\frac{1}{x})$ is discontinuous only at $x = 0$, where neither the function nor its limit exist, and is continuous at all other real numbers x.

end example 2.5.re2

Definition 2.5.re3. *The function $f(x)$ is said to be* **continuous from the right** *at a if* $\lim_{x\to a^+} f(x) = f(a)$, *and* **continuous from the left** *at a if* $\lim_{x\to a^-} f(x) = f(a)$.

A function is continuous at a iff it is continuous from both the right and the left at a.

2.5.re4. The function $g(x)$ is discontinuous at a, b, d, and e. However, g is continuous from the right at $x = d$, since the value of that function and right-sided limit are the same at d. (Both equal the altitude of the dot there.) There is no x at which g is continuous from the left.

2.5.re5. At what x-values, if any, is the function p from 2.2.re5

a. disontinuous?
b. discontinuous, but continuous from the right?
c. discontinuous, but continuous from the left?

Definition 2.5.re6. *The function $f(x)$ is said to be* **continuous** *if it is continuous at every point a in its domain.*

Fact 2.5.re7. *These functions are all continuous wherever they are defined:*

polynomials *rational functions* *power functions* *absolute value*
trig functions *inverse trig functions* *exponentials* *logarithms*

Fact 2.5.re8. *If f and g are continuous at a, then so are*

$$f + g \qquad f - g \qquad f \cdot g \qquad f \div g \text{ (assuming } g(a) \neq 0)$$

Fact 2.5.re9. *If f is continuous at a, and g is continuous at $f(a)$, then $g \circ f$ is continuous at a.*

Fact 2.5.re7 says most functions or families of function we've already given names to are continuous, and Facts 2.5.re8 and 2.5.re9 imply that when we combine any of those by addition, subtraction, multiplication, division or composition, the result is continuous at all points in their domain.

Tips from precalculus for finding a function's domain

If a function includes the expression	**then it requires**
$\frac{\text{anything}}{A}$	$A \neq 0$
$\sqrt{A}$	$A \geq 0$
$\ln A$	$A > 0$
$\tan A$ or $\sec A$	$\cos A \neq 0$
$\cot A$ or $\csc A$	$\sin A \neq 0$
$\sin^{-1} A$ or $\cos^{-1} A$	$\|A\| \leq 1$
$\sec^{-1} A$ or $\csc^{-1} A$	$\|A\| \geq 1$

2.5.re10. Where is the given function continuous?

a. $3x^2 + x - 4|x|$ b. $x \tan x$ c. $\ln\left(\dfrac{x-2}{x+3}\right)$ d. $\sqrt{\dfrac{x-2}{x+3}}$

Solution:
a. This function is composed of a polynomial and the absolute value function. Since it's defined everywhere, this function is continuous on $(-\infty, \infty)$.
b. This function is continuous everywhere except the odd multiples of $\frac{\pi}{2}$, where $\tan x$ is not defined. Since $\tan x$ has period π, we can write these points as $\frac{\pi}{2} + n\pi$ where n is any integer. See section 0.0 of these notes.
c. This function is continuous everywhere that $\frac{x-2}{x+3} > 0$. Here's a sign chart:

$$\begin{array}{rl} x+3: & - - - - - 0 + + + + + + + + \\ x-2: & - - - - - - - - - - 0 + + + + \\ \dfrac{x-2}{x+3}: & + + + + \text{ DNE } - - - - 0 + + + + \\ \hline x: & \qquad\qquad -3 \qquad\qquad 2 \end{array}$$

The function is continuous on $(-\infty, -3) \cup (2, \infty)$.
d. This function is continuous everywhere that $\frac{x-2}{x+3} \geq 0$; that is, $(-\infty, -3) \cup [2, \infty)$.

end example 2.5.re10

2.5.re11. Where is the given function continuous?

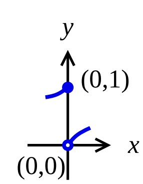
a. $\dfrac{\sin x}{|2x+1|}$ b. $\dfrac{\sin x + \cos x}{\sin x - \cos x}$ c. $e^{\sqrt{2-x}}$

A piecewise-defined function needn't be continuous at its "knots" (where its pieces join), even if it's made of continuous pieces.

2.5.re12. Where is the function $\alpha(x) = \begin{cases} e^x & \text{if } x \leq 0 \\ \ln(x+1) & \text{if } x > 0 \end{cases}$ discontinuous?

Solution:

The function e^x is continuous for all real x, and $\ln(x+1)$ is continuous for all $x > -1$. Therefore $\alpha(x)$ is continuous for all $x < 0$ (where it equals e^x) and all $x > 0$ (where it equals $\ln(x+1)$).
The only possible discontinuity is at $x = 0$. It fact, α is discontinuous at $x = 0$: because $e^0 = 1$ and $\ln(0+1) = \ln 1 = 0$ the two pieces of α don't meet continuously at $x = 0$. (See graph.)



end example 2.5.re12

2.5.re13. Where is the given function discontinuous?

a. $\beta(x) = \begin{cases} x^2 + 4x & \text{if } x \leq 0 \\ 1 - x & \text{if } 0 < x < 1 \\ x^4 - 1 & \text{if } 1 \geq x \end{cases}$ b. $\gamma(x) = \begin{cases} \frac{1}{x-2} & \text{if } x < 1 \\ x^2 + 2x - 4 & \text{if } 1 \leq x \leq 2 \\ xe^{x-2} & \text{if } 2 < x \end{cases}$

2.5.re14. Find the values of a and b that make $\gamma(x) = \begin{cases} 1-2x^2 & \text{if } x \le -1 \\ ax+b & \text{if } -1 < x \le 1 \\ 3+x^2 & \text{if } 1 < x \end{cases}$ continuous.

Intermediate Value Theorem (IVT) 2.5.re15. *If f is continuous on $[a,b]$, and if M is a number between $f(a)$ and $f(b)$, then there's at least one number c in $[a,b]$ for which $f(c) = M$.*

One consequence of the IVT is that a continuous function can change sign only where it is zero or where it doesn't exist. This allows us to build a sign chart for a continuous function by finding its zeros and discontinuities and then testing points. For instance, our solution to Example 2.5.re10b depended on the fact that $\frac{x-2}{x+3}$ can change sign only where it is zero ($x = 2$) or where it is discontinuous ($x = -3$).

IVT can often tell us that an equation has a solution, even if we're unable to produce it exactly.

2.5.re16. Show that $x \cos x = 3$ has a solution in the interval $(0, 2\pi)$.

Solution:
The function $g(x) = x \cos x$ is continuous everywhere. Since $g(0) = 0$ and $g(2\pi) = 2\pi$ and $0 < 3 < 2\pi$, by the IVT, there must be at least one x in $(0, 2\pi)$ at which $g(x) = 3$.

end example 2.5.re16

2.5.re17. Without using a calculator, find an interval (a, b) which contains a solution to the equation $-x^5 + 2x^3 + 3 = 0$.

Answers

2.5.re5a. 0, 1, 3. 2.5.re5b. none. 2.5.re5c. 3. 2.5.re11a. At all numbers except $x = -\frac{1}{2}$, where $|2x+1| = 0$. 2.5.re11b. Everywhere except the numbers where $\sin x = \cos x$, or, equivalently, $\tan x = 1$: $\frac{\pi}{4} + n\pi$ where n is any integer. 2.5.re11c. Everywhere that $2 - x \ge 0$, that is, $(-\infty, 2]$. 2.5.re13a. only at $x = 0$. 2.5.re13b. only at $x = 2$. 2.5.re14. Find a and b so that the line $y = ax + b$ goes through the points $(-1,-1)$ and $(1, 4)$. Answer is $a = 5/2$, $b = 3/2$. 2.5.re17. The function $q(x) = -x^5 + 2x^3 + 3$ is continuous everywhere. Since $q(1) = 4$ and $q(2) = -13$, the IVT promises that $q(x) = 0$ for at least one x in the interval $(1, 2)$.

2.6: Limits at Infinity and Asymptotes

Definition 2.6.re1. *The line $x = a$ is a* **vertical asymptote** *of the graph of $f(x)$ if either*

$$\lim_{x\to a^-} f(x) \text{ or } \lim_{x\to a^+} f(x)$$

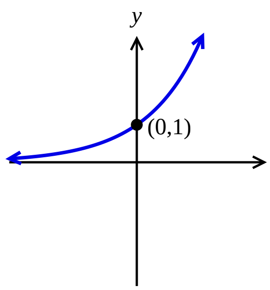
is infinite.

Definition 2.6.re2. *The line $y = a$ is a* **horizontal asymptote** *of the graph of $f(x)$ if either*

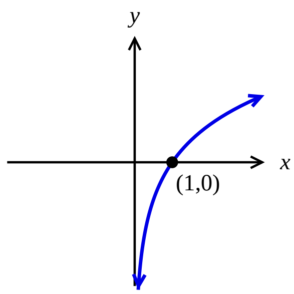
$$\lim_{x\to -\infty} f(x) = a \text{ or } \lim_{x\to \infty} f(x) = a.$$

In short, a curve has a horizontal or vertical asymptote if one variable approaches a finite number while the other blows up.

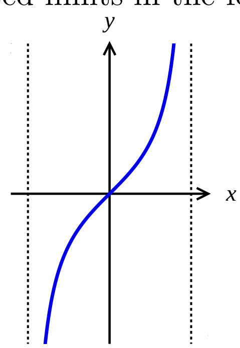
2.6.re3. Identify the asymptotes and associated limits in the following graphs.



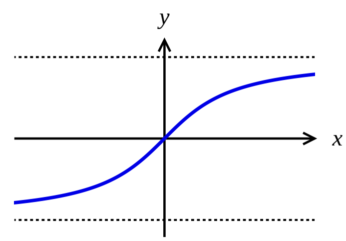
$y = e^x$



$y = \ln x$



$y = \tan x$



$y = \tan^{-1} x$

Some useful limit facts

Fact 2.6.re4. *Something goes to zero if and only if its reciprocal goes to $\pm\infty$.*

Fact 2.6.re5. *If p is a constant, then* $\lim_{x\to\infty} x^p = \begin{cases} \infty & \text{if } p > 0, \text{ and} \\ 0 & \text{if } p < 0. \end{cases}$

Fact 2.6.re6. *If $p(x)$ is a polynomial and $\ell t_p(x)$ is its lead term, then*

$$\lim_{x\to\infty} p(x) = \lim_{x\to\infty} \ell t_p(x).$$

Fact 2.6.re7. *If $p(x)$ and $q(x)$ are polynomials then*

$$\lim_{x\to\infty} \frac{p(x)}{q(x)} = \lim_{x\to\infty} \frac{\ell t_p(x)}{\ell t_q(x)}.$$

2.6.re8. Evaluate the limit.

a. $\lim_{x\to\infty} (x^{\frac{1}{3}} - x^{-4})$ b. $\lim_{x\to\infty} (4x^5 - 2x^4 + 3x + 1)$ c. $\lim_{x\to\infty} \frac{2x-1}{x+3}$

d. $\lim_{x\to\infty} \frac{3x-1}{3-x^2}$ e. $\lim_{x\to\infty} \frac{2x^2+3x-1}{3-x}$ f. $\lim_{x\to-\infty} \frac{2x^2+3x-1}{3-x}$

If the rules for polynomials and rational functions don't apply, try to factor out and cancel the dominant term in the top and bottom. When simplifying an expression, don't take any step that will change its value.

2.6.re9. Evaluate the limit.

a. $\lim_{x\to\infty} \frac{\sqrt{9x^2+1}}{x}$ b. $\lim_{x\to-\infty} \frac{\sqrt{9x^2+1}}{x}$ c. $\lim_{x\to\infty} \frac{2e^x+1}{3e^x-4}$

d. $\lim_{x\to-\infty} \frac{2e^x+1}{3e^x-4}$ e. $\lim_{x\to\infty} \sqrt{4x^2-x} - 2x$ f. $\lim_{x\to-\infty} \sqrt{4x^2-x} - 2x$

Solution:

a. This limit initially looks like "$\frac{\infty}{\infty}$", which tells us nothing about its value. Factor out x^2 from the under the radical

$$\frac{\sqrt{x^2(9+x^{-2})}}{x} = \frac{\sqrt{x^2}\sqrt{9+x^{-2}}}{x}.$$

Remember that

$$\sqrt{x^2} = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}.$$

Since $x \to \infty$, we can assume $x > 0$, and the quotient is

$$= \frac{|x|\sqrt{9+x^{-2}}}{x} = \frac{x\sqrt{9+x^{-2}}}{x} = \sqrt{9+x^{-2}}.$$

By 2.6.re7, $\lim_{x\to\infty} \sqrt{9+x^{-2}} = \sqrt{9} = 3$.

b. When $x \to -\infty$, we can assume $x < 0$, and the quotient is

$$= \frac{|x|\sqrt{9+x^{-2}}}{x} = \frac{-x\sqrt{9+x^{-2}}}{x} = -\sqrt{9+x^{-2}}.,$$

and so, $\lim_{x\to-\infty} -\sqrt{9+x^{-2}} = -3$.

c. The graph of e^x reminds us that $\lim_{x\to\infty} e^x = \infty$, so this limit looks like "$\frac{\infty}{\infty}$". Guessing e^x to be the dominant term, we factor it from numerator and denominator and cancel:

$$\frac{2e^x+1}{3e^x-4} = \frac{e^x(2+\frac{1}{e^x})}{e^x(3-\frac{4}{e^x})} = \frac{2+\frac{1}{e^x}}{3-\frac{4}{e^x}}.$$

By 2.6.re4, as $x \to \infty$, this approaches the limit

$$\frac{2+0}{3-0} = \frac{2}{3}.$$

d. As the graph of e^x shows, $\lim_{x\to-\infty} e^x = 0$, so the limit is

$$\lim_{x\to-\infty} \frac{2e^x+1}{3e^x-4} = \frac{0+1}{0-4} = -\frac{1}{4}.$$

e. By 2.6.re6, $\lim_{x\to\infty} 4x^2 - x = \lim_{x\to\infty} 4x^2 = \infty$, and so $\lim_{x\to\infty} \sqrt{4x^2-x} = \infty$. The limit in question looks like $\infty - \infty$, which tells us nothing about its value. Multiply and divide by the conjugate:

$$\frac{\sqrt{4x^2-x}-2x}{1} \cdot \frac{\sqrt{4x^2-x}+2x}{\sqrt{4x^2-x}+2x} = \frac{\sqrt{4x^2-x}^2 - (2x)^2}{\sqrt{4x^2-x}+2x} = \frac{-x}{\sqrt{4x^2-x}+2x}$$

Now, as in part a., factor x^2 out of the radical. Since $x \to \infty$, assume $x > 0$.

$$\frac{-x}{|x|\sqrt{4-x^{-1}}+2x} = \frac{-x}{x\sqrt{4-x^{-1}}+2x} = \frac{-x}{x(\sqrt{4-x^{-1}}+2)} = \frac{-1}{\sqrt{4-x^{-1}}+2}$$

the limit of which equals

$$\frac{-1}{\sqrt{4-0}+2} = -\frac{1}{4}.$$

f. The limit in question is $\infty + \infty = \infty$.

end example 2.6.re9

2.6.re10. Evaluate the limit.

a. $\lim_{x\to\infty} \frac{\sqrt{2x^2-5x}}{-3x+2}$
b. $\lim_{x\to-\infty} \frac{\sqrt{2x^2-5x}}{-3x+2}$
c. $\lim_{x\to\infty} \frac{2e^x - e^{-x}}{4e^x+7e^{-x}}$
d. $\lim_{x\to-\infty} \frac{2e^x - e^{-x}}{4e^x+7e^{-x}}$
e. $\lim_{x\to\infty} \left(x + \sqrt{x^2-3x}\right)$
f. $\lim_{x\to-\infty} \left(x + \sqrt{x^2-3x}\right)$

Answers

2.6.re3. $y = 0$ is a HA to $y = e^x$ because $\lim_{x\to-\infty} e^x = 0$. $x = 0$ is a VA to $y = \ln x$ b.c. $\lim_{x\to0^+} \ln x = -\infty$. Both $x = \frac{\pi}{2}$ and $x = -\frac{\pi}{2}$ are VAs to the graph of $y = \tan x$, b.c. $\lim_{x\to-(\pi/2)^+} \tan x = -\infty$ and $\lim_{x\to(\pi/2)^-} \tan x = \infty$. $y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$ are HAs to the graph of $y = \tan^{-1} x$, b.c. $\lim_{x\to-\infty} \tan^{-1} x = -\frac{\pi}{2}$ and $\lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}$. 2.6.re8a. "$\infty - 0$," so the limit $= \infty$. 2.6.re8b. $= \lim_{x\to\infty} 4x^5 = \infty$. 2.6.re8c. $= \lim_{x\to\infty} \frac{2x}{x} = 2$. 2.6.re8d. $= \lim_{x\to\infty} \frac{3x}{-x^2} = \lim_{x\to\infty} \frac{3}{-x} = 0$. 2.6.re8e. $= \lim_{x\to\infty} \frac{2x^2}{-x}$ $= \lim_{x\to\infty} -2x = -\infty$. 2.6.re8f. $= \lim_{x\to-\infty} \frac{2x^2}{-x} = \lim_{x\to-\infty} -2x = \infty$.

2.6.re10a. $\lim_{x\to\infty} \frac{\sqrt{x^2}\sqrt{2-\frac{5}{x}}}{x(-3+\frac{2}{x})} = \lim_{x\to\infty} \frac{|x|\sqrt{2-\frac{5}{x}}}{x(-3+\frac{2}{x})} = \lim_{x\to\infty} \frac{x\sqrt{2-\frac{5}{x}}}{x(-3+\frac{2}{x})} = \lim_{x\to\infty} \frac{\sqrt{2-\frac{5}{x}}}{(-3+\frac{2}{x})} = -\frac{\sqrt{2}}{3}$.

2.6.re10b. $\lim_{x\to-\infty} \frac{\sqrt{x^2}\sqrt{2-\frac{5}{x}}}{x(-3+\frac{2}{x})} = \lim_{x\to-\infty} \frac{|x|\sqrt{2-\frac{5}{x}}}{x(-3+\frac{2}{x})} = \lim_{x\to-\infty} \frac{-x\sqrt{2-\frac{5}{x}}}{x(-3+\frac{2}{x})} = \lim_{x\to-\infty} \frac{-\sqrt{2-\frac{5}{x}}}{(-3+\frac{2}{x})} = \frac{\sqrt{2}}{3}$.

2.6.re10c. factor out and cancel e^x. $\lim_{x\to\infty} \frac{2-e^{-2x}}{4+7e^{-2x}} = \frac{2-0}{4+0} = \frac{1}{2}$.

2.6.re10d. factor out and cancel e^{-x}. $\lim_{x\to\infty} \frac{2e^{2x}-1}{4e^{2x}+7} = \frac{0-1}{0+7} = -\frac{1}{7}$.

2.6.re10e. $\infty+\infty = \infty$. 2.6.re10f. Multiply and divide by the conjugate. $\sqrt{x^2} = |x| = -x$. Factor out and cancel x. Limit $= 3/2$.

2.7: Derivatives and Rates of Change

Here's the conclusion of section 2.1, restated with the limit notation from 2.2:
If $f(t)$ is an object's position at time t, then the difference quotient

$$\frac{f(a+h)-f(a)}{h}$$

is *both* the object's average velocity between times a and $a+h$ *and* the slope of secant line passing through the graph of f at times a and $a+h$. The limit of the difference quotient,

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

is *both* the object's velocity at time a *and* the slope of the line tangent to the graph of f at the point $\big(a, f(a)\big)$.

Definition 2.7.re1. *The* **derivative** *of the function $f(x)$ at $x=a$, denoted $f'(a)$ is*

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

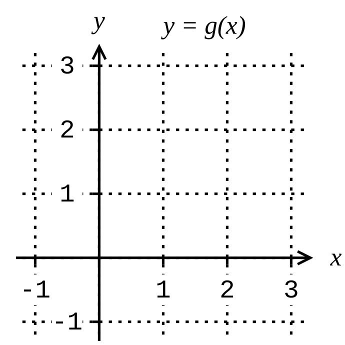
2.7.re2. Find the equation of the line tangent to $y=w(x)$ at $x=1$ if $w(1)=2$ and $w'(1)=-3$.

Solution:
The given information tells us that the graph of w passes through the point $(1,2)$ and that the line tangent to curve at that point has slope -3. In point-slope form, the line is $y-2=-3(x-1)$.

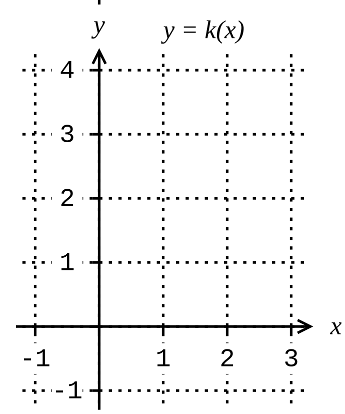
end example 2.7.re2

2.7.re3. Sketch the graph of a function g satisfying

$$g(0)=0 \qquad g'(0)=3 \qquad g(2)=2 \qquad g'(2)=-1$$



2.7.re4. Sketch $y=k(x)$ if $k(1)=-1$ and $k'(x)=\begin{cases}2 & \text{if } x>1,\\ -1 & \text{if } x<1.\end{cases}$



2.7.re5. Find $p(2)$ and $p'(2)$ if the line tangent to $y = p(x)$ at the point $(2, 5)$ passes through $(4, -7)$.

On exams and quizzes, you may be asked to find the derivative of a function using Definition 2.7.re1, as in this next example.

2.7.re6. Use the definition of derivative to find $f'(a)$ if $f(x) = \frac{2}{3-x}$:

Solution:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \left(\frac{\frac{2}{3-(a+h)} - \frac{2}{3-a}}{h} \right) \cdot \frac{(3-a-h)(3-a)}{(3-a-h)(3-a)}$$
$$= \lim_{h\to 0} \frac{2(3-a) - 2(3-a-h)}{h(3-a-h)(3-a)} = \lim_{h\to 0} \frac{6 - 2a - 6 + 2a + 2h}{h(3-a-h)(3-a)}$$
$$= \lim_{h\to 0} \frac{2h}{h(3-a-h)(3-a)} = \lim_{h\to 0} \frac{2}{(3-a-h)(3-a)} = \frac{2}{(3-a)^2}$$

end example 2.7.re6

2.7.re7. Find the derivative at a of the given function:

a. $f(x) = x^3 - 2x^2$ b. $g(x) = \sqrt{x+1}$ c. $k(x) = \dfrac{2x}{x-1}$

d. $\ell(x) = 2x^4 + 3$ e. $m(x) = \sqrt{3-2x}$ f. $n(x) = \dfrac{x-1}{2x}$

2.7.re8. A particle moving along a coordinate axis is at position $f(t) = t^3 - 2t^2$ (meters) at time t (seconds). Find the particle's velocities at times $t = 1$ and $t = 2$. What are the units in your answer?

