MATH 203–01 (Kunkle), Exam 2 Name: ______________________________

100 pts, 50 minutes Oct. 20, 2014

No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers. Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points.

Solve always means to find the general solution, if it exists.

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1 (5 pts). Define what it means for a set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ in a vector space V to be linearly independent.

2 (10 pts). Answer **either** a. **or** b. Clearly indicate which part you're answering.

a. Prove that the polynomials $\{1, (t+2), (t+2)^2\}$ are linearly independent.

b. Prove that if $T : V \to W$ is linear and if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent in V, then $\{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$ is linearly dependent in W.

3a (20 pts). Compute $\det G$, if $G = \begin{bmatrix} 1 & 1 & 2 & -2 \\ 1 & -3 & 2 & 3 \\ 1 & 5 & 2 & -2 \\ 1 & -7 & -2 & 9 \end{bmatrix}$.

3b (8 pts). Express the element in row 3, column 2 of G^{-1} in terms of determinants, but do not evaluate.

3c (5 pts). Find all solutions to $G\mathbf{x} = \mathbf{0}$.

4 (25 pts). Let H be the matrix $\begin{bmatrix} 1 & -1 & -1 & 2 \\ 2 & -2 & -2 & 4 \\ 1 & -1 & 3 & 0 \\ -1 & 1 & 5 & -4 \end{bmatrix}$.

Find the following. Work in the space provided and label your five answers a-e.

a. A basis for $\operatorname{Col} H$ b. A basis for $\operatorname{Row} H$ c. A basis for $\operatorname{Nul} H$

d. $\operatorname{rank} H$ e. $\det(GH)$, where G is the matrix in Problem 3.

5 (12 pts). Let $\mathcal{B}$ be the basis $\left\{ \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1/3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$ for $\mathbb{R}^3$. Find the coordinates of $x = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$ with respect to this basis. Hint: compute $\begin{bmatrix} 4 & -1 & 0 \\ 0 & 1/3 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/4 & 3/4 & 3/4 \\ 0 & 3 & 3 \\ 0 & 0 & 1 \end{bmatrix}$.

6 (7 pts). Suppose C is a 12×20 matrix whose rows span a vector space of dimension 7. What is the dimension of the set of all solutions to $C\mathbf{x} = \mathbf{0}$?

7 (4 pts). If $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ spans the vector space U, what, if anything, can you conclude about the dimension of U? (You are not required to give reasons for your answer.)

8 (4 pts). Suppose that S is a linearly independent set of 4 vectors in a vector space W, and that $\dim W = 4$. What else, if anything, can you conclude about S? (You are not required to give reasons for your answer.)

1. $\{v_1, v_2, v_3\}$ is linearly independent if $c_1v_1 + c_2v_2 + c_3v_3 = 0 \Rightarrow c_1 = c_2 = c_3 = 0$.

2.a. Suppose $a \cdot 1 + b(t+2) + c(t+2)^2 = 0$. plug in $t = -2 \Rightarrow a \cdot 1 = 0$.
Differentiate: $b + 2c(t+2) = 0$. plug in $t = -2 \Rightarrow b \cdot 1 = 0$.
Differentiate: $2c = 0 \Rightarrow c = 0$.
(Another solution. coordinate vectors are $\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}2\\1\\0\end{pmatrix}, \begin{pmatrix}4\\4\\1\end{pmatrix}$,

and these are linearly independent since there's a pivot in each column. 4.4.25 $\Rightarrow$ poly.s are also linearly ind.

2b. Suppose $\{v_1, v_2, v_3\}$ linearly dependent. Then there exist scalars c_1, c_2, c_3, not all zero, for which $c_1v_1 + c_2v_2 + c_3v_3 = 0$.
Take T of both: $T(c_1v_1 + c_2v_2 + c_3v_3) = c_1T(v_1) + c_2T(v_2) + cT(v_3)$ (by additivity and homogeneity), and this $= T(0) = 0$.
since $c_1T(v_1) + c_2T(v_2) + c_3T(v_3) = 0$ and c_1, c_2, c_3 are not all zero,
$\{T(v_1), T(v_2), T(v_3)\}$ is linearly dependent.

