MATH 203–01 (Kunkle), Exam 1 Name: ____________
100 pts, 50 minutes Sept. 19, 2014 

No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers. Supporting work will be required on every problem worth more than 2 points.

**Solve** always means to find the general solution, if it exists. You are expected to know the values of all trigonometric functions at multiples of $\pi/4$ and of $\pi/6$.

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1 (6 pts). Compute the following products, if they exist.

a. $\begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 5 \\ -1 & 0 \\ 3 & -2 \end{pmatrix}$ b. $\begin{pmatrix} 1 & 2 \\ -1 & 0 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 5 \\ -1 & 0 \\ 3 & -2 \end{pmatrix}$ c. $\begin{pmatrix} 1 & -1 & 3 \\ 5 & 0 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

2. If $B^{-1} = \begin{pmatrix} 2 & -3 \\ 5 & 6 \end{pmatrix}$ and $C^{-1} = \begin{pmatrix} 1 & 1 \\ -2 & 0 \end{pmatrix}$, find the following.

a (5 pts). The solution $\mathbf{x}$ to $B\mathbf{x} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ b (4 pts). $(B^T)^{-1}$ c (7 pts). $(BC)^{-1}$

3 (21 pts). Find the inverse of $Z = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 1 & 1 \\ 2 & 4 & 9 \end{pmatrix}$.

4 (10 pts). Answer **either** a. **or** b. Clearly indicate which part you're answering.

a. Prove that if $D$ is a $4 \times 4$ matrix and $\mathbf{c}$ is a vector in $\mathbb{R}^4$, and if $D\mathbf{x} = \mathbf{c}$ has no solutions, then $D^T\mathbf{x} = \mathbf{0}$ must have a nontrivial solution.

b. Prove that if $E$ is a $4 \times 4$ matrix and $\mathbf{d}$ is a vector in $\mathbb{R}^4$, and if $E\mathbf{x} = \mathbf{d}$ has exactly one solution, then the transformation $T : \mathbf{x} \mapsto E^2\mathbf{x}$ maps $\mathbb{R}^4$ *onto* $\mathbb{R}^4$.

5. Define $S$ to be the linear transformation from $\mathbb{R}^2$ into $\mathbb{R}^2$ where $S(\mathbf{x})$ is the rotation of $\mathbf{x}$ about the origin by $\pi/4$ radians in the positive (counterclockwise) direction.

a (8 pts). Find the standard matrix for $S$

b (5 pts). Determine whether $S$ is one-to-one.

6a (25 pts). Solve $A\mathbf{x} = \mathbf{b}$ if

$$A = \begin{pmatrix} 1 & -2 & -1 & -4 & -4 \\ 4 & -7 & -4 & -14 & -14 \\ 0 & 3 & 0 & 7 & 8 \end{pmatrix} \text{ and } \mathbf{b} = \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix}.$$

Write your answer in parametric vector form (or as a vector, if only one solution exists).

6b (5 pts). Find the solution to $A\mathbf{x} = \mathbf{0}$ (where $A$ is the same as in 6a).

6c (4 pts). Based on your work above, would you say that $A\mathbf{x} = \mathbf{b}$ is consistent for *all* $\mathbf{b}$ in $\mathbb{R}^3$? Briefly explain.

1\. a. $\begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 5 \\ -1 & 0 \\ 3 & -2 \end{pmatrix} = \begin{pmatrix} 1\cdot 1 - 1(-1) + 1\cdot 3 & 1\cdot 5 - 1\cdot 0 + 1\cdot(-2) \\ 2\cdot 1 + 0(-1) + 3\cdot 3 & 2\cdot 5 + 0\cdot 0 + 3(-2) \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ 11 & 4 \end{pmatrix}$

alt: $\begin{pmatrix} 1 & -1 & 1 \\ 2 & 0 & 3 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ -1 & 0 \\ 3 & -2 \end{pmatrix} = \begin{pmatrix} 5 & 2 \\ 11 & 2 \end{pmatrix}$. b. $(3\times 2)(3\times 2)$ DNE c. $(2\times 3)(2\times 4)$ DNE

2a. $x = B^{-1}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ 5 & 6 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \end{pmatrix}$. Alt: $B^{-1}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 & -3 \\ 6 & 5 \end{pmatrix}\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ 7 \end{pmatrix}$

b. $(B^T)^{-1} = (B^{-1})^T = \begin{pmatrix} 2 & 5 \\ -3 & 6 \end{pmatrix}$. Alt $(B^{-1})^T = \begin{pmatrix} 2 & 6 \\ -3 & 5 \end{pmatrix}$.

c. $(BC)^{-1} = C^{-1}B^{-1} = \begin{pmatrix} 1 & 1 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 7 & 3 \\ -4 & 6 \end{pmatrix}$. Alt $C^{-1}B^{-1} = \begin{pmatrix} 1 & 1 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ 6 & 5 \end{pmatrix} = \begin{pmatrix} 8 & 2 \\ -4 & 6 \end{pmatrix}$.