2.7.re9. Find $f(x)$ and a if the following limit is $f'(a)$. (There is more than one correct answer.)

a. $\displaystyle\lim_{h\to 0} \frac{\sqrt{16+h} - 4}{h}$ b. $\displaystyle\lim_{x\to -3} \frac{2x^2 + 4x - 6}{x+3}$

Answers

2.7.re3. Here's one such graph: https://www.desmos.com/calculator/llbjhd1fw7 Note tangent lines at $x = 0$ and $x = 2$. 2.7.re4. The graph of k is at: https://www.desmos.com/calculator/seosuy3hs2 2.7.re5. Given point on is $(2, p(2))$, so $p(2) = 5$. $p'(2)$ = the slope of the tangent line, which we calculate using the given two points to be -6. 2.7.re7a. $f'(a) = 3a^2 - 4a$ 2.7.re7b. $g'(a) = \frac{1}{2\sqrt{a+1}}$ 2.7.re7c. $k'(a) = \frac{-2}{(a-1)^2}$ 2.7.re7d. $\ell'(a) = 8a^3$ 2.7.re7e. $m'(a) = \frac{-1}{\sqrt{3-2a}}$ 2.7.re7f. $n'(a) = \frac{1}{2a^2}$. Note $n(x) = \frac{1}{k(x)}$, but $n'(a) \neq \frac{1}{k'(a)}$. 2.7.re8. Can use derivative found in 3.1.re2, part a. Velocity at time 1 is $f'(1) = -1$m/sec. Velocity at time 2 is $f'(2) = 4$m/sec. 2.7.re9a. $f(x) = \sqrt{x}$, $a = 16$.
2.7.re9b. $f(x) = 2x^2 + 4x$, $a = -3$.

2.8: The Derivative as a Function

Definition 2.7.re1, restated. *The derivative of the function $f(x)$ is the function*

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{t\to x} \frac{f(t)-f(x)}{t-x}$$

On exams and quizzes, you may be asked to find the derivative of a function using Definition 2.7.re1. On such a question, you would not receive credit for using the derivative rules of Chapter 3.

2.8.re1. Find the derivative at x of the given function.

a. $\gamma(x) = x^2 + 3x - 1$ b. $g(x) = \sqrt{2x-1}$ c. $h(x) = \dfrac{x}{4x-1}$

d. $\tau(x) = x^4 + 5x - 3$ e. $\alpha(x) = \dfrac{1}{\sqrt{3x+2}}$ f. $\beta(x) = \dfrac{2x+1}{x}$

Definition 2.8.re2. *The function $f(x)$ is said to be* **differentiable** *at a if $f'(a)$ exists.*

Fact 2.8.re3. *If $f(x)$ is differentiable at a, then it must be continuous at a.*

Reasons why $f(x)$ might fail to be differentiable at a:

1. $f(x)$ is not continuous at a. See Section 2.5.
2. The graph of $f(x)$ has a **corner** at a, meaning the one-sided limits

$$\lim_{h\to 0^-} \frac{f(a+h)-f(a)}{h} \quad \text{and} \quad \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}$$

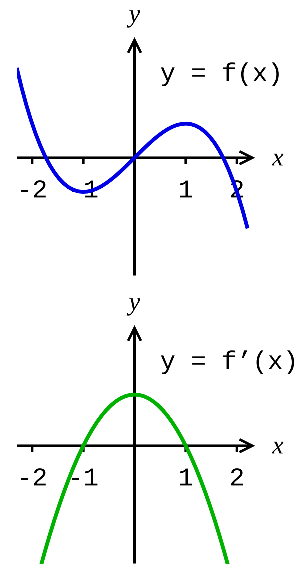
   exist but disagree. This would happen, for instance, if $f'(x)$ has a jump discontinuity at a.
3. The graph of $f(x)$ has a vertical tangent line at $x = a$, meaning that one or both of the one-sided limits above are infinite.
4. None of the above.

Sketching the derivative

You may be given the graph of a function and asked to sketch its derivative. Start by noting where the graph of the $f(x)$ is horizontal (or approaching horizontal), vertical (or approaching vertical), positively or negative sloped, and where $f'(x)$ fails to exist.

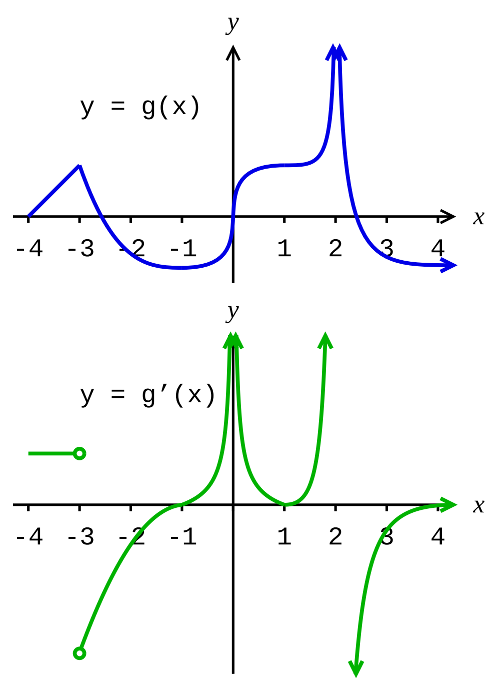
2.8.re4. Here's a sketch of a function $f(x)$ and its derivative $f'(x)$. A good sketch of the derivative will include these details, based on the behavior of f.

x-values	$f(x)$	$f'(x)$
at $-1\&1$	horizontal	0.
on $(-\infty,-1)\cup(1,\infty)$	negative sloped	negative
on $(-1,1)$	positively sloped	positive
at 0	greatest slope	maximum



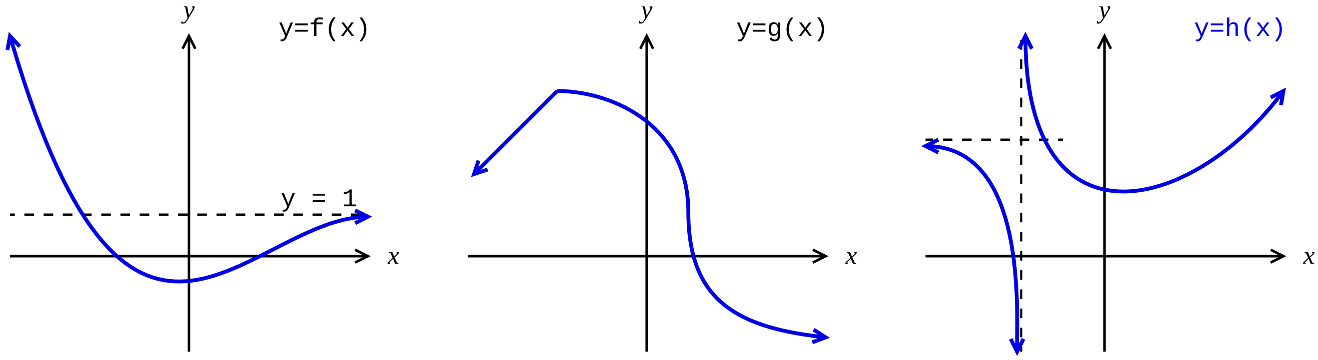
2.8.re4, continued. Here's another example and a list of necessary properties of a good sketch of the derivative.

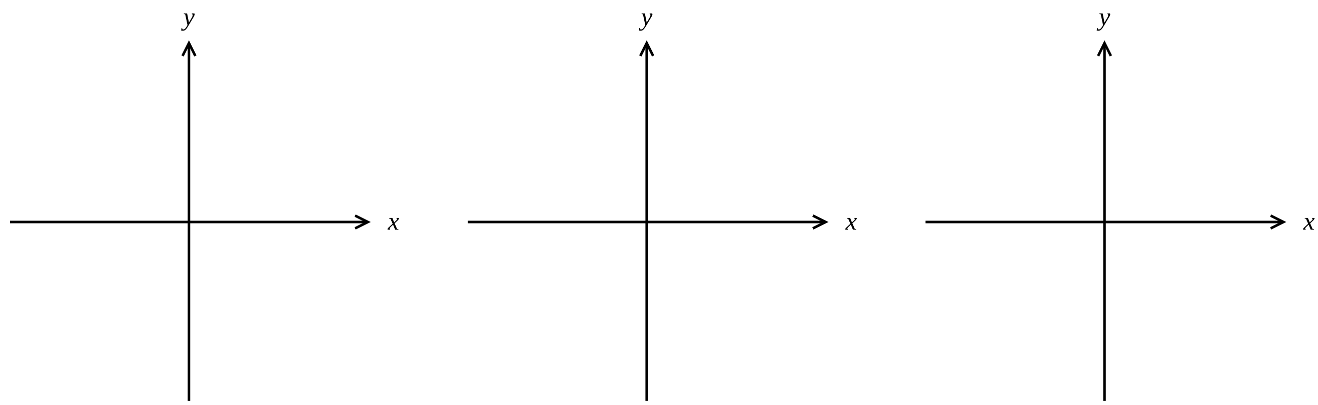
x-values	$g(x)$	$g'(x)$
at $-1\&1$	horizontal	0
on $(-3,-1)\cup(2,\infty)$	negative sloped	negative
on $(-4,-3)\cup(-1,2)$	positively sloped	positive
near 0&2	near vertical	blowing up
near 4	near horizontal	near 0
on $(-4,-3)$	linear	constant
at -3	corner	DNE; jump discontinuity



end example 2.8.re4

2.8.re5. Sketch the derivative of the function graphed.





Higher order derivatives

The derivative of a function is also called its **first derivative**.
The **second derivative** of a function is the derivative of its first derivative.
The **third derivative** of a function is the derivative of its second derivative, and so on.

Various notation for derivatives.

The symbol $\frac{d}{dx}$ means 'the derivative of', just as $\sqrt{}$ means 'the square root of'.

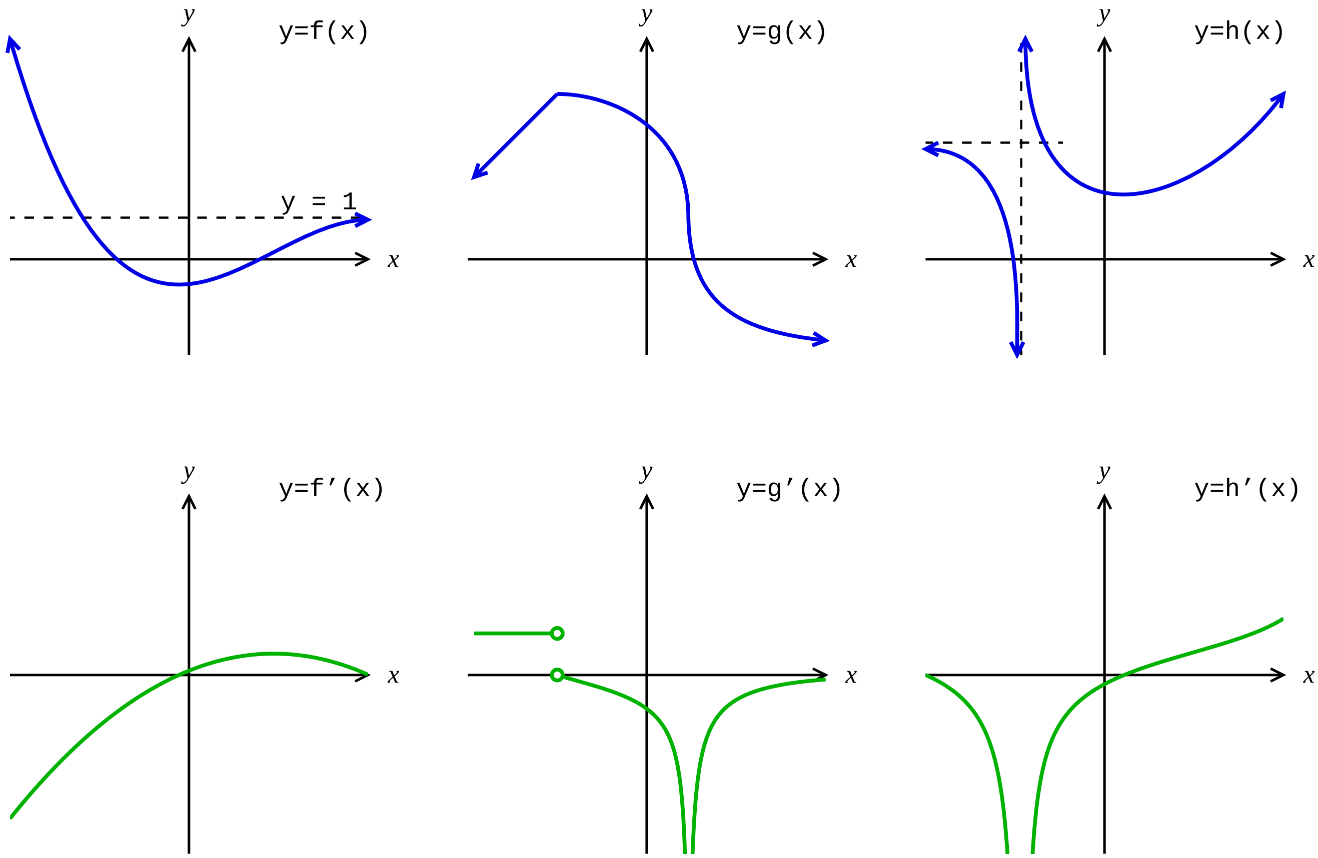
First derivative of $y = f(x)$	$f'(x)$	$\dfrac{dy}{dx}$	$\dfrac{d}{dx}y$	$\dfrac{d}{dx}f(x)$
Second derivative of $y = f(x)$	$f''(x)$	$\dfrac{d^2y}{dx^2}$	$\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$	$\left(\dfrac{d}{dx}\right)^2 f(x)$
Third derivative of $y = f(x)$	$f'''(x)$	$f^{(3)}(x)$	$\dfrac{d^3y}{dx^3}$	$\left(\dfrac{d}{dx}\right)^3 f(x)$
Fourth derivative of $y = f(x)$	$f^{(4)}(x)$	$\dfrac{d^4y}{dx^4}$		

The derivative of a function is simultaneously the slope of its tangent line and its rate of change measured in units-output per units-input.

2.8.re6. Suppose the position at time t of a particle on an axis is $s(t)$. If s is measured in meters (m) and t in seconds (s), what are the units of its first three derivatives?

Answers

2.8.re1a. $\gamma'(x) = 2x + 3$ 2.8.re1b. $g'(x) = \frac{1}{\sqrt{2x-1}}$ 2.8.re1c. $h'(x) = \frac{-1}{(4x-1)^2}$ 2.8.re1d. $\tau'(x) = 4x^3 + 5$ 2.8.re1e. $\alpha'(x) = \frac{-3}{2(3x+2)^{3/2}}$ 2.8.re1f. $\beta'(a) = 2 - \frac{1}{x^2}$ 2.8.re5. See below.



2.8.re6. Units of $s'(t)$ are meters per second, or m/s. Units of $s''(t)$ are meters per second per second, or m/s^2. Units of $s'''(t)$ are meters per second per second per second, or m/s^3.

3.1: Derivatives of Power Functions, Polynomials, and Exponentials

In this chapter we learn Differentiation Laws that allow us to find derivatives more easily than by the definition in Section 2.7. They come in two types.
Below, $f(x)$ and $g(x)$ can be any differentiable functions, and c and n can be any constants, and e is the mathematical constant mentioned in Section 1.4.

I. A catalog of elementary functions and their derivatives.

$(c)' = 0$	$(x^n)' = nx^{n-1}$	$(e^x)' = e^x$

II. Combination laws.

$\big(cf(x)\big)' = cf'(x)$
$\big(f(x) + g(x)\big)' = f'(x) + g'(x)$
$\big(f(x) - g(x)\big)' = f'(x) - g'(x)$

Warning: the value of the derivative of a function is different from the derivative of the value of the function. The latter is always 0.

3.1.re1. Find the derivative of the given function:

a. $x^4 + 5x - 3$ b. $\sqrt{2x} + \sqrt{\dfrac{9}{x}}$ c. $\sqrt[4]{x} - 3x$

d. $\dfrac{4}{x^3}$ e. πr^2 f. $\dfrac{4}{3}\pi r^3$

g. $4e^x - x^3 + 3^4$ h. $4x^{1.2} - 2e^{1.2}$ i. $(3x-1)(2x+1)$

j. $\dfrac{2\sqrt{x} + 3\sqrt[3]{x^4}}{x}$

3.1.re2. Find the second derivative of the given function:

a. $x^4 + 5x - 3$ b. $\sqrt[4]{x} - 3x$ c. $\dfrac{4}{x^3}$

d. $\dfrac{4}{3}\pi r^3$ e. $4x^{1.2} - 2e^{1.2}$ f. $\dfrac{2\sqrt{x} + 3\sqrt[3]{x^4}}{x}$

3.1.re3. Find an equation of the line tangent to the graph of $u(x) = x^4 + 2x^5$ at $x = 1$.

Solution:
The derivative is $u'(x) = 4x^3 + 10x^4$. The line must be tangent to the curve at the point $\bigl(1, u(1)\bigr) = (1, 3)$ and its slope is $u'(1) = 14$. Therefore, the point-slope form of the line is $y - 3 = 14(x - 1)$.

end example 3.1.re3

3.1.re4. Find an equation of the line tangent to $y = \dfrac{4}{x^3}$ at $x = 2$.

3.1.re5. Find numbers a, b, and c so that $h(x) = ax^2 + bx + c$ will satisfy

$$h(0) = 1 \qquad h(1) = 0 \qquad h'(1) = -2.$$

Solution:
Setting h and $h' = 2ax + b$ equal to the given values gives us a system of equations in a, b, and c:

$$\begin{aligned} h(0) &= c = 1 & & & a &= -1 \\ h(1) &= a + b + c = 0 & \implies \quad a + b &= -1 & \implies \quad b &= 0 \\ h'(1) &= 2a + b = -2 & 2a + b &= -2 & c &= 1 \end{aligned}$$

end example 3.1.re5

3.1.re6. Find numbers a, b, and c if $k(x) = ax^2 + bx + ce^x$ satisfies

$$k(0) = 2 \qquad k'(0) = 1 \qquad k''(0) = -3.$$

Answers

3.1.re1a. $4x^3 + 5$ 3.1.re1b. $\frac{1}{\sqrt{2}}x^{-1/2} - \frac{3}{2}x^{-3/2}$ 3.1.re1c. $\frac{1}{4}x^{-3/4} - 3$ 3.1.re1d. $\frac{-12}{x^4}$ 3.1.re1e. $2\pi r$
3.1.re1f. $4\pi r^2$ 3.1.re1g. $4e^x - 3x^2$ 3.1.re1h. $4.8x^{0.2}$ 3.1.re1i. $12x + 1$ 3.1.re1j. $-x^{-3/2} + x^{-2/3}$
3.1.re2a. $12x^2$ 3.1.re2b. $-\frac{3}{16}x^{-7/4}$ 3.1.re2c. $48x^{-5}$ 3.1.re2d. $8\pi r$ 3.1.re2e. $0.96x^{-0.8}$
3.1.re2f. $\frac{3}{2}x^{-5/2} - \frac{2}{3}x^{-5/3}$ 3.1.re4. $y = \frac{1}{2} - \frac{3}{4}(x - 2)$. 3.1.re6. $a = -5/2$, $b = -1$, $c = 2$.

3.2: The Product and Quotient Rules

II. Combination laws (continued).

Product Rule: If $f(x)$ and $g(x)$ are differentiable, then so is their product, and

$$\big(f(x)g(x)\big)' = f'(x)g(x) + f(x)g'(x).$$

Quotient Rule: If $f(x)$ and $g(x)$ are differentiable, then so is their quotient, and

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} \qquad \text{(provided } g(x) \neq 0\text{).}$$

Always simplify at every step to make your next step easier and more likely to be correct. When more than one differentiation rule applies, the last operation determines the first differentiation rule we must use.

3.2.re1. Find the first derivative of the given function.

a. x^2e^x b. $x^2e^x(2x-3)$ c. $\dfrac{3x^3+2}{x^2}$

d. $\dfrac{x}{3x^2+1}$ e. $(r-\sqrt{r})(r+\sqrt{r})$ f. $\dfrac{ze^z}{z+1}$

g. $\dfrac{x}{x+\frac{2}{x}}$ h. $\dfrac{x^2+4}{x^3-8}$

3.2.re2. Find the equation of the line tangent to $y = \dfrac{x}{3x^2+1}$ at $x = 1$.

3.2.re3. Find all x-values at which the slope of $y = \dfrac{x-1}{x+3}$ equals …

a. 9 b. -8 c. 0

Answers

3.2.re1a. $e^x(2x+x^2)$ 3.2.re1b. $e^x(2x^3+3x^2-6x)$ 3.2.re1c. $3-4x^{-3}$ 3.2.re1d. $\frac{1-3x^2}{(3x^2+1)^2}$
3.2.re1e. $2r-1$ 3.2.re1f. $\frac{(z^2+z+1)e^z}{(z+1)^2}$ 3.2.re1g. $\frac{4x^{-1}}{(x+2x^{-1})^2}$ 3.2.re1h. $\frac{-x^4-12x^2-16x}{(x^3-8)^2}$
3.2.re2. $y-\frac{1}{4} = -\frac{1}{8}(x-1)$ 3.2.re3a. $\frac{dy}{dx} = \frac{4}{(x+3)^2} = 9$ at $x = -3\pm\frac{2}{3}$, or $x = -\frac{11}{3}, -\frac{7}{3}$ 3.2.re3b. no solutions ($\frac{dy}{dx}$ on this curve is always positive). 3.2.re3c. no solutions.

3.3: Derivatives of the Trig Functions

In this section we add 6 more functions to our derivative catalog.

I. Catalog.

$(\sin x)' = \cos x$	$(\cos x)' = -\sin x$
$(\tan x)' = \sec^2 x$	$(\cot x)' = -\csc^2 x$
$(\sec x)' = \sec x \tan x$	$(\csc x)' = -\csc x \cot x$

It is useful to note that the last four follow from the first two and the quotient rule.

3.3.re1. Show that $\frac{d}{dx} \cot x = \csc^2 x$ using the derivatives of $\sin x$ and of $\cos x$.

Solution:

By definition, $\cot x = \frac{\cos x}{\sin x}$, and so its derivative

$$\left(\frac{\cos x}{\sin x}\right)' = \frac{(\cos x)' \sin x - \cos x(\sin x)'}{\sin^2 x} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = \frac{-1}{\sin^2 x} = -\csc^2 x.$$

end example 3.3.re1

3.3.re2. Find the derivative of the given function.

a. $\dfrac{x}{3 + \cos x}$ b. $x^2 \cos x \sin x$ c. $e^x(x \sec x + \cot x)$

d. $\dfrac{x \cos x}{x^2 + 1}$ e. $\cos x \sin x \sec x \cot x$ f. $\dfrac{(\sin x + 1)}{\tan x}$

g. $3e^x + x^5 \csc x$ h. $e^x \cos x \sin x$

3.3.re3. Find the equation of the line tangent to the given function at the given x-value.

a. $2 \sin x - \cos x; \quad x = \pi/2$ b. $\tan x + \cot x; \quad x = \pi/4$

Answers

3.3.re2a. $\frac{3+\cos x + x \sin x}{(3+\cos x)^2}$ 3.3.re2b. $2x \cos x \sin x - x^2 \sin^2 x + x^2 \cos^2 x$ 3.3.re2c. $e^x(x \sec x + \sec x + x \sec x \tan x + \cot x - \csc^2 x)$ 3.3.re2d. $\frac{(\cos x - x \sin x)(x^2+1) - 2x^2 \cos x}{(x^2+1)^2}$ 3.3.re2e. $-\sin x$ 3.3.re2f. $-\sin x - \csc^2 x$ 3.3.re2g. $3e^x + 5x^4 \csc x - x^5 \csc x \cot x$ 3.3.re2h. $e^x(\cos x \sin x - \sin^2 x + \cos^2 x)$ 3.3.re3a. $y - 2 = (x - \pi/2)$ 3.3.re3b. $y = 2$

3.4: The Chain Rule

Our final Combination law will allow us to differentiate the *composition* of two functions.
II. Combination laws (continued).

Chain Rule: If y is a differentiable function of u, and u is a differentiable function of x, then y is a differentiable function of x, and

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

If function notation, if f and g are differentiable functions, and if the range of g lies in the domain of f, then $f \circ g(x) = f\big(g(x)\big)$ is differentiable, and

$$\Big[f\big(g(x)\big)\Big]' = f'\big(g(x)\big)g'(x)$$

3.4.re1. Find the derivative of the given function.

a. $\cos x^2$

b. $\tan^2 x$

c. $x^2 e^{-\cos x}$

d. $\sqrt{x^2 - 5e^x}$

e. $(2x-1)^3(x^3-4x)^4$

f. $e^{x^2-e^x}$

g. $\sqrt{\dfrac{x+1}{x-1}}$

h. $\sin(\tan^3 x)$

i. $e^{2e^{3e^{4x}}}$

j. $e^{2+e^{3+e^{4+x}}}$

3.4.re2. At what x-values is the graph of $y = e^{\tan^2 x}$ horizontal?

3.4.re3. Use these values of $f(x)$ and $g(x)$ and their derivatives at $x = 1$, 2, and 3

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	3	-3	2	6
2	1	-4	3	7
3	2	-5	1	4

to evaluate the **derivatives** of the following functions at $x = 1$:

a. $g(f(x))$

b. $f(g(x))$

c. $f(x)g(x)$

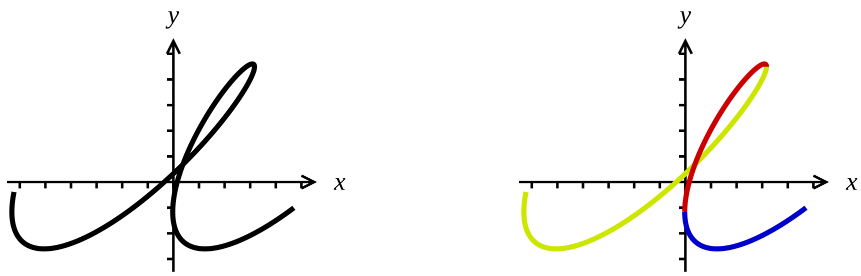
d. $f(x)\big(g(x)\big)^2$

Answers

3.4.re1a. $-2x\sin(x^2)$ 3.4.re1b. $2\tan x\sec^2 x$ 3.4.re1c. $e^{-\cos x}(2x + x^2\sin x)$ 3.4.re1d. $\frac{2x-5e^x}{2\sqrt{x^2-5e^x}}$
3.4.re1e. $6(2x-1)^2(x^3-4x)^4 + 4(2x-1)^3(x^3-4x)^3(3x^2-4)$, or
$(2x-1)^2(x^3-4x)^3\Big(6(x^3-4x) + 4(2x-1)(3x^2-4)\Big)$ 3.4.re1f. $(2x-e^x)e^{x^2-e^x}$
3.4.re1g. $\left(\frac{-1}{(x-1)^2}\right)\left(\frac{x-1}{x+1}\right)^{1/2}$ 3.4.re1h. $\cos(\tan^3 x)3\tan^2 x\sec^2 x$ 3.4.re1i. $\left(e^{2e^{3e^{4x}}}\right)\left(2e^{3e^{4x}}\right)\left(3e^{4x}\right)4$
3.4.re1j. $\left(e^{2+e^{3+e^{4+x}}}\right)\left(e^{3+e^{4+x}}\right)\left(e^{4+x}\right)$ 3.4.re2. Solution: set the derivative $2e^{\tan^2 x}\tan x\sec^2 x = 0$. Since $e^{\tan^2 x}$ and $\sec x$ are never zero, the only solutions are where $\tan x = 0$, that is, $x = n\pi$ for any integer n. 3.4.re3a. $g'(f(1))f'(1) = g'(3)f'(1) = 4(-3) = -12$ 3.4.re3b. $f'(g(1))g'(1) = f'(2)g'(1) = -4\cdot 6 = -24$ 3.4.re3c. $f'(1)g(1)+f(1)g'(1) = -3\cdot 2+3\cdot 6 = 12$ 3.4.re3d. $f'(1)(g(1))^2+f(1)2g(1)g'(1) = -3\cdot 4+3\cdot 4\cdot 6 = 60$

3.5: Implicit Differentiation and the Inverse Trig Functions

Implicit differentiation is a method to find $\frac{dy}{dx}$ along the graph of an equation. It's based on the fact that, even if the graph of an equation fails to be the graph of a function, it can



still consist of the graphs of *several* functions. So, if we differentiate both sides of an x-y equation, it's safe to assume that y is some unspecified function of x. The result is an equation in x, y, and $\frac{dy}{dx}$ that, regardless of the original x-y equation, is linear in $\frac{dy}{dx}$.

3.5.re1. Find $\frac{dy}{dx}$ along the curve $x^3 + xy - y^4 = 1$.

Solution:
Differentiate both sides with respect to x, remembering that y is some unspecified function of x. Wherever the product or chain rule says the derivative of y should appear, write $\frac{dy}{dx}$:

$$3x^2 + y + x\frac{dy}{dx} - 4y^3\frac{dy}{dx} = 0$$

Now solve for the unknown $\frac{dy}{dx}$:

$$\begin{aligned} 3x^2 + y &= -x\frac{dy}{dx} + 4y^3\frac{dy}{dx} = (4y^3 - x)\frac{dy}{dx} \\ \frac{3x^2 + y}{4y^3 - x} &= \frac{dy}{dx} \end{aligned}$$

end example 3.5.re1

3.5.re2. Determine which of the points $(1, 1)$ and $(0, -1)$ are on the curve in 3.5.re1. Then find the equation of the line tangent to the curve at that point.

Solution:
Test which of $x = 1$, $y = 1$ and $x = 0$, $y = -1$ is a solution to the equation in 3.5.re1:

$(1, 1):$

$$1^3 + 1 \cdot 1 - 1^4 = 1$$

$(0, -1):$

$$0^3 + 0(-1) - (-1)^4 = -1 \neq 1$$

So $(1, 1)$ in on the curve but $(0, -1)$ is not.
To find the slope the line, evaluate $\frac{dy}{dx} = \frac{3x^2+y}{4y^3-x}$ at $(1, 1)$ to find $\frac{dy}{dx} = \frac{4}{3}$. In point-slope form, the line is $y - 1 = \frac{4}{3}(x - 1)$.

end example 3.5.re2

3.5.re3. Find $\frac{dy}{dx}$ along the given curve.

a. $x^3 - 3x^2y - y^3 = 3$

b. $\dfrac{x+y}{x^2+y^2} = \sin y$

c. $e^{xy} = \cos(x^2+y^2)$

d. $\sqrt{x+2y} = \tan(2x+y)$

e. $x^2 + e^y = y$

Finding $\frac{d^2y}{dx^2}$ implicitly

3.5.re4. Find $\frac{d^2y}{dx^2}$ along the curve $\cos x + \cos y = 1$. Express your answer solely in terms of x and y.

Solution:
Differentiate implicitly and solve for $\frac{dy}{dx}$:

$$-\sin x - \sin y \frac{dy}{dx} = 0 \quad \Longrightarrow \quad \frac{dy}{dx} = -\frac{\sin x}{\sin y} = -\sin x \csc y$$

Differentiate implicitly again:

$$\frac{d^2y}{dx^2} = -\cos x \csc y - \sin x(-\csc y \cot y)\frac{dy}{dx}$$

To express $\frac{d^2y}{dx^2}$ in terms of x and y, replace $\frac{dy}{ds}$ with $-\sin x \csc y$:

$$\begin{aligned}\frac{d^2y}{dx^2} &= -\cos x \csc y - \sin x(-\csc y \cot y)(-\sin x \csc y) \\ &= -\cos x \csc y - \sin^2 x \csc^2 y \cot y\end{aligned}$$

end example 3.5.re4

3.5.re5. Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ along the given curve.

a. $x^2 + 2xy - 2y^2 = 1$

b. $e^x + 1 = e^y + y$

c. $e^{x+y} - e^y = x - 1$

Derivatives of the inverse trig functions

I. Catalog.

$(\sin^{-1} x)' = \dfrac{1}{\sqrt{1-x^2}}$	$(\cos^{-1} x)' = -\dfrac{1}{\sqrt{1-x^2}}$
$(\tan^{-1} x)' = \dfrac{1}{1+x^2}$	$(\cot^{-1} x)' = -\dfrac{1}{1+x^2}$
$(\sec^{-1} x)' = \dfrac{1}{x\sqrt{x^2-1}}$	$(\csc^{-1} x)' = -\dfrac{1}{x\sqrt{x^2-1}}$

3.5.re6. Find the derivative of the given function.

a. $\tan^{-1}(\sin^{-1} x)$ b. $(\tan^{-1} x)(\sin^{-1} x)$ c. $e^x \arccos x^3$

d. $\sec(\sec^{-1} x^2)$ e. $\cot(\arctan e^x)$

Answers

3.5.re3a. $\frac{dy}{dx} = \frac{x^2-2xy}{x^2+y^2}$ 3.5.re3b. $\frac{dy}{dx} = \frac{1-2x\sin y}{(x^2+y^2)\cos y+2y\sin y-1}$ 3.5.re3c. $\frac{dy}{dx} = -\frac{ye^{xy}+2x\sin(x^2+y^2)}{2y\sin(x^2+y^2)+xe^{xy}}$

3.5.re3d. $\frac{dy}{dx} = \frac{1-4\sec^2(2x+y)\sqrt{x+2y}}{-2+2\sec^2(2x+y)\sqrt{x+2y}}$ 3.5.re3e. $\frac{dy}{dx} = \frac{2x}{1-e^y}$.

3.5.re5a. $\frac{dy}{dx} = \frac{x+y}{2y-x}$; $\frac{d^2y}{dx^2} = \frac{(1+\frac{x+y}{2y-x})(2y-x)-(x+y)(2\frac{x+y}{2y-x}-1)}{(2y-x)^2}$.

3.5.re5b. $\frac{dy}{dx} = \frac{e^x}{1+e^y}$; $\frac{d^2y}{dx^2} = \frac{e^x(1+e^y)-e^x e^y \frac{e^x}{1+e^y}}{(1+e^y)^2}$.

3.5.re5c. $\frac{dy}{dx} = \frac{1-e^{x+y}}{e^{x+y}-e^y} = \frac{e^{-y}-e^x}{e^x-1}$. $\frac{d^2y}{dx^2} = \frac{(-e^{-y}\frac{e^{-y}-e^x}{e^x-1}-e^x)(e^x-1)-(e^{-y}-e^x)e^x}{(e^x-1)^2}$.

3.5.re6a. $(1+(\sin^{-1} x)^2)^{-1}(1-x^2)^{-1/2}$ 3.5.re6b. $\frac{\sin^{-1} x}{1+x^2} + \frac{\tan^{-1} x}{\sqrt{1-x^2}}$ 3.5.re6c. $e^x(\arccos x^3 - \frac{3x^2}{\sqrt{1-x^6}})$.

3.5.re6d. $2x$ 3.5.re6e. $-e^{-x}$

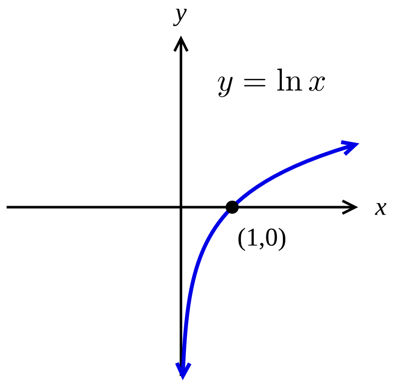
3.6: Derivatives of Logarithmic Functions

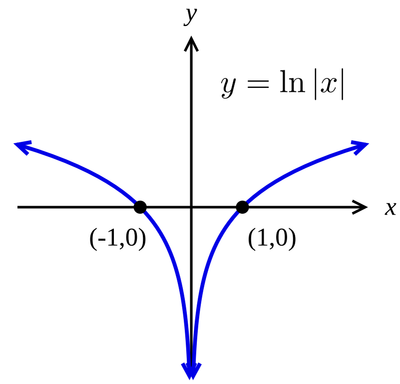
Two more entries for our catalog of functions and their derivatives.

I. Catalog.

$(\ln x)' = \dfrac{1}{x}$	$(\ln \lvert x\rvert)' = \dfrac{1}{x}$

The derivative of $\ln|x|$ has exactly the same algebraic formula as the derivative of $\ln x$, but these two functions are different. $\ln x$ and $\ln|x|$ agree on $(0,\infty)$, but, of the two, only $\ln|x|$ is defined on $(-\infty, 0)$.





3.6.re1. Find the derivative of the given function.

a. $\ln 2$ b. $\ln|\sin x|$ c. $\ln\big((2x^2+5x-3)^3\big)$

d. $\log_3 x$ e. 4^{2x-1} f. $(x+1)^{(x-1)}$

Solution:
a. $\ln 2$ is a constant, so its derivative is 0.
b. By the chain rule, $\frac{d}{dx}\ln|\sin x| = \frac{1}{\sin x}(\sin x)' = \frac{\cos x}{\sin x}$, or $\cot x$.
c. Simplify before differentiation.

$$\frac{d}{dx}\ln\big((x+3)^3(2x-1)^3\big) = \frac{d}{dx}\left(3\ln(x+3)+3\ln(2x-1)\right)$$
$$= 3\frac{1}{x+3} + 3\frac{1}{2x-1}2 = \frac{3}{x+3} + \frac{6}{2x-1}$$

d. Use the change of base formula to rewrite $\log_3 x = \frac{1}{\ln 3}\ln x$. The derivative of this is $\frac{1}{\ln 3}\frac{1}{x}$, or $\frac{1}{x\ln 3}$.
e. Any exponential function can be rewritten with base e using properties of logs:

$$y = 4^{2x-1} = e^{\ln 4^{2x-1}} = e^{(2x-1)\ln 4}$$

Now differentiate, remembering that $\ln 4$ is a constant:

$$\frac{dy}{dx} = e^{(2x-1)\ln 4}\big((2x-1)\ln 4\big)' = e^{(2x-1)\ln 4} 2\ln 4,$$

or, if you prefer, $4^{2x-1}\ln 16$.

f. When x appears in the exponent, rewrite with base e as above.

$$y = (x+1)^{(x-1)} = e^{\ln(x+1)^{(x-1)}} = e^{(x-1)\ln(x+1)}$$

Now differentiate using the chain and product rules:

$$\frac{dy}{dx} = e^{(x-1)\ln(x+1)}\left(\ln(x+1) + \frac{x-1}{x+1}\right).$$

end example 3.6.re1

3.6.re2. Find the derivative of the given function.

a. $\ln|x^3+5|$ b. $\ln|2x|$ c. $\ln(e^{-x}5x^2\sin x)$

d. $\ln|\sec x|$ e. $\log_2 3$ f. $e^{\frac{1}{2}\ln x}$

g. $\ln(e^{\cos x})$ h. $\ln\left(\frac{(3x-1)^4}{\sqrt{x+1}}\right)$ i. $\log_{10} x^2$

j. $x^{x/2}$ k. $(x^2+1)^{\sin x}$

Answers

3.6.re2a. $\frac{3x^2}{x^3+5}$ 3.6.re2b. $\frac{1}{x}$ 3.6.re2c. $-1+\frac{2}{x}+\cot x$ 3.6.re2d. $\tan x$ 3.6.re2e. 0 3.6.re2f. $\frac{1}{2}x^{-1/2}$
3.6.re2g. $-\sin x$ 3.6.re2h. $\frac{12}{3x-1} - \frac{1}{2}\ln(x+1)$ 3.6.re2i. $\frac{2}{x\ln 10}$ 3.6.re2j. $e^{\frac{1}{2}x\ln x}(\frac{1}{2}\ln x + \frac{1}{2})$
3.6.re2k. $e^{\sin x\ln(x^2+1)}(\cos x\ln(x^2+1) + \frac{2x\sin x}{x^2+1})$

3.7: Rates of Change in the Sciences

$\frac{dy}{dx}$ stands for the derivative, or rate of change, of the function y with respect to x. If it were constant, $\frac{dy}{dx}$ would equal the change in y corresponding to a 1 unit increase in x. The units of $\frac{dy}{dx}$ are the units of y per units of x.

3.7.re1. Find the units of the derivative, given those of x and y.

If the units of x are:	and the units of y are:	then the units of dy/dx are:
sec	m	m/sec
sec	m/sec	
century	g	
in^2	lb	
fortnight	hectacres	

3.7.re2. An object moving along a coordinate axis is at position $p(t) = (t^2 - 4)^2$ meters at time t seconds. Assume that $-\infty < t < \infty$.

a. Express the object's velocity, speed, and acceleration as functions of t. Label your answers and include the correct units.

b. At what time(s), if any, is velocity zero?

c. At what time(s), if any, is acceleration zero?

d. Over what interval(s) of time is the object moving in the positive direction?

e. Sketch a schematic diagram of the object's motion between times $t = -2$ and 3.

f. What is the object's *net* distance travelled between $t = -2$ and $t = 3$?

g. Find the total distance the object travels between times $t = -2$ and $t = 3$.

Solution:

a. Position $= p = (t^2 - 4)^2$ m; velocity $= v = p' = 2(t^2 - 4)2t = 4t(t^2 - 4) = 4t^3 - 16t$ m/sec; speed is defined as the absolute value of velocity $= |4t^3 - 16t|$ m/sec; acceleration $= a = p'' = 12t^2 - 16$ m/sec^2.

b. Factor v and set equal zero: $v = 2(t^2 - 4)2t = 4t(t - 2)(t + 2) = 0$ at $t = -2$, 0, and 2.

c. $a = 12t^2 - 16 = 4(3t^2 - 4) = 4(\sqrt{3}t - 2)(\sqrt{3}t + 2) = 0$ at $t = \pm 2/\sqrt{3}$.

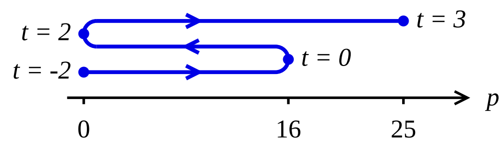
d. The object is moving in the positive direction when position is increasing. This is when the derivative of position, or velocity, is positive. Make a sign chart for v as in precalculus:

$t + 2$:	− − − − − − − − − − −−0 + + + +
t :	− − − − − − −−0 + + + + + + + +
$t - 2$:	− − − − −0 + + + + + + + + + + + +
$4t(t - 2)(t + 2)$:	− − −−0 + + + +0 − − − −0 + + + +
t :	−2 0 2

So, the object is moving forward for t in $(-2, 0)$ and in $(2, \infty)$.

e. The object moves forward from $t = -2$ to $t = 0$, then backward between $t = 0$ and $t = 2$, and then forward from $t = 2$ to $t = 3$. Calculate p at these times and draw the object's motion on a schematic diagram like so:

t	-2	0	2	3
$p(t)$	0	16	0	25



f. Net distance traveled, or **displacement**, between $t = -2$ and $t = 3$ is the change in position: $p(3) - p(-2) = 25 - 0 = 25\text{m}$.

g. Total distance traveled, as the diagram illustrates, between $t = -2$ and $t = 3$ is 16 forward +16 backward +25 forward = 57m.

end example 3.7.re2

3.7.re3. The given function is the position s at time t of an object moving along an axis. Assume $-\infty < t < \infty$. Find the following.

(1). The velocity, speed and acceleration of the object at time t.

(2). When the object is moving forward and when it is moving backward.

(3). A schematic diagram of the object's motion on $-\infty < t < \infty$.

(4). The net and total distance traveled by the object between times 0 and 3.

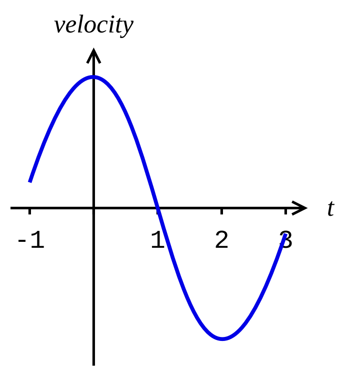
a. $t^4 - 8t^2 + 16$ b. $\frac{4}{3}t^3 - 36t$ c. $\frac{t^2}{t^2+4}$ d. $\frac{t}{t^2+1}$

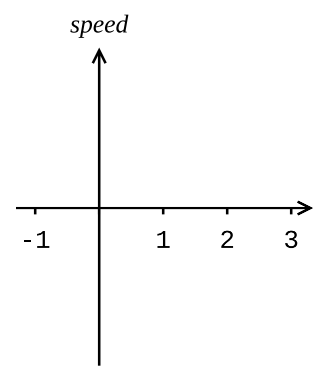
Be careful to distinguish between velocity and speed, the absolute value of velocity.

3.7.re4. An object moves along a coordinate axis from time $t = -1$ to $t = 3$. The figure on the left shows the graph of the object's velocity.

a. On the axes provided, sketch the graph of the object's speed.

b. Over what interval(s) of time is the object speeding up?