3\. $(Z \,\vdots\, I) = \begin{pmatrix} 0 & 1 & 3 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 2 & 4 & 9 & 0 & 0 & 1 \end{pmatrix} \underset{r_1 \leftrightarrow r_2}{\sim} \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 1 & 0 & 0 \\ 2 & 4 & 9 & 0 & 0 & 1 \end{pmatrix} \underset{r_3 \leftarrow r_3 - 2r_1}{\sim} \begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 1 & 0 & 0 \\ 0 & 2 & 7 & 0 & -2 & 1 \end{pmatrix}$

$r_3 \leftarrow r_3 - 2r_2$: $\begin{pmatrix} 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & -2 & 1 \end{pmatrix} \underset{\substack{r_2 \leftarrow r_2 - 3r_3 \\ r_1 \leftarrow r_1 - r_3}}{\sim} \begin{pmatrix} 1 & 1 & 0 & 2 & 3 & -1 \\ 0 & 1 & 0 & 7 & 6 & -3 \\ 0 & 0 & 1 & -2 & -2 & 1 \end{pmatrix} \underset{r_1 \leftarrow r_1 - r_2}{\sim} \begin{pmatrix} 1 & 0 & 0 & -5 & -3 & 2 \\ 0 & 1 & 0 & 7 & 6 & -3 \\ 0 & 0 & 1 & -2 & -2 & 1 \end{pmatrix}$

So $Z^{-1} = \begin{pmatrix} -5 & -3 & 2 \\ 7 & 6 & -3 \\ -2 & -2 & 1 \end{pmatrix}$,

Alt:

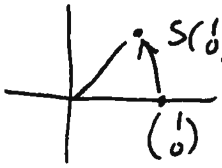
$\begin{pmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 1 & 4 & 0 & 1 & 0 \\ 3 & 1 & 9 & 0 & 0 & 1 \end{pmatrix} \underset{r_2 \leftrightarrow r_1}{\sim} \begin{pmatrix} 1 & 1 & 4 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 3 & 1 & 9 & 0 & 0 & 1 \end{pmatrix} \underset{r_3 \leftarrow r_3 - 3r_1}{\sim} \begin{pmatrix} 1 & 1 & 4 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -2 & -3 & 0 & -3 & 1 \end{pmatrix} \underset{r_3 \leftarrow r_3 + 2r_2}{\sim} \begin{pmatrix} 1 & 1 & 4 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & -3 & 1 \end{pmatrix}$

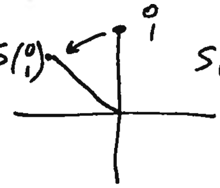
$\substack{r_2 \leftarrow r_2 - 2r_3 \\ r_1 \leftarrow r_1 - 4r_3}$: $\begin{pmatrix} 1 & 1 & 0 & -8 & 13 & -4 \\ 0 & 1 & 0 & -3 & 6 & -2 \\ 0 & 0 & 1 & 2 & -3 & 1 \end{pmatrix} \underset{r_1 \leftarrow r_1 - r_2}{\sim} \begin{pmatrix} 1 & 0 & 0 & -5 & 7 & -2 \\ 0 & 1 & 0 & -3 & 6 & -2 \\ 0 & 0 & 1 & 2 & -3 & 1 \end{pmatrix}$, so $Z^{-1} = \begin{pmatrix} -5 & 7 & -2 \\ -3 & 6 & -2 \\ 2 & -3 & 1 \end{pmatrix}$.

4a. $Dx = c$ has no solutions $\Rightarrow$ Columns of $D$ fail to span $\mathbb{R}^4$ $\Rightarrow$ $D$ not invertible $\Rightarrow$ $D^T$ not invertible $\Rightarrow$ $D^Tx = 0$ has nontrivial solutions.

4b. $Ex = d$ has exactly one solution $\Rightarrow$ $E$ has a pivot in every column $\Rightarrow$ $E$ is invertible $\Rightarrow$ $E\cdot E = E^2$ is invertible $\Rightarrow$ $x \mapsto E^2x$ is onto.

4: Note. We sometimes say that the columns of a matrix are linearly independent. We NEVER say that a matrix is linearly independent. We sometimes say that the columns of a matrix span $\mathbb{R}^m$, but we do NOT say that a matrix spans $\mathbb{R}^m$.