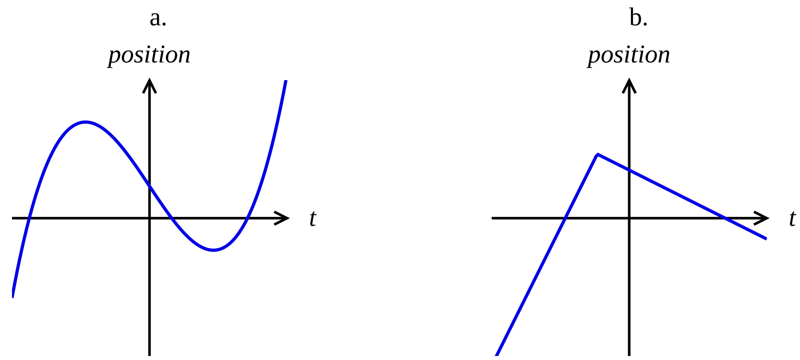




3.7.re5. A ball launched upward is at $s = 140 + 60t - 5t^2$ meters at time t seconds after launch. Find the following.

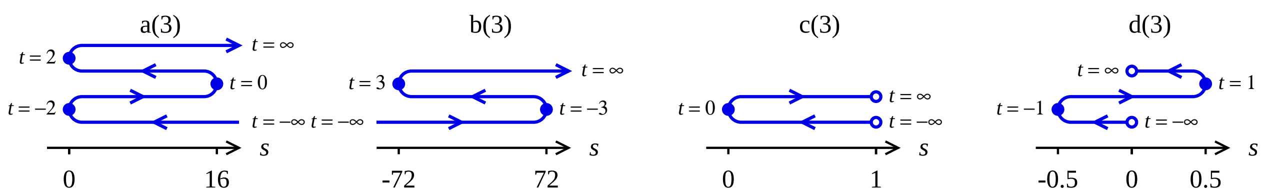
a. The ball's velocity and acceleration at time t, and the units of each.
b. When the ball is traveling upward and when it is traveling downward.
c. When the ball was launched, and when it hit the ground?
d. A schematic diagram of the ball's motion
e. The graph of the ball's velocity and speed.
f. When the ball's velocity is increasing.
g. When the ball is speeding up and when it is slowing down.

3.7.re6. Graph velocity and speed for the given position function.

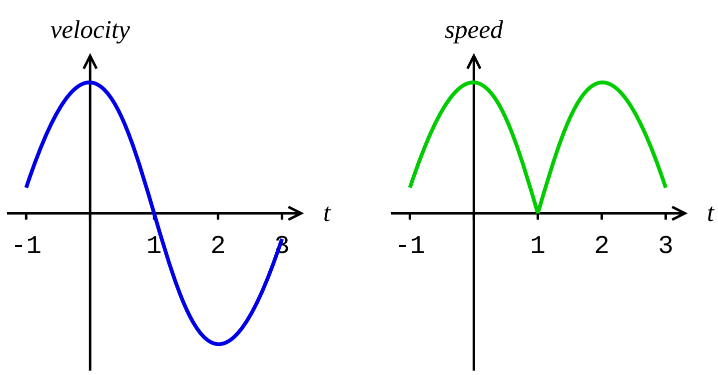


Answers

3.7.re1. m/sec^2, g/century, lb/in^2, hectacres/fortnight 3.7.re3a. (1) $v = 4t^3 - 16t$. $|v| = |4t^3 - 16t|$. $a = 12t^2 - 16$. (2) Forward when $-2 < t < 0$ or $2 < t$. Backward when $-\infty < t < -2$ or $0 < t < 2$. (3) below. (4) Net $= s(3) - s(0) = 9$. Total $= 41$. 3.7.re3b. (1) $v = 4(t^2 - 9)$. $|v| = 4|t^2 - 9|$. $a = 8t$. (2) Forward when $t < -3$ or $3 < t$. Backward when $-3 < t < -3$. (3) below. (4) Net $= s(3) - s(0) = -72$. Total $= 72$. 3.7.re3c. (1) $v = \frac{8t}{(t^2+4)^2}$. $|v| = \frac{8|t|}{(t^2+4)^2}$. $a = \frac{32-24t^2}{(t^2+4)^3}$. (2) Forward when $0 < t$. Backward when $t < 0$. (3) below. (4) Net $= s(3) - s(0) = \frac{9}{13}$ = Total. 3.7.re3d. (1) $v = \frac{1-t^2}{(t^2+1)^2}$. $|v| = \frac{|1-t^2|}{(t^2+1)^2}$. $a = \frac{2t^3-6t}{(t^2+1)^3}$. (2) Forward when $-1 < t < 1$. Backward when $t < -1$ or $1 < t$. (3) below. (4) Net $= s(3) - s(0) = \frac{3}{10}$. Total $= \frac{7}{10}$.

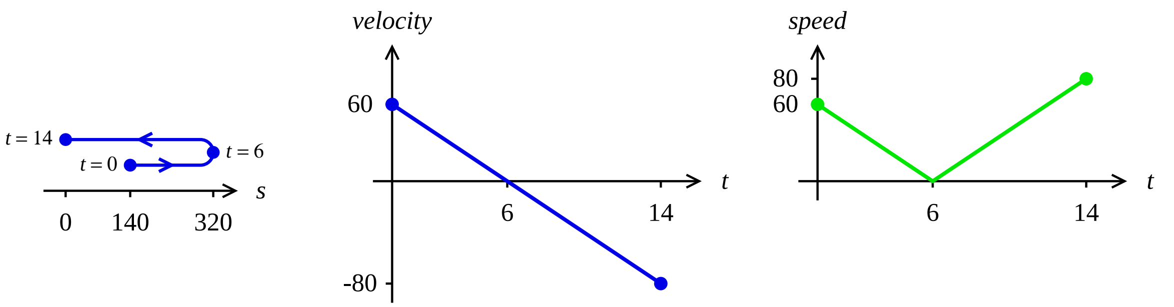


3.7.re4. a. speed = |velocity|. See graph:



b. The object is speeding up when speed is increasing. That is, when $-1 \le t \le 0$ and $1 \le t \le 2$. 3.7.re5a. $v = 60 - 10t$ m/sec. $a = -10$ m/sec^2. 3.7.re5b. Up when $t < 6$. Down when $t > 6$. 3.7.re5c. Question implies launch is at time $t = 0$. Ball reaches ground level when $0 = 140 + 60t - 5t^2 = -5(t-14)(t+2)$. Since $t = -2$ is before time of launch, the only relevant solution is $t = 14$ sec.

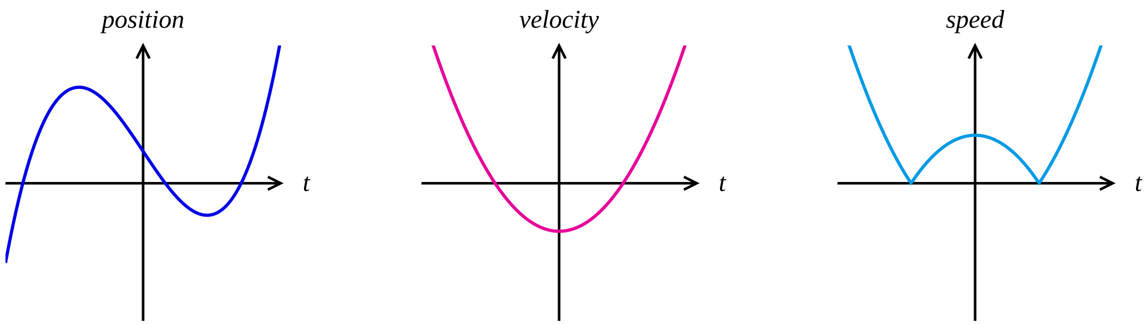
3.7.re5d. Assuming the ball stops when it strikes the ground, schematic diagram looks like the one below.



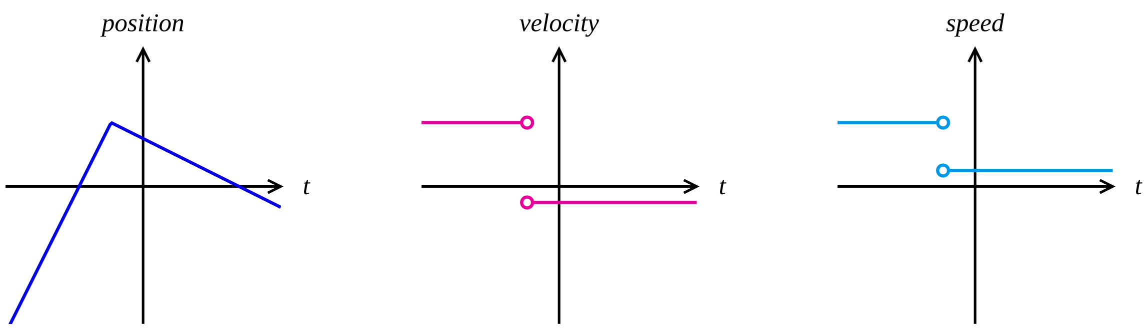
3.7.re5e. Above 3.7.re5f. Velocity is never increasing 3.7.re5g. Ball is slowing down when its speed is decreasing, or $0 < t < 6$. It is speeding up when its speed is increasing, or $6 < t < 14$.

3.7.re6.

a.



b.



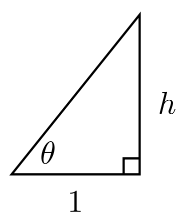
3.9: Related Rates

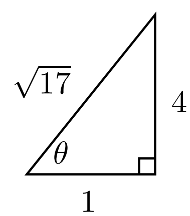
Tips for solving a related rates problem:

1. Draw a picture that's true for all time, not just one moment in time. To do this, you must read carefully and distinguish between quantities in your picture that are variable and those that are fixed. Watch out for particular values of variables that are true at only one moment.
2. Identify all rates referred to in the problem as derivatives of variable quantities in your picture (lengths, areas, volumes, angle measurements, etc.) and make those quantities your variables. Find an equation* that relates just those variables **and no others**.
3. Differentiate implicitly with respect to time. *Then*, plug in any given values of variables at the moment in question and solve for the desired derivative.

3.9.re1. A rocket flies vertically away from its launchpad as an observer standing 1 km from the launchpad records the rocket's flight. How fast is the angle of elevation from the observer to the rocket increasing when the rocket is 4 km above the launchpad and traveling 5 km/sec?

Solution:
See below, left. Given $\frac{dh}{dt} = 5$ km/sec, we're asked for $\frac{d\theta}{dt}$ when $h = 4$.





We need an equation relating h and θ, and it comes from the definition of the tangent:

$$\frac{h}{1} = \tan\theta.$$

Now differentiate both sides implicitly with respect to the variable time t:

$$\frac{dh}{dt} = \sec^2\theta\frac{d\theta}{dt}$$

Use Pythagorus to find hypotenuse when $h = 4$, as above, right. At that moment, $\sec\theta = \frac{1}{\cos\theta} = \sqrt{17}$. Plug this and $\frac{dh}{dt} = 5$ into the equation immediately above and solve for $\frac{d\theta}{dt}$:

$$5 = \sqrt{17}^2\frac{d\theta}{dt} \quad\Longrightarrow\quad \frac{d\theta}{dt} = \frac{5}{17}.$$

That is, at the moment in question, the angle of elevation from the observer to the rocket is increasing $\frac{5}{17}$ radians per second.

end example 3.9.re1

* The most common sources of equations in these problems is the pythagorean theorem, similar triangles, right-triangle trigonometry, 3-D geometry formulas, and, occasionally, the law of cosines. See the trig review earlier in these notes.

3.9.re2. A 10-ft long ladder is leaning against a vertical wall on horizontal ground. The bottom of the ladder is pulled away from the wall at a rate of 1 ft/sec.
a. How fast is the ladder sliding down the wall when the bottom is 6 ft from the wall?
b. How fast is the area of the triangle formed by the ladder, wall and ground changing at that moment? Is the area increasing or deceasing?

(See an animation of this at https://www.desmos.com/calculator/akotvavges Click the play button on line 2. Notice that if the bottom of the ladder moves at a constant rate, the top does not.)

3.9.re3. A duck flying at altitude 50 m is moving horizontally 3 m/sec away from a hunter on the ground.
a. How fast is the distance between the hunter and her prey increasing when the duck is 130 m away from her?
b. At that moment, how fast is the angle of elevation from the hunter to the duck decreasing in radians/sec?

3.9.re4. A conical paper cup has height 3 in and radius 1.5 in. Water is leaking from the bottom of the cup $\frac{1}{18}$ in^3/sec. Find how fast the radius of the water's surface decreasing when the water is 2 in deep.
Note: the volume of a cone with height h and radius r is $\frac{1}{3}\pi r^2 h$.

3.9.re5. Two sides of a triangle have lengths 2 and 3 cm. The angle between the two sides is increasing 2 radian/sec.
a. How fast is the length of the third side changing when that angle is 60°? Is the length increasing or decreasing at that time?
b. How fast is the area of the triangle changing when that angle is 30°? Is the area increasing or decreasing at that time?

3.9.re6. A lighthouse stands on a small island 1 mile from a point P on a straight shoreline. If the light atop the light house rotates 2 revolutions per minute, how fast is the beam of light moving along the shore when it's shining on a point $\frac{3}{4}$ mile from P? (Animation at https://www.desmos.com/calculator/d09fdiiwah)

Answers

3.9.re2a. The height of the ladder is decreasing $\frac{3}{4}$ ft/sec 3.9.re2b. $\frac{7}{4}$ ft^2/sec. Since $dA/dt > 0$, the area is increasing. 3.9.re3a. $\frac{36}{13}$ m/sec 3.9.re3b. 3/338 radians/sec 3.9.re4. At the moment in question, the radius decreases $-1/(36\pi)$ in/sec. 3.9.re5a. Increasing $6\sqrt{3/7}$ cm/sec. 3.9.re5b. Increasing $3\sqrt{3}$ cm^2/sec. 3.9.re6. $25\pi/4$ mi/min.

3.10: Linear Approximation

Definition 3.10.re1. *If f is differentiable at a, then the* **linearization** *of f at a is the function*

$$L(x) = f(a) + f'(a)(x - a).$$

$L(x)$ is the function whose graph is the line tangent to $y = f(x)$ at $x = a$.

3.10.re2. The linearization of the $f(x) = x^{1/3}$ at 27 is

$$\begin{aligned} L(x) &= 27^{1/3} + \frac{1}{3}27^{-2/3}(x - 27) \\ &= 3 + \frac{1}{27}(x - 27) \end{aligned} \tag{3.10.re3}$$

end example 3.10.re2

3.10.re4. Find the linearization of the function at the given point.

a. $\frac{1}{1+2x}$, $a = 0$ b. $\ln x$, $a = 1$ c. $\sin x$, $a = \pi$

d. $\cos x$, $a = 0$ e. $\sqrt{x}$, $a = 16$ f. $e^{2x} + 4e^{3x}$, $a = 0$

Generally, **linear approximation** refers to the approximation of the function $f(x)$ by the function $L(x)$ for x near a, which we vaguely indicate by

$$f(x) \approx L(x) \qquad (x \approx a).$$

We can use linear approximation to approximate a number if (1) we recognize that number as $f(x)$ for some function f at some x-value, and (2) that x-value is near another x-value a at which we know the value of f and f'.

3.10.re5. We can estimate $\sqrt[3]{27.3}$ by replacing the cube root function with its linearization $L(x)$ at the nearby $x = 27$ (3.10.re3). That is, if $f(x) = x^{1/3}$, then

$$\sqrt[3]{27.3} = f(27.3) \approx L(27.3) = 3 + \frac{1}{27}(27.3 - 27) = 3 + \frac{1}{27}\frac{3}{10} = 3 + \frac{1}{90}.$$

end example 3.10.re5

3.10.re6. Use linear approximation to approximate the given number. You are not required to state your answer in decimal form. What function are you approximating, at what x is the function evaluated, and what a did you use?

a. $\sqrt{25.25}$ b. $\ln(1.05)$ c. $\cos(89°)$

Definition 3.10.re7. *The* **differential** *of* y *is* $\dfrac{dy}{dx}\,dx$*, which we write as* dy*.*

3.10.re8. If $y = \sin^2 x$, then $dy = 2\sin x \cos x\,dx$.

end example 3.10.re8

The differential is of great importance in Chapter 5.

3.10.re9. Find the differential of the function.

a. $\ln|\sec x|$ b. $e^{-x}\sin x$ c. $\ln(\sqrt{x})$ d. $\dfrac{x-1}{2x+1}$

Answers

3.10.re4a. $1-2x$ 3.10.re4b. $x-1$ 3.10.re4c. $-(x-\pi)$ 3.10.re4d. 1 3.10.re4e. $4+\frac{1}{8}(x-16)$.
3.10.re4f. $5+14x$ 3.10.re6a. $\sqrt{25.25} \approx 5 + \frac{1}{10}(25.25-25) = 5.025$, using $f(x)=\sqrt{x}$, $x = 25.25$, and $a = 25$.
3.10.re6b. $\ln(1.05) \approx 0.05$, using $f(x) = \ln x$, $x = 1.05$, and $a = 1$. 3.10.re6c. $\cos(89^\circ) = \cos(\frac{89\pi}{180}\text{ rad}) \approx \frac{\pi}{180}$, using $f(x) = \cos x$, $x = \frac{89}{180}\pi$, and $a = \frac{1}{2}\pi$. 3.10.re9a. $\tan x\,dx$ 3.10.re9b. $e^{-x}(\cos x - \sin x)\,dx$
3.10.re9c. $\frac{dx}{2x}$ 3.10.re9d. $3(2x+1)^{-2}\,dx$

4.1: Extrema

Vocabulary:

maxima/minima: plural of maximum/minimum
extremum (*pl* extrema): a maximum or minimum

We study two types of extrema in calculus: absolute and local.

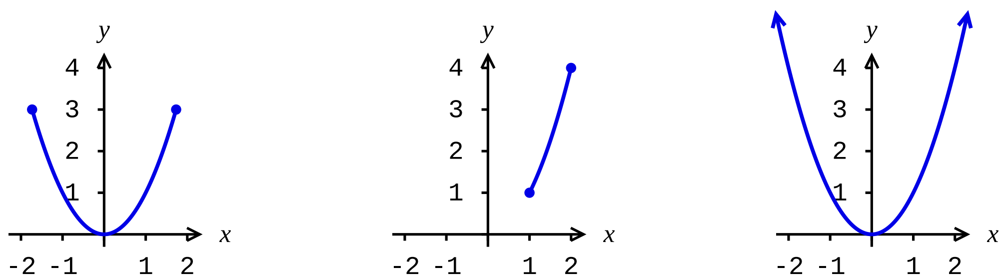
Definition 4.1.re1. *Suppose a function f is defined on some interval D. The* **absolute maximum** *of f on D is the largest value that f attains on D, if such a value exists.*

The **absolute minimum** *of f on D is its smallest value on D, if it exists.*

Definition 4.1.re2. *The function f has a* **local maximum** *at $x = c$ if $f(c)$ is the absolute maximum of f on some (possibly small) open interval containing c.*

f has a **local minimum** *at $x = c$ if $f(c)$ is the absolute minimum of f on some open interval containing c.*

4.1.re3. On the interval $[-\sqrt{3}, \sqrt{3}]$, the absolute maximum value of x^2 is 3 and the absolute maximum is 0 (below left).



On $[1, 2]$, the absolute maximum of x^2 is 4 and the absolute maximum is 1 (above center). The function x^2 has a local minimum at $x = 0$ but no local maximum (above right).

end example 4.1.re3

Notes:

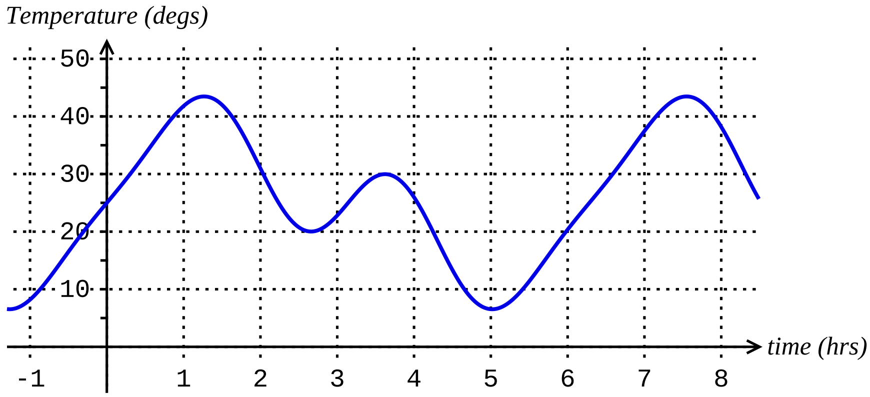
Extrema are function values, not input values.

The *local* extrema of a function are the altitudes of the mountain tops and valleys in its graph.

The *absolute* extrema depend on the function *and* the interval.

On any given interval, the absolute maximum, if it exists, is a unique y-value, although it may occur at more than one x-value. Similarly, the absolute minimum is unique.

4.1.re4. In the graph below, temperature T is a function of time t.



Estimate the absolute extrema of T on the given intervals of t.

	$[0,3]$	$[2,4]$	$[0,8]$	$[6,8]$	$(6,8]$
abs max T					
abs min T					

4.1.re5. Identify the t-values at which the function T in 4.1.re4 attains a local max or local min.

Definition 4.1.re6. *c is a* **critical number** *for f if $f'(c)$ either $= 0$ or does not exist*

Theorem 4.1.re7. *A function can attain local extrema only at its* **critical numbers**.

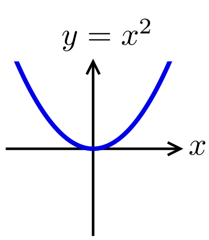
4.1.re8. Not every critical point is the location of a local extremum. For example, each of these functions

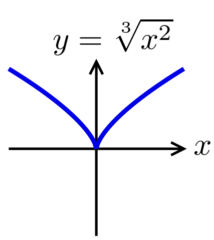
$$f(x) = x^2 \qquad g(x) = x^{2/3} \qquad h(x) = x^{1/3}$$

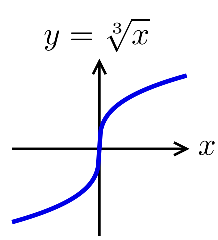
has a critical point at $x = 0$. To see why, take their derivatives:

$$f'(x) = 2x \qquad g'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}} \qquad h'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$$

and observe that $f'(0) = 0$ and that $g'(0)$ and $h'(0)$ are undefined. Both x^2 and $x^{2/3}$ have local minima at the critical point $x = 0$, but $x^{1/3}$ has no local extremum (see below).







end example 4.1.re8

In section 4.2, we'll see how to determine whether f has a local extrema at a critical point.

4.1.re9. Find the critical points of the given function.

a. $4x^3 - 15x^2 - 18x$ b. x^2e^x c. $x^{2/3} - x^{8/3}$ d. $e^x \sin x$ on $[0, 2\pi]$ only

Finding absolute extrema on a closed interval

Theorem 4.1.re10. *If the function f is continuous on the closed interval $[a, b]$, then f attains an absolute maximum and an absolute minimum on that interval. Furthermore, the absolute extrema of $f(x)$ on $[a, b]$ can only occur at the endpoints (a and b) or critical points in the interior (a, b).*

4.1.re11. Find the absolute exrema of the function $u(t) = 4t - 3t^{1/3}$ on the interval $[0, 8]$.

Solution:
First, find the critical points of $u(t)$ in $(0, 8)$. These are those t-values where $u'(t) = 4 - t^{-2/3}$ is either zero or undefined. $t = 0$ is a critical point because $t^{-2/3}$ is undefined there. To find any others, set $u'(t) = 0$ and solve:

$$0 = 4 - t^{-2/3} \implies t^{-2/3} = 4 \implies t^{2/3} = \frac{1}{4}.$$

Take $\pm$ the square root of both sides and cube:

$$t^{1/3} = \pm\frac{1}{2} \implies t = \pm\frac{1}{8}.$$

Therefore, the critical points of $u(t)$ are $-\frac{1}{8}$, 0, and $\frac{1}{8}$. Of these, only $\frac{1}{8}$ lies in $(0, 8)$.

The absolute maximum and minimum values of $u(t)$ on $[0, 8]$ can occur only at the endpoints or at critical points interior to the the interval, so we only need to compute and compare the values of $u(t)$ at 0, $\frac{1}{8}$, and 8:

t	$4t - 3t^{1/3}$
0	0
$\frac{1}{8}$	$\frac{4}{8} - 3\sqrt[3]{\frac{1}{8}} = \frac{1}{2} - \frac{3}{2} = -1$
8	$4 \cdot 8 - 3\sqrt[3]{8} = 32 - 6 = 26.$

Therefore the absolute maximum of u on $[0, 8]$ is 26 and the absolute minimum is -1.

end example 4.1.re11

4.1.re12. Find the absolute extrema of the function on the given interval.

a. $4x^3 - 15x^2 - 18x \quad [-1, 1]$

b. $x^{2/3} - x^{8/3} \quad [-1, 1]$

c. $\frac{1}{4}x + x^{-1} \quad [1, 3]$

d. $\cos x + \cos^2 x \quad [0, 2\pi]$

Answers

4.1.re4.

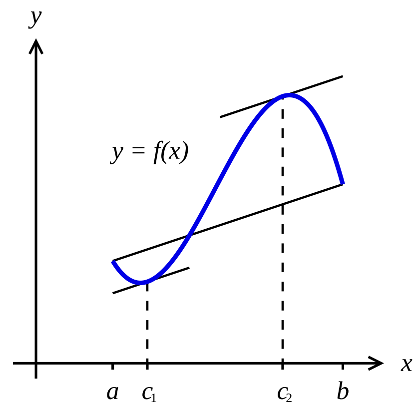
	$[0,3]$	$[2,4]$	$[0,8]$	$[6,8]$	$(6,8]$
abs max T	44	30	44	44	44
abs min T	20	20	8	20	(dne)

4.1.re5. T has local max at (approximately) $t = 1.3$, 3.6, and 7.5, and a local min at approximately $t = 2.7$ and 5. The values of the local maxima are (approximately) 44, 30, and 44. The local minima are approximately 20 and 8. 4.1.re9a. $x = -1/2$, $x = 3$. 4.1.re9b. $x = 0$, $x = -2$. 4.1.re9c. $x = 0$ (where $\frac{dy}{dx}$ dne) and $x = \pm 1/2$ (where $\frac{dy}{dx} = 0$). 4.1.re9d. Critical numbers are those x in $[0, 2\pi]$ at which $\sin x = -\cos(x)$. These are $x = \frac{3}{4}\pi$ and $\frac{7}{4}\pi$. 4.1.re12a. abs. max. $= \frac{19}{4}$. abs. min. $= -29$. 4.1.re12b. abs. min. $= 0$. abs. max. $= \frac{1}{\sqrt[3]{2^2}} - \frac{1}{\sqrt[3]{2^8}}$ 4.1.re12c. abs. max. $= 5/4$. abs. min. $= 1$. 4.1.re12d. abs. max. $= 2$. abs. min. $= -\frac{1}{4}$.

4.2: The Mean Value Theorem

Mean Value Theorem (MVT) 4.2.re1. *If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists at least one number c in (a, b) for which*

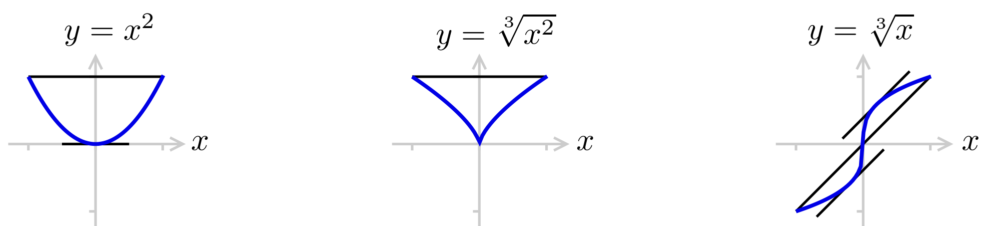
$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



The conclusion of the MVT says that, for some c in (a, b), the line tangent to $y = f(x)$ at $x = c$ is parallel to the secant line joining the points $(a, f(a))$ and $(b, f(b))$.
In the special case that $f(a) = f(b)$, the MVT guarantees c in (a, b) at which $f'(c) = 0$. This is known as **Rolle's Theorem**.

4.2.re2. The function $f(x) = x^2$ satisfies the hypotheses of the MVT on the interval $[-1, 1]$, since x^2 is continuous on $[-1, 1]$ and differentiable on $(-1, 1)$. (In fact, it is continuous and differentiable everywhere.) The number $c = 0$ satisfies the conclusion, since

$$f'(0) = \frac{f(1) - f(-1)}{2} = 0.$$



On the same interval, $g(x) = x^{2/3}$ violates the hypotheses, since $g'(x)$ does not exist at 0. Indeed, there is no $c \in (-1, 1)$ as in the conclusion, since $\frac{g(1)-g(-1)}{2} = 0$ but

$$g'(c) = \frac{2}{3c^{1/3}} = 0$$

has no solutions.
The function $h(x) = x^{1/3}$ also violates the hypotheses of the MVT, since $h'(0)$ does not exist. However, the conclusion of the MVT is still true. Find the two numbers $c \in (-1, 1)$ satisfying

$$\frac{1}{3}c^{-2/3} = \frac{1^{1/3} - (-1)^{1/3}}{1 - (-1)}.$$

Why does this not violate the MVT?

end example 4.2.re2

4.2.re3. Check whether the function satisfies the hypothesis of the MVT on the given interval. If it does, find the number(s) c as in the conclusion.

a. $f(x) = x^3 + x + 2 \quad [0, 2]$

b. $g(x) = 2x^{2/3} - x^{5/3} \quad [0, 2]$

c. $h(x) = \sqrt[3]{\cos x} \quad [0, \pi]$

d. $\ell(x) = \sqrt{x+1} \quad [1, 3]$

Answers

4.2.re2. $c = \pm\left(\frac{1}{3}\right)^{3/2}$. In general, when its hypotheses are not satisfied, MVT makes no conclusion; c may or may not exist. 4.2.re3a. $c = 2/\sqrt{3}$. 4.2.re3b. $c = 4/5$.
4.2.re3c. fails to satisfy hypothesis: $h'(x)$ does not exist at $\pi/2$. 4.2.re3d. $c = -1 + (2 - \sqrt{2})^{-2}$.

4.3: What the Derivative Tells about the Graph of a Function

Definition 4.3.re1.

$$f \textbf{ increasing } \textit{means:} \quad a < b \implies f(a) < f(b)$$
$$f \textbf{ decreasing } \textit{means:} \quad a < b \implies f(a) > f(b)$$

Theorem 4.3.re2. *Suppose f is differentiable on an interval I. Then:*

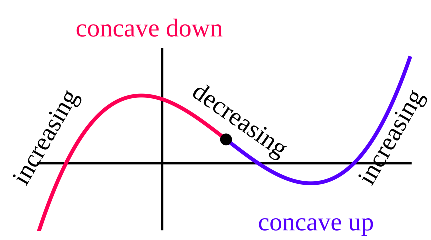
a. $f' > 0$ *on* $I \implies f$ *is increasing on* $I \implies f' \geq 0$ *on* I.
b. $f' < 0$ *on* $I \implies f$ *is decreasing on* $I \implies f' \leq 0$ *on* I.

That is, the sign of the first derivative determines a function's **monotonicity** (increasingness and decreasingness). The *second* derivative determines a function's **concavity**:

Definition 4.3.re3. *The graph of f is* **concave up** *when f' is increasing (or $f'' > 0$), and* **concave down** *when f' is decreasing (or $f'' < 0$).*
An **inflection point** *is a point on the graph of f where f changes concavity (and is continuous).*

Monotonicity and concavity are independent, in that a function can be any of

increasing and concave up,	increasing and concave down,
decreasing and concave up, or	decreasing and concave down.



Generally, a curve is concave up where its tangent lines lie below the curve, and concave down where its tangent lines lie above the curve. You can see this on an interactive graph of the above function at https://www.desmos.com/calculator/necjpvhc6j

1st Derivative Test for Local Extrema 4.3.re4. *Suppose c is a critical point of f. If f' changes sign at c:*

$$\begin{array}{rc} f'(x): & - - - - + + + + \\ \hline x: & c \end{array} \implies f \text{ has a local minimum at } c.$$

$$\begin{array}{rc} f'(x): & + + + + - - - - \\ \hline x: & c \end{array} \implies f \text{ has a local maximum at } c.$$

If f' does not change sign at c, then f does not have a local extremum at c.

2nd Derivative Test for Local Extrema 4.3.re5. *Suppose $f'(c) = 0$.*

If $f''(c) > 0$, then $f(c)$ is a local minimum.

If $f''(c) < 0$, then $f(c)$ is a local maximum.

Observe that the Second Derivative Test is inconclusive if $f''(c) = 0$.

In general, the First Derivative Test and Second Derivative Test are not useful in determining *absolute* extrema.

4.3.re6. We can determine the monotonicity, concavity, local extrema, and inflection points of $g(x) = xe^{-x}$ by analyzing its first and second derivatives.

Make a sign chart for $g'(x) = e^{-x} + x(-e^{-x}) = e^{-x}(1-x)$:

$$\begin{array}{rc} (1-x): & + + + + + 0 - - - - - \\ e^{-x}: & + + + + + + + + + + + \\ e^{-x}(1-x): & + + + + + 0 - - - - - \\ \hline x: & 1 \end{array}$$

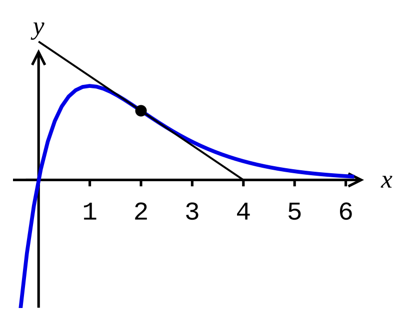
Consequently, $g(x)$ is increasing on $(-\infty, 1)$ and decreasing on $(1, \infty)$. The first derivative test tells us that g has a local max at $x = 1$ and no other local extrema.

Now make the sign chart for $g''(x) = -e^{-x}(1-x) + e^{-x}(-1) = e^{-x}(x-2)$:

$$\begin{array}{rc} (x-2): & - - - - - 0 + + + + + \\ e^{-x}: & + + + + + + + + + + + \\ e^{-x}(x-2): & - - - - - 0 + + + + + \\ \hline x: & 2 \end{array}$$

Therefore the graph of $g(x)$ is concave down on $(-\infty, 2)$ and concave up on $(2, \infty)$ and has an inflection point at $x = 2$, $y = g(2) = 2e^{-2}$.

Here's a graph of $g(x)$. Note the relative max at $x = 1$ and the change in concavity at $x = 2$. The line tangent to the curve at the inflection point lies above the curve on one side and below the curve on the other.



end example 4.3.re6

4.3.re7. Determine the intervals of increase and decrease and the local extrema for the given function.

a. $f(x) = x(x-3)^2$

b. $g(x) = \dfrac{1}{2}x + \cos x,\ [0, 2\pi]$ only

c. $\ell(x) = \dfrac{1}{x^2 - 4}$

d. $k(x) = x^{1/2} - x^{3/2}$

4.3.re8. Determine where the function is concave up and down. What are its inflection points?

a. $f(x) = x(x-3)^2$

b. $g(x) = \dfrac{1}{2}x + \cos x,\ [0, 2\pi]$ only

c. $\ell(x) = \dfrac{1}{x^2 - 4}$

d. $k(x) = x^{1/2} - x^{3/2}$

4.3.re9. Sketch the graph of the functions in examples 4.3.re7 and 4.3.re8, showing intervals of monotonicity and concavity and local extrema and inflection points. Watch out for vertical asymptotes or tangent lines. Check your results against graphs produced by Desmos.

4.3.re10. a. Sketch the graph of function g defined on $(-\infty, \infty)$ satisfying

$g'(x) < 0$ if $x > 1$

$g'(x) > 0$ if $x < 1$

$g'(x) \to \infty$ as $x \to 1^-$

$g'(x) \to -\infty$ as $x \to 1^+$

$g''(x) > 0$ if $x \neq 1$

$g''(x)$ dne at $x = 1$

4.3.re10, continued. b. Sketch the graph of function f satisfying

$f'(x) < 0$ if $x < 0$

$f'(x) > 0$ if $0 < x < 4$ or $4 < x$

$f'(x) = 0$ at $x = 0, 4$

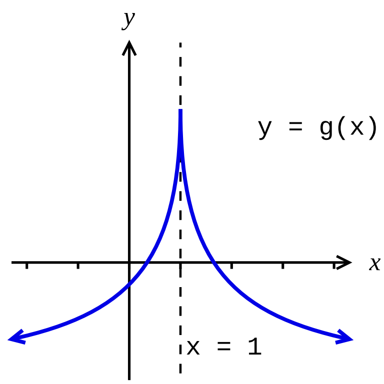
$f''(x) < 0$ if $x < -2$ or $2 < x < 4$

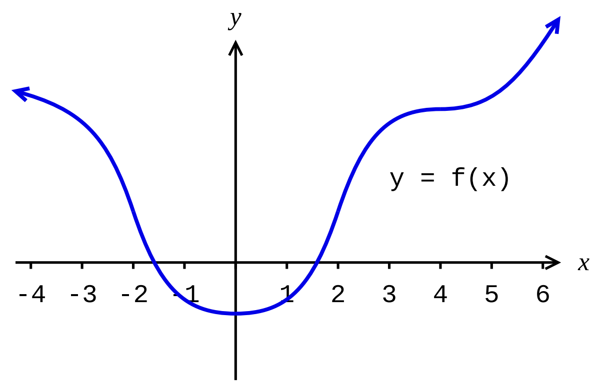
$f''(x) > 0$ if $-2 < x < 2$ or $4 < x$

$f''(x) = 0$ at $x = -2, 2, 4$

Answers

4.3.re7a. increases on $(-\infty, 1]$ and on $[3, \infty)$. deceases on $[1, 3]$. local max is 4 (at $x = 1$) and local min is 0 (at $x = 3$). 4.3.re7b. increases on $[0, \pi/6]$ and on $[5\pi/6, 2\pi]$. deceases on $[\pi/6, 5\pi/6]$. local max is $\pi/12 + \sqrt{3}/2$ at $x = \pi/6$; local min is $5\pi/12 - \sqrt{3}/2$ at $x = 5\pi/6$. 4.3.re7c. Curve has vertical asymptotes at $x = \pm 2$. ℓ increases on $(\infty, -2)$ and on $(-2, 0]$. deceases on $[0, 2)$ and on $(2, \infty)$. local max $= -1/4$ at $x = 0$. no local min. 4.3.re7d. increases on $[0, 1/3]$. deceases on $[1/3, \infty)$. local max is $2/(3^{3/2})$ at $x = 1/3$. no local min. 4.3.re8a. concave up on $(2, \infty)$, concave down on $(-\infty, 2)$. inflection point is at $x = 2$. 4.3.re8b. concave down on $[0, \pi/2]$ and on $[3\pi/2, 2\pi]$. concave up on $[\pi/2, 3\pi/2]$. inflection points at $x = \pi/2$ and $x = 3\pi/2$ 4.3.re8c. concave up on $(-\infty, -2)$ and on $(2, \infty)$. concave down on $(-2, 2)$ no inflection points (since $x = \pm 2$ are not points on the curve).
4.3.re8d. concave down on its entire domain, $[0, \infty)$. no inflection point.
4.3.re10.





4.4: l'Hospital's Rule and Indeterminant Forms.

Limit values and Indeterminate forms

4.4.re1. The function $\frac{\cos(1+h)-\cos 1}{h}$ does not exist at $h = 0$ because $\frac{0}{0}$ is not a number. However, as seen in 2.2 of these notes, $\lim_{h\to 0} \frac{\cos(1+h)-\cos 1}{h}$ exists and is approximately 0.841. We say that this limit has the form "$\frac{0}{0}$" because 0 is the limit of both the numerator and denominator. But there's no way for us to deduce value the limit (0.841) from the initial form of the limit ($\frac{0}{0}$).

4.4.re2. $\lim_{x\to\infty} \frac{2x-1}{x+3}$ initially has the form "$\frac{\infty}{\infty}$", but, as seen in 2.6.re10c, the value of the limit is 2. It's not possible to deduce the value of the limit from this initial form.

$$\frac{0}{0} \qquad \frac{\infty}{\infty} \qquad 0\cdot\infty \qquad \infty-\infty$$

are called **indeterminate forms** of limits, since they tell us nothing about the *value* or even the *existence* of such limits.
That's not simply because these forms are not numbers. For example, $\frac{\text{nonzero}}{0}$ and $\frac{\text{finite}}{\infty}$ are not numbers, but these forms tell us something about the limit value:

$$\text{A limit of the form } \frac{\text{nonzero}}{0} \text{ can only equal } \pm\infty;$$

$$\text{A limit of the form } \frac{\text{finite}}{\infty} \text{ must equal zero.}$$

Many interesting limits in calculus are of an indeterminate form. l'Hôpital's Rule is a tool to help us evaluate some of these.

l'Hospital's Rule 4.4.re3. *If* $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$ *has the form* $\dfrac{0}{0}$ *or* $\dfrac{\infty}{\infty}$, *and if* $\lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)} = L$, *then* $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$ *exists and also equals* L.

Notes on l'Hôpital's Rule

1. $x \to a$ can be replaced throughout by $x \to \infty$, $x \to -\infty$, $x \to a^-$, or $x \to a^+$.
2. L can be ∞, $-\infty$, or any real number.
3. l'Hospital's Rule applies only to the limits of the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Limits of the form $0\cdot\infty$ or $\infty-\infty$ would have to be rewritten as quotients for l'Hôpital's Rule to be of any use.
4. Don't read l'Hospital's Rule backwards. $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)} = L$ doesn't imply $\lim\limits_{x\to a} \dfrac{f'(x)}{g'(x)} = L$.

4.4.re4. Evaluate the limit.

a. $\lim_{x\to 0} \dfrac{e^{2x}-1-2x}{1-\cos x}$ b. $\lim_{x\to\infty} x^2 e^{-x^2}$ c. $\lim_{x\to\pi/2} \dfrac{1-\sin x}{1-2\sin x}$

Solution:

a. Two applications of l'Hôpital's Rule:

$$\frac{e^{2x}-1-2x}{1-\cos x} = \text{“}\frac{0}{0}\text{”} \overset{HR}{\hookrightarrow} \frac{2e^{2x}-2}{\sin x} = \text{“}\frac{0}{0}\text{”} \overset{HR}{\hookrightarrow} \frac{4e^{2x}}{\cos x} \longrightarrow \frac{4}{1}.$$

Therefore, the original limit must also equal 4.

b. Rewrite $x^2e^{-x^2}$ as a quotient so that l'Hospital's Rule is applicable:

$$\frac{x^2}{e^{x^2}} = \text{“}\frac{\infty}{\infty}\text{”} \overset{HR}{\hookrightarrow} \frac{2x}{2xe^{x^2}} = \frac{1}{e^{x^2}} = \text{“}\frac{1}{\infty}\text{”} = 0.$$

Therefore, the original limit must also equal 0.

c. $\sin\frac{\pi}{2} = 1$, so $\lim_{x\to 0} \dfrac{e^{2x}-1-2x}{1-\cos x} = \dfrac{1-1}{1-2} = 0$. l'Hôpital's Rule does not apply.

end example 4.4.re4

4.4.re5. Evaluate the limit.

a. $\lim_{x\to 0} \dfrac{e^{-x}-e^{-3x}}{x}$ b. $\lim_{x\to 4} \dfrac{2-\sqrt{x}}{16-x^2}$ c. $\lim_{x\to 0} \dfrac{e^{x}-e^{-x}-2x}{x-\sin x}$

d. $\lim_{x\to 1} \dfrac{x^{1/2}-x^{1/3}}{x^{1/3}-x^{1/4}}$ e. $\lim_{x\to 8} \dfrac{\sqrt{1+x}-x+5}{\sqrt{x+17}-x+3}$ f. $\lim_{x\to 0} \dfrac{x}{x-\sin x}$

g. $\lim_{x\to 0^-} \dfrac{x}{\cos x-1}$ h. $\lim_{x\to 0} \dfrac{x}{\cos x-1}$ i. $\lim_{x\to 2^+} \left(\dfrac{x}{x-2} - \dfrac{x^2}{x^2-4}\right)$

j. $\lim_{x\to\infty} \left(3x-\sqrt{9x^2-2x}\right)$ k. $\lim_{x\to\infty} \left(\ln(6x+1)-\ln(3x-1)\right)$ l. $\lim_{x\to\infty} \left(\ln x^3 - \ln x^2\right)$

m. $\lim_{x\to\infty} \left(\dfrac{\ln x^3}{\ln x^2}\right)$ n. $\lim_{x\to\infty} x^2e^{-x}$ o. $\lim_{x\to 0^+} x^3\ln x$

When $\lim_{x\to a}\frac{f(x)}{g(x)}$ is indeterminate but $\lim_{x\to a}\frac{f'(x)}{g'(x)}$ is not an easier problem, perhaps l'Hospital's Rule is not the way to go. Instead, try factoring out (what you believe to be) the dominant terms in the top and bottom, as in 2.6.re9.

p. $\lim_{x\to\infty} \dfrac{\sqrt{x^2+x}}{1+2x}$ q. $\lim_{x\to-\infty} \dfrac{\sqrt{x^2+x}}{1+2x}$

r. $\lim_{x\to\infty} \dfrac{2e^x-e^{-x}}{e^x+3e^{-x}}$ s. $\lim_{x\to-\infty} \dfrac{2e^x-e^{-x}}{e^x+3e^{-x}}$

Answers

4.4.re5a. 2. 4.4.re5b. 1/32. 4.4.re5c. 2. 4.4.re5d. 2. 4.4.re5e. 25/27. 4.4.re5f. ∞.
4.4.re5g. ∞. 4.4.re5h. does not exist. 4.4.re5i. ∞. 4.4.re5j. 1/3. 4.4.re5k. $\ln 2$.
4.4.re5l. ∞. 4.4.re5m. 3/2. 4.4.re5n. 0. 4.4.re5o. 0. 4.4.re5p. 1/2. 4.4.re5q. $-1/2$.
4.4.re5r. 2. 4.4.re5s. $-1/3$.