5a.  $S\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\cos \pi/4\\ \sin \pi/4\end{pmatrix} = \begin{pmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{pmatrix}$

 $S\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-\sin \pi/4\\ \cos \pi/4\end{pmatrix} = \begin{pmatrix}-1/\sqrt{2}\\ 1/\sqrt{2}\end{pmatrix}$.

standard matrix $= \left[S\begin{pmatrix}1\\0\end{pmatrix} \;\; S\begin{pmatrix}0\\1\end{pmatrix}\right] = \begin{pmatrix}1/\sqrt{2} & -1/\sqrt{2}\\ 1/\sqrt{2} & 1/\sqrt{2}\end{pmatrix}$.

5b. Solution 1: matrix $\sim \begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix} \sim \begin{pmatrix}1 & -1\\ 0 & 2\end{pmatrix}$. pivot in each col. $\Rightarrow$ 1-1.

Solution 2: Because $S(u)$ is obtained by rotating $u$ $\pi/4$ radians, $u$ can be obtained by rotating $S(u)$ $-\pi/4$ radians. So, if $S(u) = S(v)$, then $u = v$. By definition, $S$ is one-to-one.

6a. $b = (1 \;\; 4 \;\; -1)^T$. Alt: $b = (4 \;\; 15 \;\; -4)^T$. Here's the solution to both:

$(A \,\vdots\, b \,\vdots\, \text{alternate } b)$

$$= \left(\begin{array}{ccccc|c|c} 1 & -2 & -1 & -4 & -4 & 1 & 4\\ 4 & -7 & -4 & -14 & -14 & 4 & 15\\ 0 & 3 & 0 & 7 & 8 & -1 & -4 \end{array}\right) \underset{r_2 \leftarrow r_2 - 4r_1}{\sim} \begin{pmatrix} 1 & -2 & -1 & -4 & -4 & 1 & 4\\ 0 & 1 & 0 & 2 & 2 & 0 & -1\\ 0 & 3 & 0 & 7 & 8 & -1 & -4 \end{pmatrix}$$

$$\underset{r_3 \leftarrow r_3 - 3r_2}{\sim} \begin{pmatrix} 1 & -2 & -1 & -4 & -4 & 1 & 4\\ 0 & 1 & 0 & 2 & 2 & 0 & -1\\ 0 & 0 & 0 & 1 & 2 & -1 & -1 \end{pmatrix} \underset{r_2 \leftarrow r_2 - 2r_3}{\sim} \begin{pmatrix} 1 & -2 & -1 & -4 & -4 & 1 & 4\\ 0 & 1 & 0 & 0 & -2 & 2 & 1\\ 0 & 0 & 0 & 1 & 2 & -1 & -1 \end{pmatrix}$$

$$r_1 \leftarrow r_1 + 4r_3: \begin{pmatrix} 1 & -2 & -1 & 0 & 4 & -3 & 0\\ 0 & 1 & 0 & 0 & -2 & 2 & 1\\ 0 & 0 & 0 & 1 & 2 & -1 & -1 \end{pmatrix} \underset{r_1 \leftarrow r_1 + 2r_2}{\sim} \left(\begin{array}{ccccc|c|c} 1 & 0 & -1 & 0 & 0 & 1 & 2\\ 0 & 1 & 0 & 0 & -2 & 2 & 1\\ 0 & 0 & 0 & 1 & 2 & -1 & -1 \end{array}\right).$$

Solution $Ax = (1 \; 4 \; -1)^T$:

$x_1 = 1 + x_3$
$x_2 = 2 + 2x_5$
$x_3 = \text{free}$
$x_4 = -1 - 2x_5$
$x_5 = \text{free}$

$$x = \begin{pmatrix}1\\2\\0\\-1\\0\end{pmatrix} + x_3\begin{pmatrix}1\\0\\1\\0\\0\end{pmatrix} + x_5\begin{pmatrix}0\\2\\0\\-2\\1\end{pmatrix}.$$

Solution to $Ax = (4 \;\; 15 \; -4)^T$:

$$x = \begin{pmatrix}2\\1\\0\\-1\\0\end{pmatrix} + x_3\begin{pmatrix}1\\0\\1\\0\\0\end{pmatrix} + x_5\begin{pmatrix}0\\2\\0\\-2\\1\end{pmatrix}.$$

6b. Sol'n to $Ax = 0$ is $x = x_3\begin{pmatrix}1\\0\\1\\0\\0\end{pmatrix} + x_5\begin{pmatrix}0\\2\\0\\-2\\1\end{pmatrix}$.

6c. Yes. A has a pivot in each row.