4.5: Summary of Curve Sketching

In this section, we'll graph functions without calculating more than a few points, using ideas from calculus and precalculus:

- Domain (the set of all x-values at which $f(x)$ exists)
- Intercepts
  - y-axis: $f(0) =?$
  - x-axis: $f(?) = 0$
- Symmetry of even and odd functions
  - $f(x)$ is even if $f(-x) = f(x)$.

    The graph of of an even function is symmetric across the y-axis.
  - $f(x)$ is odd if $f(-x) = -f(x)$.

    The graph of of an odd function is symmetric through the origin.
- Asymptotes
  - $x = c$ is a Vertical Asymptote if $|y| \to \infty$ as $x \to c$.
  - $y = c$ is a Horizontal Asymptote if $y \to c$ as either $x \to \infty$ or $x \to -\infty$.
- Monotonicity
  - $f(x)$ is increasing when $f'(x)$ is positive.
  - $f(x)$ is decreasing when $f'(x)$ is negative.
  - The local extrema of $f(x)$ are where its graph changes monotonicity.
- Concavity
  - The graph of $f(x)$ is concave up when $f''(x)$ is positive .
  - The graph of $f(x)$ is concave down when $f''(x)$ is negative.
  - The infection points of $f(x)$ are where its graph changes concavity.

See "Tips from precalculus for finding a function's domain," section 2.5 of these notes.

4.5.re1. Sketch the graph of the function $\rho(x) = \dfrac{x^2+4x+4}{x}$.

Solution:

Domain: $\rho(x)$ is defined for all x other than $x = 0$. That is, its domain $= (-\infty, 0) \cup (0, \infty)$.

Symmetry: $\rho(-x) = \dfrac{x^2-4x+4}{-x}$, which equals neither $\rho(x)$ nor $-\rho(x)$, so ρ is neither even nor odd.

Intercepts: No y-intercept, since $x \neq 0$ along this curve.
$y = 0 \Longrightarrow 0 = x^2 - 4x + 4 = (x-2)^2 \Longrightarrow x = 2$. That is, $(2, 0)$ is the only x-intercept.

Asymptotes: Because $\frac{\text{nonzero}}{0}$ indicates an infinite limit, $\lim_{x\to 0^\pm} \rho(x)$ is infinite, so the graph has a vertical asymptote at $x = 0$.
The graph has no horizontal asymptote because the limits

$$\lim_{x\to\pm\infty} \frac{x^2+4x+4}{x} = \text{“}\,\frac{\infty}{\infty}\,\text{”} \overset{HR}{\hookrightarrow} \lim_{x\to\pm\infty} \frac{2x+4}{1} = \pm\infty$$

are not finite. What's more,

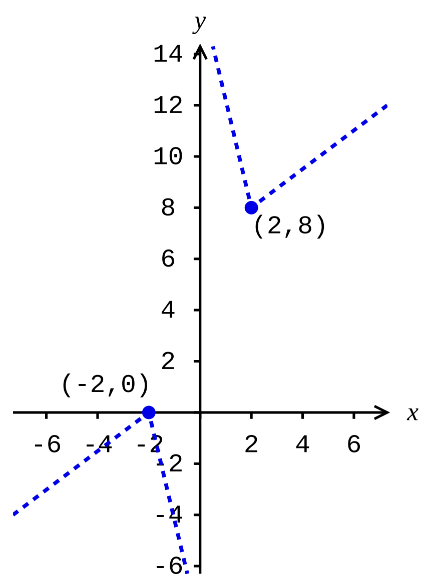
$$\begin{aligned} y \to \infty \quad &\text{as} \quad x \to \infty, \quad \text{and} \\ y \to -\infty \quad &\text{as} \quad x \to -\infty \end{aligned}$$

mean that, when we draw the graph of ρ on a finite screen, the graph should leave the screen in the upper right and lower left corners.

Monotonicity: To make differentiation easier, rewrite $\rho(x) = x + 4 + 4x^{-1}$. Then $\rho'(x) = 1 - 4x^{-2}$. Factor and make a sign chart for $\rho'(x) = x^{-2}(x^2-4) = \frac{(x+2)(x-2)}{x^2}$:

$x-2$:	$- - - - - - - - - - - - - - - 0 + + + +$
x^2 :	$+ + + + + + + 0 + + + + + +$
$x+2$:	$- - - - 0 + + + + + + + + + + + +$
$\frac{(x-2)(x+2)}{x^2}$:	$+ + + + 0 - - - -$ DNE $- - - - 0 + + + +$
x :	$-2 \qquad 0 \qquad 2$

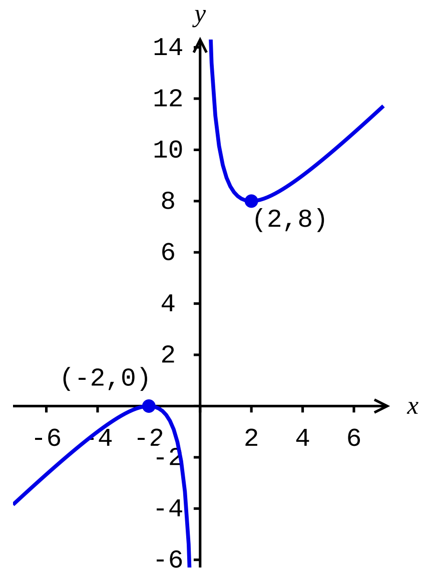
So, ρ is increasing on the intervals $(-\infty, -2)$ and $(2, \infty)$ and decreasing on the intervals $(-2, 0)$ and $(0, 2)$. ρ has a local maximum at $x = -2$ and a local minimum at $x = 2$. Evaluate ρ at ± 2, plot those points, and rough in the graph based what we know about its intercepts, asymptotes, and monotonicity.



Concavity: $\rho''(x) = 8x^{-3} = \frac{8}{x^3}$. Sign chart:

$$\begin{array}{rc} x^3: & - - - - - - 0 + + + + + \\ \frac{8}{x^3}: & - - - - \text{DNE} + + + + \\ \hline x: & 0 \end{array}$$

The graph of ρ is concave up on $(0, \infty)$ and concave down on $(-\infty, 0)$. (Since $\rho(x)$ is undefined at $x = 0$, it has no inflection point at $x = 0$.) Use this concavity information to smooth out your sketch of of the graph. Be careful not to violate the properties of ρ discovered earlier.



end example 4.5.re1

4.5.re2. For each of the following functions, determine the domain, intercepts, symmetry, asymptotes, monotonicity, concavity, and the x-values where the function has a local extremum or point of inflection. Sketch the function and compare your graph with the one produced by Desmos.

a. $x^4 - 9x^2$

b. $x^3 - x^2 - 6x$

c. $\dfrac{x}{x^2 - 9}$

d. $\dfrac{x^2}{x^2 + 4}$

e. $\dfrac{x - 2}{x + 3}$

f. $x^{-1}e^x$

g. $4e^x - e^{2x}$

h. e^{-x^2}

i. $x - 2\sin x$ on $[0, 2\pi]$

j. $\sec x - \cos x$ on $[0, 2\pi]$

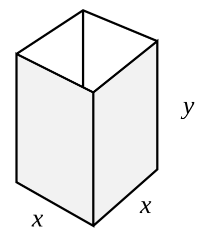
Answers

4.5.re2a. **domn** $(-\infty,\infty)$. **intrcpt** $(0,0)$, $(3,0)$, $(-3,0)$. **symm** across the y-axis. **asymp** none. **mono** increases on $(-3/\sqrt{2},0)$ and $(3/\sqrt{2},\infty)$. decreases elsewhere. **concav** up on $(-\infty,-\sqrt{3/2})$ and $(\sqrt{3/2},\infty)$. down elsewhere. **loc extrm at** $x=0$ (max) and $x=\pm 3/\sqrt{2}$ (min). **infl pts at** $x=\pm\sqrt{3/2}$. 4.5.re2b. **domn** $(-\infty,\infty)$. **intrcpt** $(0,0)$, $(3,0)$, $(-2,0)$. **symm** none. **asymp** none. **mono** increases on $(-\infty,\frac{1}{3}(1-\sqrt{19}))$ and $(\frac{1}{3}(1+\sqrt{19}),\infty)$. decreases elsewhere. **concav** up on $(\frac{1}{3},\infty)$. down elsewhere. **loc extrm at** $x=\frac{1}{3}(1-\sqrt{19})$ (max) and $x=\frac{1}{3}(1+\sqrt{19})$ (min). **infl pts at** $x=1/3$. 4.5.re2c. **domn** $(-\infty,-3)\cup(-3,3)\cup(3,\infty)$. **intrcpt** $(0,0)$. **symm** through the origin. **asymp** $x=-3$, $x=3$, $y=0$. **mono** decreases on $(-\infty,-3)$ and on $(-3,3)$ and on $(3,\infty)$. increases nowhere. **concav** up on $(-3,0)$ and $(3,\infty)$. down elsewhere. **loc extrm at** none. **infl pts at** $x=0$. 4.5.re2d. **domn** $(-\infty,\infty)$. **intrcpt** $(0,0)$. **symm** across the y-axis. **asymp** $y=1$. **mono** increases on $(0,\infty)$. decreases elsewhere. **concav** up on $(-2/\sqrt{3},2/\sqrt{3})$. down elsewhere. **loc extrm at** $x=0$ (min). **infl pts at** $x=\pm 2/\sqrt{3}$. 4.5.re2e. **domn** $(-\infty,-3)\cup(-3,\infty)$. **intrcpt** none. **symm** none. **asymp** $y=1$. **mono** increases on $(-\infty,-3)$ and on $(-3,\infty)$. decreases nowhere. **concav** up on $(-\infty,-3)$. down elsewhere. **loc extrm at** none. **infl pts at** none. 4.5.re2f. **domn** $(-\infty,0)\cup(0,\infty)$. **intrcpt** none. **symm** none. **asymp** $x=0$, $y=0$ (as $x\to-\infty$). **mono** decreases on $(-\infty,0)$ and on $(0,1)$. increases on $(1,\infty)$. **concav** up on $(0,\infty)$. down elsewhere. **loc extrm at** $x=1$ (min). **infl pts at** none. 4.5.re2g. **domn** $(-\infty,\infty)$. **intrcpt** $(0,3)$, $(\ln 4,0)$. **symm** none. **asymp** $y=0$ (as $x\to-\infty$). **mono** increases on $(-\infty,\ln 2)$. decreases elsewhere. **concav** up on $(-\infty,0)$. down elsewhere. **loc extrm at** $x=\ln 2$ (max). **infl pts at** $x=0$. 4.5.re2h. **domn** $(-\infty,\infty)$. **intrcpt** $(0,1)$. **symm** about y-axis. **asymp** $y=0$ (as $x\to\pm\infty$). **mono** increases on $(-\infty,0)$. decreases nowhere. **concav** up on $(-\infty,-1/\sqrt{2})$ and on $(1/\sqrt{2},\infty)$. down elsewhere. **loc extrm at** $x=0$ (max). **infl pts at** $x=\pm 1/\sqrt{2}$. 4.5.re2i. **domn** $[0,2\pi]$. **intrcpt** $(0,0)$. Finding the other intercept at about $(1.895,0)$ requires a calculator and could not appear on an exam. **symm** through the origin, but none on the given interval. **asymp** none. **mono** increases on $(\frac{1}{3}\pi,\frac{5}{3}\pi)$. decreases elsewhere. **concav** up on $(0,\pi)$. down elsewhere. **loc extrm at** $x=\frac{1}{3}\pi$ (min), $x=\frac{5}{3}\pi$ (max). **infl pts at** $x=\pi$. 4.5.re2j. **domn** $[0\pi/2)\cup(\pi/2,3\pi/2)\cup(3\pi/2,,2\pi]$. **intrcpt** $(0,0)$, $(\pi,0)$, $(2\pi,0)$. **symm** across the y-axis, but none on the given interval. **asymp** $x=\pi/2$, $x=3\pi/2$. **mono** increases on $(0,\pi/2)$ and on $(\pi/2,\pi)$. decreases on $(\pi,3\pi/2)$ and on $(3\pi/2,2\pi)$. **concav** up $(0,\pi/2)$ and on $(3\pi/2,2\pi)$ down elsewhere. **loc extrm at** $x=0$ and $x=2\pi$ (min), $x=\pi$ (max). **infl pts at** none.

4.7: Optimization

4.7.re1. A rectangular box with a square base and an open top is constructed from two different grades of cardboard. The material for the sides costs 1 ¢/in^2 and the material for the base costs 5 ¢/in^2. If the cost of materials of the box is limited to \$7.35, find the dimensions that will maximize the volume of the box.

Solution:



Draw a picture. Since the base is a square, we can label the side-lengths x, x, and y. Identify your **objective function:** the quantity to be maximized or minimized, in this case volume:

$$V = x^2 y$$

As often happens, our objective V is a function of two variables. To rewrite V in terms of one variable, find the **constraint equation** that ties the variables together. In this case, x and y can't be just anything, because the cost of the box—that's the cost of the $x \times x$ base plus the coast of the 4 $x \times y$ sides—must be \$7.35:

$$5x^2 + 4xy = 735 \tag{4.7.re2}$$

Solve for y in the constraint and substitute this expression into V:

$$y = \frac{735 - 5x^2}{4x} \tag{4.7.re3}$$

$$V = x^2 \left(\frac{735 - 5x^2}{4x} \right) = \frac{1}{4} x(735 - 5x^2) = \frac{1}{4}(735x - 5x^3) \tag{4.7.re4}$$

The maximum of V will occur as its critical point—the x-value where $V'(x) = 0$:

$$0 = \frac{1}{4}(735 - 15x^2) \Longrightarrow x^2 = \frac{735}{15} = 49 \Longrightarrow x = 7.$$

(We can ignore the solution $x = -7$ since x is a distance.) Make sure you answer the question. Did it ask for the maximum volume, or the dimensions that will produce it? At $x = 7$, $y = (735 - 5 \cdot 7^2) \div (4 \cdot 7) = \frac{35}{2}$and so the desired dimensions are $7 \times 7 \times \frac{35}{2}$.

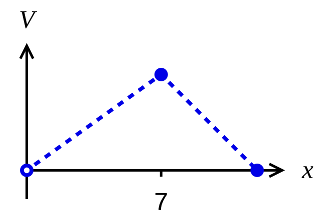
end example 4.7.re1

Optimization problems require you to express the objective as a function of one variable over some interval and then find either its absolute max or absolute min. Problems often don't ask for both, because one of these is trivial or nonexistent. Be careful not to confuse the objective with the constraint (if there is one).

4.7.re5. How do we know that the critical point we found in 4.7.re1 is an absolute max?

Solution:
First identify the interval of x for which our formula for V is valid. Since it is a distance, $x \geq 0$. In this problem, the constraint (4.7.re2) rules out $x = 0$, so $0 < x$. Likewise, y is a distance and must be nonnegative, so (4.7.re3) implies that $|x| \leq \sqrt{\frac{735}{5}} = 7\sqrt{3}$. So, (4.7.re4) is valid for all x in $(0, 7\sqrt{3}]$.
Next, check the behavior of V at the critical point and at the endpoints of the interval.

$x = 0$	$\lim_{x\to 0} V(x) = V(0) = 0$
$x = 7$	$V(7) = \frac{1}{4}7(735 - 5 \cdot 7^2) > 0$
$x = 7\sqrt{3}$	$V(7\sqrt{3}) = 0$



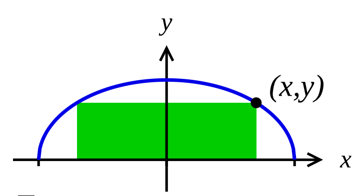
Since $x = 7$ is the only critical point inside the interval, the graph of V must look roughly like the accompanying graph. Therefore, V must have an absolute max at $x = 7$.

end example 4.7.re5

4.7.re6. A right circular cylinder is to be inscribed in a sphere of radius $\sqrt{3}$. Find the radius and height of the cylinder of maximum volume. Hint: the volume of a cylinder with radius r and height h is $\pi r^2 h$.

4.7.re7. A closed, rectangular box with a square base is have volume 16 cubic inches. If the material for the base costs \$3 per square inch and the material for the sides and top costs \$1 per square inch, what are the dimensions of the least expensive box?

4.7.re8. A rectangle is inscribed in the region bounded beneath the ellipse $2x^2 + 8y^2 = 1$ and above the x-axis. See figure. Find the dimensions of the rectangle of largest possible area.

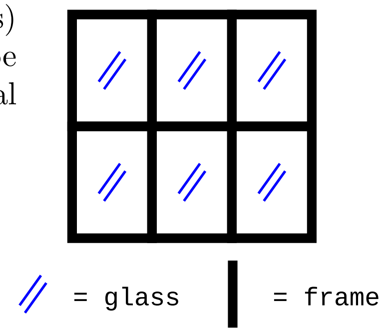


4.7.re9. A right circular cone is inscribed in a sphere of radius $\sqrt{5}$. Find the maximum possible volume of the cone. Hint: the volume of a cone of height h and radius r (measured at its base) is $\frac{1}{3}\pi r^2 h$.

4.7.re10. A rectangular plot of land will be bounded on one side by a straight river and on the other three sides by a wire fence. With 144 meters of fence at your disposal, what are the dimensions of the largest area you can enclose?

4.7.re11. Find the dimensions of the cylinder of largest possible volume that can be generated by rotating a rectangle of perimeter 24 about one of its sides.

4.7.re12. A window consists of six identical panes (rectangles) of glass separated by a frame. If the area of each pane must be 2 ft^2, find the dimensions of the panes that minimize the total length of frame in the window.



Answers

4.7.re6. $r = \sqrt{2}$, $h = 2$ 4.7.re7. $2 \times 2 \times 4$. 4.7.re8. Base $= 1$, height $= \frac{1}{4}$. 4.7.re9. Maximum volume $8\pi/3$ occurs at $r = h = 2$. 4.7.re10. 36×72, with the longer side parallel the river. 4.7.re11. $r = 8$, $h = 4$. 4.7.re12. Pane has base $4/3$ and height $3/2$.

4.9: Antiderivatives

An **antiderivative** for $f(x)$ is a function whose derivative is $f(x)$.
If $F(x)$ is an antiderivative of $f(x)$ on an interval, then *every* antiderivative of f on that interval equals $F(x)+C$ for some constant C.
We say that $F(x)+C$ is the **general antiderivative** of f on that interval. The general antiderivative of $f(x)$ is denoted $\int f(x)\,dx$, although this notation doesn't appear in our text until section 5.4.

4.9.re1. x^2 is an antiderivative of $2x$ because $(x^2)'=2x$. The general antiderivative of $2x$ is x^2+C.

I. A catalog of elementary functions and their antiderivatives.

A table of antiderivatives is practically the same as a table of derivatives read backwards:

$f(x)$	derivative of f
$f(x)$	$f'(x)$
x^n	nx^{n-1}
$\ln\lvert x\rvert$	x^{-1}
e^x	e^x
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\csc^2 x$
$\sec x$	$\sec x\tan x$
$\csc x$	$-\csc x\cot x$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\sec^{-1} x$	$\frac{1}{x\sqrt{x^2-1}}$

$f(x)$	antiderivative of f
$f(x)$	$\int f(x)\,dx$
x^n	$\begin{cases} \frac{1}{n+1}x^{n+1}+C & \text{if } n\neq -1 \\ \ln\lvert x\rvert + C & \text{if } n=-1 \end{cases}$
e^x	e^x+C
$\cos x$	$\sin x+C$
$\sin x$	$-\cos x+C$
$\sec^2 x$	$\tan x+C$
$\csc^2 x$	$-\cot x+C$
$\sec x\tan x$	$\sec x+C$
$\csc x\cot x$	$-\csc x+C$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x+C$
$\frac{1}{1+x^2}$	$\tan^{-1} +C$
$\frac{1}{x\sqrt{x^2-1}}$	$\sec^{-1} x+C$

II. Combination laws.

If F and G are antiderivatives for f and g, and if c is a constant, then

1. $F+G$ is an antiderivative for $f+g$.
2. $F-G$ is an antiderivative for $f-g$.
3. cF is an antiderivative for cf.

It is important to note that FG is **not** an antiderivative for fg, and F/G is not an antiderivative for f/g.

4.9.re2. x^2 is an antiderivative for $2x$, and e^x is an antiderivative for itself, but x^2e^x is **not** an antiderivative for $2xe^x$, because

$$(x^2e^x)' = 2xe^x + x^2e^x, \text{ not } 2xe^x.$$

4.9.re3. $\sin x$ is an antiderivative of $\cos x$ and x is an antiderivative of 1, but $\frac{\sin x}{x}$ is not an antiderivative for $\frac{\cos x}{1}$, because

$$\left(\frac{\sin x}{x}\right)' = \frac{x\cos x - \sin x}{x^2}, \text{ not } \frac{\cos x}{1}$$

4.9.re4. Find the general antiderivative of the function:

a. $5x^3 - 2\sqrt{x} + 6$

b. $6x^{2/3} + 3x^{5/3} - e^x$

c. $\sqrt{x^3} - 5\sqrt[3]{x^2}$

d. $\dfrac{9}{4} - \dfrac{2}{x} + \dfrac{3}{x^2}$

e. $\dfrac{x^2 - 2x + 2}{x}$

f. $(x-2)(x^{1/2} - x^{-1/2})$

g. $(\sec x)(2\sec x - 3\tan x)$

h. $3\sin x + 4\cos x$

Given the derivative of f, we can only hope to find f up to a constant.

4.9.re5. If $f'(x) = 2x + e^x$, then

$$f(x) = x^2 + e^x + C \qquad (4.9.re6)$$

for some constant C. To find this constant, we'd also have to know the value of $f(x)$ at one x. For instance, if $f(0) = -2$, then evaluate (4.9.re6) at $x = 0$ to find C:

$$\begin{aligned} f(x) &= x^2 + e^x + C \\ f(0) &= 0^2 + e^0 + C \\ -2 &= 1 + C \end{aligned}$$

so $C = -3$, and $f(x) = x^2 + e^x - 3$.

4.9.re7. Use the given information to find $f(x)$.

a. $f'(x) = 4x^3 - 2e^x + \sec x \tan x; \quad f(0) = 9$

b. $f'(x) = x(3 - 2x^2); \quad f(2) = -3$

c. $f'(x) = \dfrac{5x^2 - 3x + 1}{\sqrt{x}}; \quad f(1) = -6$

d. $f''(x) = -6; \quad f'(1) = 2; \quad f(1) = 0$

e. $f''(x) = \sin x; \quad f(0) = 1; \quad f(\pi/2) = -2$

4.9.re8. At time $t = 0$, a cannonball is fired upward from the ground with an velocity of 300 ft/sec. Assuming its upward acceleration due to gravity is -32 ft/sec^2, express the altitude of the cannon ball as a function of time t.

Solution:
Let s denote the cannonball's altitude. Then

$$s'' = -32 \implies s' = -32t + C.$$

Use the given $s' = 300$ at time $t = 0$ to find C:

$$\begin{aligned} s' = 300 = &-32 \cdot 0 + C &\implies& \quad C = 300 \\ s' = &-32t + 300 &\implies& \quad s = -16t^2 + 300t + D \end{aligned}$$

Use the given $s = 0$ at $t = 0$ to find D:

$$\begin{aligned} s = 0 = &-16 \cdot 0^2 + 300 \cdot 0 + D \implies D = 0 \\ s = &-16t^2 + 300t \end{aligned}$$

end example 4.9.re8

4.9.re9. At time $t = 0$, a marble is tossed from a 100 ft tower with an upward velocity of 30 ft/sec. Assuming its acceleration due to gravity is -32 ft/sec^2, find the marble's altitude at time t.

Answers

4.9.re4a. $\frac{5}{4}x^4 - \frac{4}{3}x^{3/2} + 6x + C$ 4.9.re4b. $\frac{18}{5}x^{5/3} + \frac{9}{8}x^{8/3} - e^x + C$ 4.9.re4c. $\frac{2}{5}x^{5/2} - 3x^{5/3} + C$ 4.9.re4d. $\frac{9}{4}x - 2\ln|x| - 3x^{-1} + C$ 4.9.re4e. $\frac{1}{2}x^2 - 2x + 2\ln|x| + C$ 4.9.re4f. $\frac{2}{5}x^{5/2} - 2x^{3/2} + 4x^{1/2} + C$ 4.9.re4g. $2\tan x - 3\sec x + C$ 4.9.re4h. $-3\cos x + 4\sin x + C$ 4.9.re7a. $f(x) = x^4 - 2e^x + \sec x + 10$. 4.9.re7b. $f(x) = \frac{3}{2}x^2 - \frac{1}{2}x^4 - 1$ 4.9.re7c. $f(x) = 2x^{5/2} - 2x^{3/2} + 2x^{1/2} - 8$ 4.9.re7d. $f(x) = -3x^2 + 8x - 5$ 4.9.re7e. $f(x) = -\sin x - \frac{4}{\pi}x + 1$ 4.9.re9. Solve $s''(t) = -32$, $s'(0) = 30$, $s(0) = 100$ to find altitude $s = -16t^2 + 30t + 100$

5.1: The Area and Distance Problems

The **Area Problem**: how can we find the area between the the graph of a function $f(x)$ and the x-axis, between $x = a$ and $x = b$?

The **Distance Problem**: how can we use the velocity $v(t)$ of an object to determine its the distance traveled between times $t = a$ and $t = b$?

Sigma notation for sums

The symbol $\sum_{i=1}^{n}$ means "the sum of values of the following, for i equal $1, 2, \ldots, n$."

5.1.re1. $\sum_{i=1}^{5}(i^2+1) = (1^2+1) + (2^2+1) + (3^2+1) + (4^2+1) + (5^2+1) = 60.$

5.1.re2. Calculate the sum.

a. $\displaystyle\sum_{i=0}^{3}(i+3)$ b. $\displaystyle\sum_{j=1}^{5}2^{j-1}$ c. $\displaystyle\sum_{k=3}^{7}(-1)^k$

Approximating area and distance by Riemann sums

Solutions to the area and distance problems can both be estimated by a Riemann sum:

Definition 5.1.re3. *Divide the interval $[a, b]$ into n subintervals of equal length Δx, and select a number x_i^* in each $(1 \le i \le n)$. If the function $f(x)$ is defined on $[a, b]$, then*

$$f(x_1^*)\Delta x + f(x_2^*)\Delta x + f(x_3^*)\Delta x + \cdots + f(x_n^*)\Delta x = \sum_{i=1}^{n} f(x_i^*)\Delta x \qquad (5.1.re4)$$

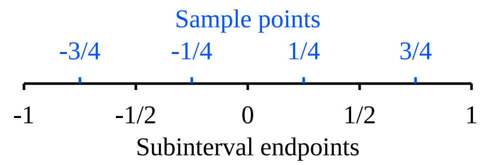
is called a **Riemann Sum** *for $f(x)$ on $[a, b]$.*

When evaluating the Riemann sum (5.1.re4), it's handy to take out the common factor:

$$\Delta x\big(f(x_1^*) + f(x_2^*) + f(x_3^*) + \cdots + f(x_n^*)\big) = \Delta x \sum_{i=1}^{n} f(x_i^*)$$

5.1.re5. Approximate the area between the $y = x^2$ and the x-axes from $x = -1$ to $x = 1$ using $n = 4$ subintervals and their midpoints.

Solution:
Divide interval $[-1, 1]$ into 4 subintervals of equal length $\Delta x = \frac{1}{2}$. Midpoints of these are $-\frac{3}{4}$, $-\frac{1}{4}$, $\frac{1}{4}$, and $\frac{3}{4}$. Sum the values of x^2 at these four point and multiply the results by Δx:



$$\text{Riemann sum} = \frac{1}{2}((-3/4)^2 + (-1/4)^2 + (1/4)^2 + (3/4)^2).$$

Check with your instructor whether an answer in this form is sufficient on exams. This Riemann sum simplifies to $\frac{5}{8}$.

end example 5.1.re5

5.1.re6. Approximate the net signed area under the given curve, on the given interval, with a Riemann sum as described.

a. $y = x^3 - 2$, $[0, 1]$, $n = 4$ subintervals and their midpoints.

b. $y = x^3 - 2$, $[0, 6]$, $n = 6$ subintervals and their left endpoints.

c. $y = e^x$, $[-1, 1]$, $n = 5$ subintervals and their right endpoints.

d. $y = \frac{1}{1+x}$, $[6, 12]$, $n = 3$ subintervals and their midpoints.

5.1.re7. Use the prescribed Riemann sum to estimate the distance traveled between times $t = 60$ and $t = 140$ by the object whose velocity is recorded in the following table.

t(sec)	60	70	80	90	100	110	120	130	140
v(m/sec)	1	3	7	12	14	16	12	7	2

a. $n = 8$ subintervals and their left endpoints.

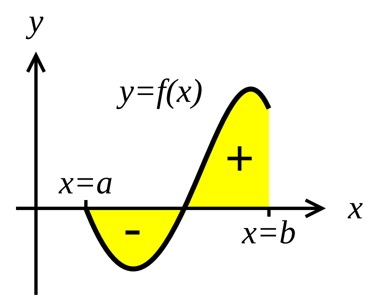
b. $n = 4$ subintervals and their right endpoints.

c. $n = 4$ subintervals and their midpoints.

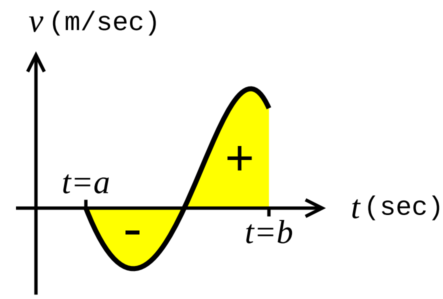
d. $n = 2$ subintervals and their midpoints.

Solutions to the area and distance problems

The limit as $n \to \infty$ of the Riemann sums for $f(x)$ on $a \le x \le b$ is the **net signed area** between the graph of f and the x-axis between $x = a$ and $x = b$. Areas above the x-axis are counted as positive and areas below the x-axis are counted as negative.

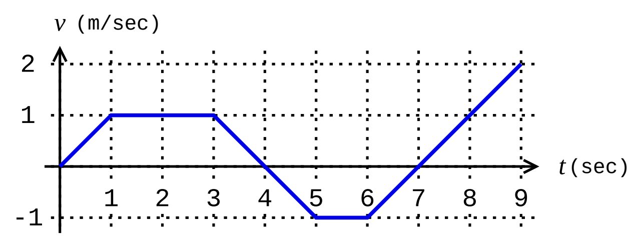


The limit as $n \to \infty$ of the Riemann sums for $v(t)$ on $a \le t \le b$ is the **net distance traveled** by the object whose velocity at time t is $v(t)$. The object's net distance traveled is also called its **displacement** or **change in position**.



That means that the net distance traveled by the object from time a to time b is the same as the net signed area between its velocity curve $y = v(t)$ and the t-axis over $[a, b]$ (since both are limits of the same Riemann sums). Areas above the t-axis, where velocity is positive, measure motion in the positive direction and areas below the t-axis, when velocity is negative, measure motion in the negative direction.

5.1.re8. The graph shows the velocity $v(t)$ of an object moving along an axes from time $t = 0$ to time $t = 9$. Find the following.



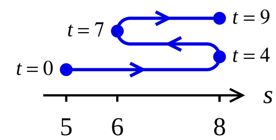
a. The times when the object is moving forward, and when it's moving backward.
b. The distance traveled between times 0 and 4.
c. The object's net and total distances traveled between times 0 and 9.
d. A schematic diagram of the object's motion from times 0 to 9, assuming its position at time 0 is 5m.

Solution:
Answers are based on the following distances found by calculating areas between the velocity curve and t-axis. Areas above [below] the t-axis are counted as positive [negative] and represent forward [backward] motion.

t-intervals (sec):	$[0, 4]$	$[4, 7]$	$[7, 9]$
distances (m):	3	-2	2

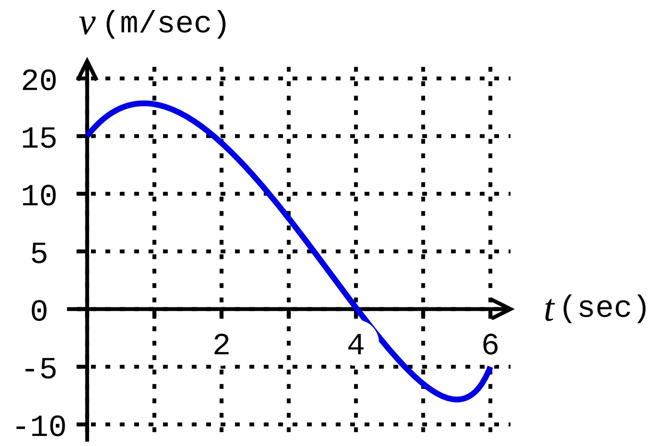
a. Object moves forward when velocity is positive, for t in $[0, 4]$ and again in $[7, 9]$. Object moves backward for times $[4, 7]$, when velocity is negative.
b. 3m.

c. Net distance: $3 - 2 + 2 = 3$m. Total distance: $3 + 2 + 2 = 7$m.
d. Starting from position 5, object moves forward 3, backward 2, then forward 2.



end example 5.1.re8

5.1.re9. The velocity of a car, recorded for six seconds, is graphed in the below.



a. When (between times 0 and 6 sec) is the car moving forward and when is it moving backward?

b. Ignoring any backward distance, approximate (not necessary with a Riemann sum) the forward distance traveled by the car between time $t = 0$ and time $t = 6$ sec.

c. Ignoring any forward distance, approximate the backward distance traveled by the car between time $t = 0$ and time $t = 6$ sec.

d. Using these approximations, what do you estimate to be the car's total and net distances traveled between time 0 and 6?

Answers

5.1.re2a. (0+3)+(1+3)+(2+3)+(3+3) = 18. 5.1.re2b. 1+2+4+8+16 = 31. 5.1.re2c. $-1+1-1+1-1 = -1$. 5.1.re6a. $\frac{1}{4}\Big((\frac{1}{8})^3 - 2 + (\frac{3}{8})^3 - 2 + (\frac{5}{8})^3 - 2 + (\frac{7}{8})^3 - 2\Big)$. 5.1.re6b. $0^3 - 2 + 1^3 - 2 + 2^3 - 2 + 3^3 - 2 + 4^3 - 2 + 5^3 - 2$ (or 213). 5.1.re6c. $0.4(e^{-1} + e^{-0.6} + e^{-0.2} + e^{0.2} + e^{0.6})$. 5.1.re6d. $2(\frac{1}{8} + \frac{1}{10} + \frac{1}{12})$. 5.1.re7a. 720 m 5.1.re7b. 700 m 5.1.re7c. 760 m 5.1.re7d. 760 m 5.1.re9a. Forward when $0 < t < 4$ and backward when $4 < t < 6$. 5.1.re9b. Area of each square is (5m/sec)×(1 sec) = 5m. Area above the t axis appears to be approximately 48 m. 5.1.re9c. Area below the t axis appears to be approximately 12m.
5.1.re9d. Using the above, the estimated total distance is $48 + 12 = 60$m, and the estimated net distance is $48 - 12 = 36$ m. That is, at $t = 6$, the car is about 36 m ahead of its position at time 0, but is traveled 60 m total to get there.

5.2: The Definite Integral

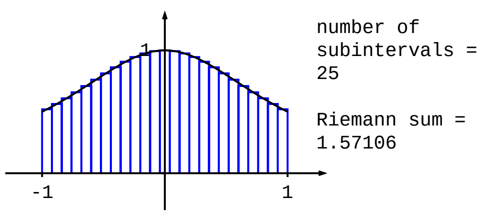
Definition 5.2.re1. *If $f(x)$ is defined on the interval $[a,b]$, then the limit of its Riemann sums on $[a,b]$ as $n \to \infty$, if it exists, is called the* **definite integral** *(or* **Riemann integral***)* **from a to b of** $f(x)\,dx$, *written $\int_a^b f(x)\,dx$. That is,*

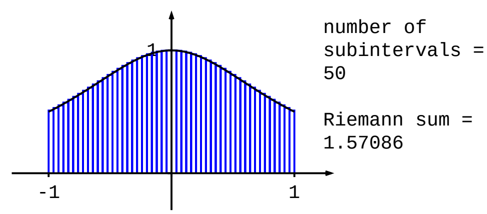
$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x,$$

When this limit exists we say that $f(x)$ is **integrable** *on $[a,b]$.*

Note that $\int_a^b f(x)\,dx$, when it exists, is a number.
The definite integral $\int_a^b f(x)\,dx$ can be interpreted as the:
1) net signed area between the graph of $f(x)$ and the x-axis between $x=a$ and $x=b$, and
2) net distance traveled from time a to b by an object whose velocity is $f(x)$ at time x .

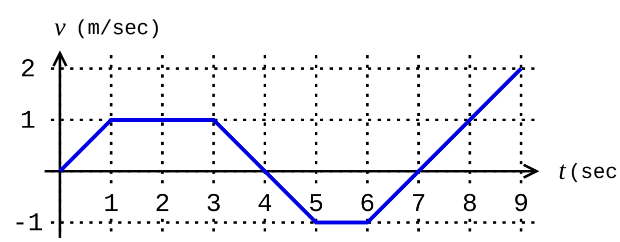
5.2.re2. Here are two Riemann sums for the function $\dfrac{1}{x^2+1}$ on the interval $[-1,1]$:





Based on these two Riemann sums, $\displaystyle\int_{-1}^{1} \frac{1}{x^2+1}\,dx$ appears to equal about 1.5708.

5.1.re8, continued. The areas we calculated earlier are integrals of the velocity function v, seen again here.



$$\int_0^4 v(t)\,dt = 3 \qquad \int_7^9 v(t)\,dt = 2$$

$$\int_4^7 v(t)\,dt = -2 \qquad \int_0^9 v(t)\,dt = 3-2+2 = 3$$

end example 5.1.re8

Not every function is integrable, but a great many are:

Theorem 5.2.re3. *If $f(x)$ is continuous on $[a,b]$ (with the possible exception of finitely many jump or removable discontinuities) then f is integrable on $[a,b]$.*

Some useful algebraic properties of the definite integral:

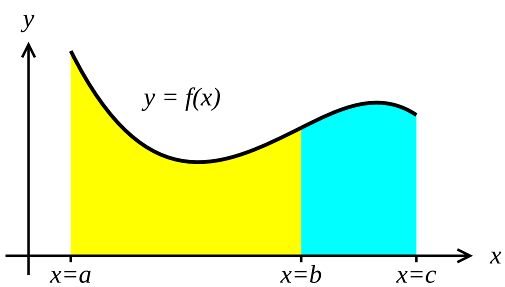
If $f(x)$ and $g(x)$ are integrable on $[a,b]$ and c is a constant, then

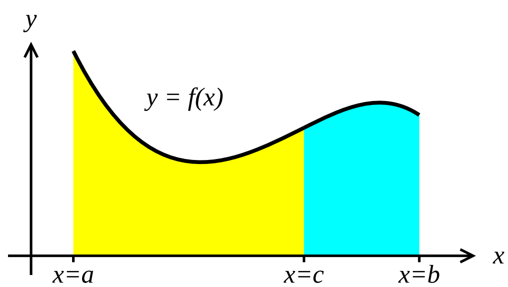
1. $\int_a^b c\,dx = c(b-a)$
2. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$
3. $\int_a^b \big(f(x)+g(x)\big)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
4. $\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$

A convenient definition:

5. $\int_b^a f(x)\,dx \stackrel{\text{def}}{=} -\int_a^b f(x)\,dx$
6. $\int_a^a f(x)\,dx \stackrel{\text{def}}{=} 0$

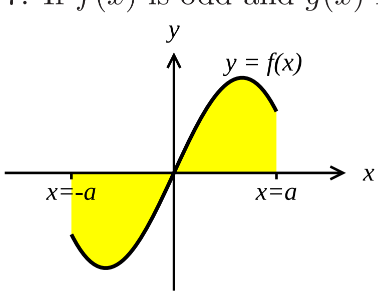
As a consequence of 5, properties 1-4 are valid even when the upper endpoint of an integral is less than its lower endpoint. For example, think about what 4 says in these two pictures:



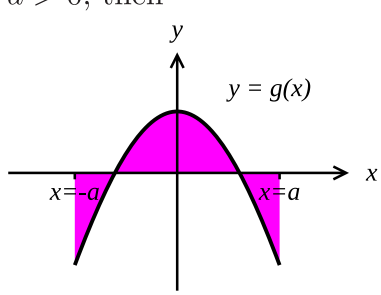


Along similar lines, here's 7 from section 5.4:

7. If $f(x)$ is odd and $g(x)$ is even, and if $a>0$, then



$\int_{-a}^a f(x)\,dx = 0$



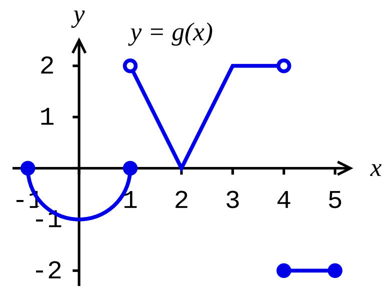
$\int_{-a}^a g(x)\,dx = 2\int_0^a g(x)\,dx$

5.2.re4. If $\int_1^3 f(x)\,dx = 7$, $\int_3^5 f(x)\,dx = -3$, and $\int_1^5 g(x)\,dx = 2$, and $\int_5^3 g(x)\,dx = 6$, find the following:

a. $\displaystyle\int_1^3 g(x)\,dx$ b. $\displaystyle\int_5^3 f(x)\,dx$ c. $\displaystyle\int_1^5 2f(x)\,dx$

d. $\displaystyle\int_1^5 \big(f(x) - 2g(x)\big)\,dx$ e. $\displaystyle\int_1^5 \big(6 + g(x)\big)\,dx$ f. $\displaystyle\int_3^1 \big(3g(x) - 5\big)\,dx$

5.2.re5. Let $g(x)$ be the function graphed here. Express g as a piecewise-defined function of x, (its first piece is a semicircle) and find the following.



a. $\displaystyle\int_1^{-1} g(x)\,dx$ b. $\displaystyle\int_0^3 g(x)\,dx$

c. $\displaystyle\int_1^4 g(x)\,dx$ d. $\displaystyle\int_1^5 g(x)\,dx$

e. $\displaystyle\int_1^5 \big(6 + g(x)\big)\,dx$ f. $\displaystyle\int_3^1 \big(3g(x) - 5\big)\,dx$

5.2.re6. Suppose that $g(x)$ from 5.2.re5 is the velocity at time x of an object moving along a number line. Find the following.

a. When the object is traveling forward, and when it's traveling backward.

b. The net distance traveled (displacement) by the object between times 0 and 4.

c. The total distance traveled by the object between times 0 and 4.

d. The net and total distances traveled of the object between time 1 and time 3.

5.2.re7. Find the following. (A sketch will help.)

a. $\displaystyle\int_{-2}^4 |x|\,dx$ b. $\displaystyle\int_{-2}^2 1 + \sqrt{4 - x^2}\,dx$ c. $\displaystyle\int_{-1}^3 2 - \sqrt{4 - (x-1)^2}\,dx\,dx$

d. $\displaystyle\int_{-2}^2 x\,dx$ e. $\displaystyle\int_1^3 4\,dx$

Answers

5.2.re4a. 8 5.2.re4b. 3 5.2.re4c. 8 5.2.re4d. 0 5.2.re4e. 26 5.2.re4f. -14

5.2.re5. $g(x) = \begin{cases} -\sqrt{1-x^2} & -1 \le x \le 1 \\ -2(x-2) & 1 < x \le 2 \\ 2(x-2) & 2 < x \le 3 \\ 2 & 3 \le x < 4 \\ -2 & 4 \le x \le 5 \end{cases}$ 5.2.re5a. $\pi/2$ 5.2.re5b. $2 - \frac{1}{4}\pi$ 5.2.re5c. 4 5.2.re5d. 2

5.2.re5e. 26 5.2.re5f. 4 5.2.re6a. Backward when $-1 < t < 1$ and when $4 < t < 5$. Forward when $1 < t < 4$. 5.2.re6b. $4 - \frac{1}{4}\pi$ 5.2.re6c. $4 + \frac{1}{4}\pi$ 5.2.re6d. Because the object does not change direction between these times, both equal 2 5.2.re7a. 10 5.2.re7b. $4 + 2\pi$ 5.2.re7c. $8 - 2\pi$ 5.2.re7d. 0 5.2.re7e. 8

5.3: The Fundamental Theorem of Calculus

Fundamental Theorem of Calculus I (FTC.I) 5.3.re1. *If the function f is integrable on the interval $[a,b]$, and if $F(x)$ is defined as*

$$F(x) \stackrel{\text{def}}{=} \int_a^x f(t)\,dt,$$

then, wherever $f(x)$ is continuous, $F(x)$ is differentiable and $F'(x) = f(x)$.

5.3.re2. Find the derivative of each of integral-defined functions.

a. $\displaystyle\int_4^x \frac{\sin t}{t^2+1}\,dt$ b. $\displaystyle\int_x^1 \frac{\sin t}{t^2+1}\,dt$ c. $\displaystyle\int_1^{2x^3} \frac{\sin t}{t^2+1}\,dt$ d. $\displaystyle\int_x^{2x^3} \frac{\sin t}{t^2+1}\,dt$

Solution:

a. $\displaystyle\frac{d}{dx}\int_4^x \frac{\sin t}{t^2+1}\,dt = \frac{\sin x}{x^2+1}$, since this is continuous at every x-value.

b. $\displaystyle\frac{d}{dx}\int_x^1 \frac{\sin t}{t^2+1}\,dt = \frac{d}{dx}\left(-\int_1^x \frac{\sin t}{t^2+1}\,dt\right) = -\frac{\sin x}{x^2+1}.$

c. A chain rule problem: Call the integral I and let $u = 2x^3$. Then

$$\frac{dI}{dx} = \frac{dI}{du}\frac{du}{dx} = \frac{\sin u}{u^2+1}(2x^3)' = \frac{\sin(2x^3)}{4x^6+1}(6x^2)$$

d. $\int_x^{2x^3} \frac{\sin t}{t^2+1}\,dt = \int_x^1 \frac{\sin t}{t^2+1}\,dt + \int_1^{2x^3} \frac{\sin t}{t^2+1}\,dt$, so $\frac{d}{dx}\int_x^{2x^3} \frac{\sin t}{t^2+1}\,dt = -\frac{\sin x}{x^2+1} + \frac{\sin(2x^3)}{4x^6+1}(6x^2)$.

end example 5.3.re2

5.3.re3. Find the derivative of each of integral-defined functions.

a. $\displaystyle\int_0^x e^{t^2}\,dt$ b. $\displaystyle\int_1^x e^{t^2}\,dt$ c. $\displaystyle\int_x^1 e^{t^2}\,dt$

d. $\displaystyle\int_x^4 \sin(\cos(\theta))\,d\theta$ e. $\displaystyle\int_0^{2x} \sqrt{1+t^{-1}}\,dt$ f. $\displaystyle\int_{3x}^{2x} \frac{1}{1+t^4}\,dt$

g. $\displaystyle\int_{\ln x}^{2x} \frac{1}{t}\,dt$ h. $\displaystyle\int_1^{x^2} \frac{1+t}{1+t^3}\,dt$ i. $\displaystyle\int_{e^x}^{\cos x} \ln(t+5)\,dt$

j. $\displaystyle\int_1^{\tan^{-1} x} \tan\theta\,d\theta$ k. $\displaystyle\int_0^{\ln x} e^{2u}\,du$

Fundamental Theorem of Calculus II (FTC.II) 5.3.re4. *If $F(x)$ is any antiderivative of $f(x)$ and it f is continuous on $[a,b]$, then*

$$\int_a^b f(t)\,dt = F(b) - F(a) \stackrel{\text{def}}{=} F(x)\Big|_a^b$$

5.3.re5. Evaluate the definite integral.

a. $\displaystyle\int_1^9 \left(\frac{x-2}{\sqrt{x}} - \frac{3}{x}\right) dx$ b. $\displaystyle\int_{1/2}^1 \frac{1}{\sqrt{1-x^2}}\,dx$

Solution:

a. $\displaystyle\frac{x-2}{\sqrt{x}} - \frac{3}{x} = x^{1/2} - 2x^{-1/2} - 3\frac{1}{x} = \frac{d}{dx}\left(\frac{2}{3}x^{3/2} - 4x^{1/2} - 3\ln|x|\right)$, so

$$\begin{aligned}\int_1^9 \left(x^{1/2} - 2x^{-1/2} - 3\frac{1}{x}\right) dx &= \left(\frac{2}{3}x^{3/2} - 4x^{1/2} - 3\ln|x|\right)\Big|_1^9 \\ &= \left(\frac{2}{3}9^{3/2} - 4\cdot 9^{1/2} - 3\ln 9\right) - \left(\frac{2}{3} - 4\right) = \frac{28}{3} - 3\ln 9\end{aligned}$$

(because $\ln 1 = 0$, and $9^{1/2} = 3$, and $9^{3/2} = 3^3 = 27$).

b. $\displaystyle\frac{1}{\sqrt{1-x^2}} = \frac{d}{dx}\sin^{-1}x$, so

$$\int_{1/2}^1 \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x\Big|_{1/2}^1 = \sin^{-1}1 - \sin^{-1}\frac{1}{2} = \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}.$$

end example 5.3.re5

5.3.re6. Evaluate the definite integral.

a. $\displaystyle\int_0^2 e^t\,dt$ b. $\displaystyle\int_{\pi/4}^{\pi} (\sin\theta + \cos\theta)\,d\theta$ c. $\displaystyle\int_{\pi/4}^{\pi} (\sin^2\theta + \cos^2\theta)\,d\theta$

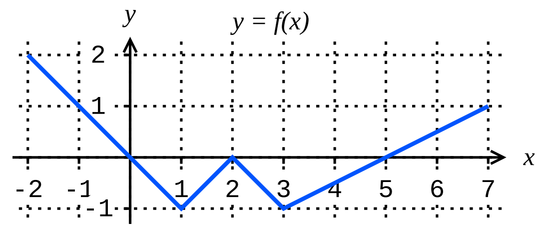
d. $\displaystyle\int_{-1}^1 (3x-2)(x+1)\,dx$ e. $\displaystyle\int_{-3}^{-1} \frac{xe^x+1}{x}\,dx$ f. $\displaystyle\int_1^4 \frac{2x^2+1}{\sqrt{x}}\,dx$

g. $\displaystyle\int_1^3 \left(\frac{2-t}{t}\right)^2 dt$ h. $\displaystyle\int_{-\pi/4}^{\pi/4} (\sec^2 x - \cos x)\,dx$

5.3.re7. Find the area of the "triangular" region between the curves $y = e^x - 1$, $y = 0$, and $x = 2$.

5.3.re8. The velocity of a particle moving along a number line is $v(t) = \cos t - \sin t$ at time t. Find the displacement of the object between times $t = -\pi/2$ and $t = \pi/2$.

5.3.re9. Let $f(x)$ be the function graphed here and define $g(x) = \int_{-2}^{x} f(t)\, dt$.



a. On what intervals is $g(x)$ increasing? On what intervals is g decreasing?
b. Find the critical numbers of g. Where does g have a local max or min?
c. Find the (x-coordinates of) inflection points of $g(x)$. Where is $g(x)$ concave up and concave down?
d. Find the absolute extrema of g on the interval $[-2, 7]$.

Answers

5.3.re3a. e^{x^2} 5.3.re3b. e^{x^2} 5.3.re3c. $-e^{x^2}$ 5.3.re3d. $-\sin(\cos(x))$ 5.3.re3e. $2\sqrt{1 + (2x)^{-1}}$ 5.3.re3f. $\frac{2}{1+16x^4} - \frac{3}{1+81x^4}$ 5.3.re3g. $\frac{1}{x} - \frac{1}{x \ln x}$ 5.3.re3h. $\frac{(1+x^2)2x}{1+x^6}$ 5.3.re3i. $-\sin x \ln(5 + \cos x)$ $-e^x \ln(e^x + 5)$ 5.3.re3j. $\frac{x}{1+x^2}$ 5.3.re3k. x 5.3.re6a. $e^2 - 1$ 5.3.re6b. 1 5.3.re6c. $3\pi/4$ 5.3.re6d. -2 5.3.re6e. $e^{-1} - e^{-3} - \ln 3$ 5.3.re6f. $\frac{134}{5}$ 5.3.re6g. $\frac{14}{3} - 4\ln 3$ 5.3.re6h. $2 - \sqrt{2}$ 5.3.re7. $e^2 - 3$ 5.3.re8. 2 5.3.re9a. g is increasing on $[-2, 0]$ and $[5, 7]$ and decreasing on $[0, 5]$. 5.3.re9b. Critical numbers of g are $x = 0$, 2, and 5. g has a local max at $x = 0$ and local min at $x = 5$. 5.3.re9c. The graph of g is concave up where $g''(x) = f'(x) > 0$, on $[1, 2]$ and $[3, 7]$. The graph of g is concave down on $[-2, 1]$ and on $[2, 3]$. $g''(x)$ is never zero, but $g''(x)$ changes sign at $x = 1$, 2, and 3, so the inflection points of g are at these x-values. 5.3.re9d. Compute g at critical points and endpoints:

x	-2	0	2	5	7
$g(x)$	0	2	1	$-1/2$	$1/2$

Conclusion: the absolute maximum of $g(x)$ on $[-2, 7]$ is 2, and the absolute minimum is $-1/2$.
Graph of $g(x)$ at https://www.desmos.com/calculator/humnqkgqrd

5.4: Indefinite Integrals and Net Change

The word "integral" and the integral sign $\int$ are used two different ways:

Definition 5.4.re1. *The* **definite integral** *from a to b of $f(x)\,dx$, written*

$$\int_a^b f(x)\,dx,$$

is the limit of the Riemann sums of f on the interval $[a, b]$ as the number of subintervals goes to infinity.

The **indefinite integral** *of $f(x)\,dx$, written*

$$\int f(x)\,dx,$$

is the collection of all antiderivatives of $f(x)$.

$$\begin{aligned}\int_a^b f(x)\,dx &= \text{a number.}\\ &= \text{the net signed area trapped between the graph of } f(x) \text{ and the } x\text{-axis from } x = a \text{ to } x = b.\\ &= \text{the net distance traveled from time } x = a \text{ to time } x = b \text{ by an object whose velocity at time } x \text{ is } f(x).\end{aligned}$$

and

$$\begin{aligned}\int f(x)\,dx &= \text{a collection of functions.}\\ &= \text{the set of all antiderivatives of } f(x).\\ &= F(x) + C, \text{ where } F'(x) = f(x)\end{aligned}$$

(The **value** of an integral, whether definite or indefinite, refers to the right side above.)

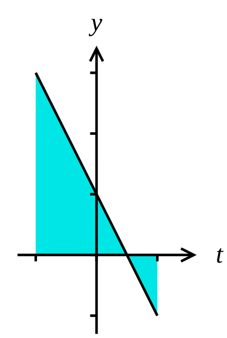
5.4.re2. The indefinite integral $\int(1 - 2x)\,dx = x - x^2 + C$, because $\frac{d}{dx}(x - x^2) = 1 - 2x$.

end example 5.4.re2

5.4.re3. The definite integral $\int_{-1}^{1}(1 - 2x)\,dx$ equals

$$(x - x^2)\Big|_{-1}^{1} = (1 - 1) - (-1 - (-1)^2) = 2.$$

The definite integral is positive because, on the interval $[-1, 1]$, the line $y = 1 - 2x$ traps more area above the x-axis than below.



end example 5.4.re3

To restate the results from section 4.9, page 90, here's a list of the indefinite integrals we know so far, and the combination rules for integration:

$f(x)$	$\int f(x)\,dx$
x^n	$\begin{cases} \frac{1}{n+1}x^{n+1} + C & \text{if } n \neq -1, \text{ and} \\ \ln\lvert x\rvert + C & \text{if } n = -1. \end{cases}$
e^x	$e^x + C$
$\cos x$	$\sin x + C$
$\sin x$	$-\cos x + C$
$\sec^2 x$	$\tan x + C$
$\csc^2 x$	$-\cot x + C$
$\sec x \tan x$	$\sec x + C$
$\csc x \cot x$	$-\csc x + C$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x + C$
$\frac{1}{1+x^2}$	$\tan^{-1} x + C$
$\frac{1}{x\sqrt{x^2-1}}$	$\sec^{-1} x + C$

If f and g are functions and c is a constant, then

$$\int \big(f(x) + g(x)\big)\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int cf(x)\,dx = c\int f(x)\,dx$$

5.4.re4. Evaluate the definite or indefinite integrals:

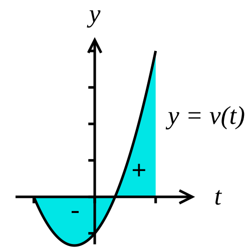
a. $\displaystyle\int_1^3 \frac{x^3+8}{2}\,dx$ b. $\displaystyle\int_0^{\pi/4} (\sin t - 2\cos t)\,dt$ c. $\displaystyle\int \frac{r^3-7r^2+1}{2r}\,dr$

d. $\displaystyle\int (x^2-1)^3\,dx$ e. $\displaystyle\int (\sec t\tan t - \csc t\cot t)\,dt$ f. $\displaystyle\int_0^{1/2} \frac{du}{\sqrt{1-u^2}}$

g. $\displaystyle\int (2-t)(3-2t)\,dt$ h. $\displaystyle\int_{-1}^{1} \sqrt[3]{s^2}\,ds$ i. $\displaystyle\int_0^2 (e^x - e)\,dx$

j. $\displaystyle\int x(5\sqrt{x} - 7\sqrt[3]{x})\,dx$ k. $\displaystyle\int \frac{v^4-1}{v^2-1}\,dv$ l. $\displaystyle\int_{-2}^{2} |x|\,dx$

5.4.re5. The velocity of an object moving along a number line is $v(t) = 3t^2 + 2t - 1$.



a. When is the object moving backward and when it is moving forward?

b. Find the *net* distance the object traveled between times -1 and 1.

c. Find the *total* distance it traveled between times -1 and 1.

d. Suppose the object is at position 0 at $t = 1$. Find the object's position $s(t)$ at time t.

e. Draw a schematic diagram of the object's motion for $-\infty < t < \infty$.

5.4.re6. Suppose an object's acceleration at time t is -2 m/sec^2 and that its velocity at time 1 sec is 4 m/sec.

a. Find the net and total distances traveled by the object between times 0 and 4 sec.

b. If the object is at position 12 m at time 3 sec, find its position at time 4 sec.

The definite integral of a rate of change

The Fundamental Theorem, part II (5.3.re4), says if $F(t)$ is any quantity that depends on time t, and if $F'(t)$ is continuous, then

$$\int_a^b F'(t)\,dt = F(b) - F(a).$$

That is,

$$\int_a^b (\text{rate of change of } F \text{ w.r.t. } t)\,dt = \text{change in } F \text{ from time } a \text{ to time } b..$$

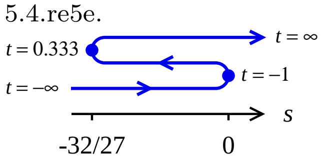
5.4.re7. If $g'(t)$ is the rate at which money in my bank account is changing with respect to time t, measured in days, what does $\int_0^{30} g'(t)\,dt$ represent?

5.4.re8. If $r(t)$ is the altitude of an elevator in the Willis Tower in Chicago and time t is measured in hours, what do the integrals $\int_0^{24} r'(t)\,dt$ and $\int_0^{24} |r'(t)|\,dt$ represent?

5.4.re9. The number of catfish in a farm pond increases at the rate of $\frac{300}{t^2}$ fish per month. Here, t is time measured in months, with $t = 1$ representing January 1. By how many fish will the population have increased by June 1?

Answers

5.4.re4a. 18 5.4.re4b. $-\frac{3}{\sqrt{2}} + 1$ 5.4.re4c. $\frac{1}{6}r^3 - \frac{7}{4}x^2 + \frac{1}{2}\ln|r| + C$ 5.4.re4d. $\frac{1}{7}x^7 - \frac{3}{5}x^5 + x^3 - x + C$ 5.4.re4e. $\sec t + \csc t + C$ 5.4.re4f. $\pi/6$ 5.4.re4g. $6t - \frac{7}{2}t^2 + \frac{2}{3}t^3 + C$ 5.4.re4h. 6/5 5.4.re4i. $e^2 - 2e - 1$ 5.4.re4j. $2x^{5/2} - 3x^{7/3} + C$ 5.4.re4k. $\frac{1}{3}v^3 + v + C$ 5.4.re4l. 4 5.4.re5a. Factor and make a sign chart for v. Forward $t < -1$ and $t > 1/3$. Backward: $-1 < t < 1/3$. 5.4.re5b. net distance traveled $= \int_{-1}^{1} v(t)\,dt = 0$. 5.4.re5c. Total distance $= \int_{-1}^{1} |v(t)|\,dt = -\int_{-1}^{1/3} v(t)\,dt + \int_{1/3}^{1} v(t)\,dt = \frac{64}{27}$.
5.4.re5d. $s(t) = t^3 + t^2 - t - 1$.
5.4.re5e.



5.4.re6a. 8 m net; 10 m total. 5.4.re6b. 11 m. 5.4.re7. The difference between my bank balance in 30 days and my bank balance today. 5.4.re8. The first is the net distance traveled by the elevator in 24 hours—its stopping position minus its starting position. The second is the total distance traveled by the elevator in the same time period. 5.4.re9. 250 fish.

5.5: Substitution

The **Substitution Rule** is a mechanism using the chain rule that helps us to simplify an integral. It says that an integral of the form $\int f\big(u(x)\big)u'(x)\,dx$ (for some function u of x) can be rewritten

$$\int f(u)\frac{du}{dx}\,dx = \int f(u)\,du$$

and thenceforth be treated as if u were the variable of integration. For this reason, substitution is also called a **change of variable**.

5.5.re1. Evaluate the integral: $\displaystyle\int \sec^2 x\sqrt{2+\tan x}\,dx$.

Solution:
Let $u = 2+\tan x$. Then $du = \frac{du}{dx}\,dx = \sec^2 x\,dx$. Now carefully rewrite every part of the integral, including the dx, in terms of u and du:

$$\int \sqrt{2+\tan x}\,\sec^2 x\,dx = \int \sqrt{u}\,du = \int u^{1/2}\,du \qquad (5.5.re2)$$

Now integrate:

$$= \frac{2}{3}u^{3/2} + C$$

and rewrite the answer in terms of the original variable x;

$$= \frac{2}{3}(2+\tan x)^{3/2} + C.$$

If you check your answer in a substitution problem, you should see the chain rule:

$$\frac{d}{dx}\left(\frac{2}{3}(2+\tan x)^{3/2}\right) = \frac{3}{2}\cdot\frac{2}{3}(2+\tan x)^{1/2}(2+\tan x)' = \sqrt{2+\tan x}\cdot\sec^2 x$$

end example 5.5.re1

5.5.re3. Evaluate the integral:

a. $\displaystyle\int (x^2-1)\cos(x^3-3x)\,dx$ b. $\displaystyle\int x^3\sqrt{x^4-1}\,dx$ c. $\displaystyle\int \frac{e^x}{e^x+1}\,dx$

d. $\displaystyle\int \frac{\sec(\ln x)\tan(\ln x)}{x}\,dx$ e. $\displaystyle\int x^2(x^3+1)^9\,dx$ f. $\displaystyle\int (3x+4)^{10}\,dx$

g. $\displaystyle\int \frac{1}{\sqrt{x}(1+\sqrt{x})^2}\,dx$ h. $\displaystyle\int \frac{\sin x}{\cos x}\,dx$ i. $\displaystyle\int e^{2x-1}\,dx$

Substitutions in definite integrals

5.5.re4. Evaluate the definite integral: $\int_{\pi/4}^{\pi/3} \sec^2 x\sqrt{2+\tan x}\,dx$.

Solution:
We're found the antiderivative in 5.5.re1, so now the integral is easy to evaluate:

$$= \frac{2}{3}(2+\tan x)^{3/2}\Big|_{\pi/4}^{\pi/3} = \frac{2}{3}\big((2+\tan\frac{\pi}{3})^{3/2} - (2+\tan\frac{\pi}{4})^{3/2}\big)$$
$$= \frac{2}{3}\big((2+\sqrt{3})^{3/2} - (2+1)^{3/2}\big)$$

Alternate solution:
Begin as in 5.5.re1. Once we succeed in rewriting the indefinite integral in terms of u, calculate the u-limits:

$$x = \frac{\pi}{3} \quad\Longrightarrow\quad u = 2+\tan\frac{\pi}{3} = 2+\sqrt{3}$$
$$x = \frac{\pi}{4} \quad\Longrightarrow\quad u = 2+\tan\frac{\pi}{3} = 2+1 = 3$$

Use these limits in the u-integral in (5.5.re2). Then, when we integrate, we can skip the step where we rewrite the antiderivative back in terms of x:

$$\int_{\pi/4}^{\pi/3} \sec^2 x\sqrt{2+\tan x}\,dx = \int_3^{2+\sqrt{3}} u^{1/2}\,du$$
$$= \frac{2}{3}u^{3/2}\Big|_3^{2+\sqrt{3}} = \frac{2}{3}\big((2+\sqrt{3})^{3/2} - (2+1)^{3/2}\big)$$

end example 5.5.re4

5.5.re5. Evaluate the integral:

a. $\int_0^1 \sin(\frac{\pi}{2}x)\,dx$ b. $\int_1^4 \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$ c. $\int_{-1}^1 x^2\sin(x^3)\,dx$

d. $\int_{-2}^2 x^2\cos(x^3)\,dx$ e. $\int_0^\pi (\sin x)e^{\cos x}\,dx$ f. $\int_{-1}^1 (2x+3)^5\,dx$

Answers

5.5.re3a. $\frac{1}{3}\sin(x^3-3x)+C$ 5.5.re3b. $\frac{1}{6}(x^4-1)^{3/2}+C$ 5.5.re3c. $\ln(e^x+1)+C$ 5.5.re3d. $\sec(\ln x)+C$
5.5.re3e. $\frac{1}{30}(x^3+1)^{10}+C$ 5.5.re3f. $\frac{1}{33}(3x+4)^{11}+C$ 5.5.re3g. $-2(1+\sqrt{x})^{-1}+C$
5.5.re3h. $-\ln|\cos x|+C$, or $\ln|\sec x|+C$. 5.5.re3i. $\frac{1}{2}e^{2x-1}+C$ 5.5.re5a. $\frac{2}{\pi}$ 5.5.re5b. $2e(e-1)$
5.5.re5c. 0 5.5.re5d. $\frac{2}{3}\sin 8$ 5.5.re5e. $e-e^{-1}$ 5.5.re5f. $\frac{1}{12}(5^6-1)$