3a Compute $|G|$ by row reduction.

$$\begin{vmatrix} 1 & 1 & 2 & -2 \\ 1 & -3 & 2 & \boxed{3} \\ 1 & 5 & 2 & -2 \\ 1 & -7 & -2 & 9 \end{vmatrix} \overset{1}{=} \begin{vmatrix} 1 & 1 & 2 & -2 \\ 0 & -4 & 0 & \boxed{5} \\ 0 & 4 & 0 & 0 \\ 0 & -8 & -4 & 11 \end{vmatrix} \overset{2}{=} - \begin{vmatrix} 1 & 1 & 2 & -2 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & \boxed{5} \\ 0 & 0 & -4 & 11 \end{vmatrix} \overset{3}{=} \begin{vmatrix} 1 & 1 & 2 & -2 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & -4 & 11 \\ 0 & 0 & 0 & \boxed{5} \end{vmatrix}$$

$\overset{4}{=} 1 \cdot 4(-4) \cdot 5 = -80$.

1. $r_2 \leftarrow r_2 - r_1$, $r_3 \leftarrow r_3 - r_1$, $r_4 \leftarrow r_4 - r_1$
2. $r_2 \leftrightarrow r_3$, then $r_3 \leftarrow r_3 + r_2$, $r_4 \leftarrow r_4 + 2r_2$
3. $r_3 \leftrightarrow r_4$
4. det. of triangular matrix

Alt form: $\boxed{3} \leftarrow 1$, then $\boxed{5} \leftarrow 3$.
answer $= 4(-4) \cdot 3 = -48$.

3b. $G^{-1} = \frac{1}{\det G} \operatorname{adj}(G)$, so $G^{-1}_{32} = \frac{1}{\det G}(\text{cofactor}_{23}) = \frac{-1}{\det G} \cdot \begin{vmatrix} 1 & 1 & -2 \\ 1 & 5 & -2 \\ 1 & -7 & 9 \end{vmatrix}$.

Alt: $G^{-1}_{23} = \frac{1}{\det G}(\text{cofactor}_{3,2}) = \frac{-1}{\det G} \cdot \begin{vmatrix} 1 & 2 & -2 \\ 1 & 2 & 1 \\ 1 & -2 & 9 \end{vmatrix}$

3c. G is invertible b.c. $\det G \neq 0$. Therefore $\operatorname{Nul} G = \{0\}$.
That is, only sol'n to $Gx = 0$ is $x = 0$.

4. $H \sim \begin{pmatrix} 1 & -1 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & -2 \\ 0 & 0 & 4 & -2 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & -1 & 2 \\ 0 & 0 & 4 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & -1 & 0 & 3/2 \\ 0 & 0 & 1 & -1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$

a. Pivot cols of H $= \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -2 \\ 3 \\ 5 \end{pmatrix} \right\}$

b. Pivot Rows of $\underset{\text{any}}{\wedge}$ echelon form $= \{(1\ -1\ -1\ 2), (0\ 0\ 4\ -2)\}$

c. Solve $HX=0$. $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_2 - 3/2\, x_4 \\ x_2 \\ 1/2\, x_4 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -3/2 \\ 0 \\ 1/2 \\ 0 \end{pmatrix}.$

Basis $= \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3/2 \\ 0 \\ 1/2 \\ 0 \end{pmatrix} \right\}$. (other correct ans. include $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$.)

d. Rank H = dim Col H = dim Row H = 2.

e. $\det(GH) = \det G \cdot \det H = (\det G) \cdot 0$ (b.c. H is not invertible) $= 0$.

5. Check: product in hint $= I$, so these matrices are inverses.

Coord. vector c satisfies $\begin{pmatrix} 4 & -1 & 0 \\ 0 & 1/3 & -1 \\ 0 & 0 & 1 \end{pmatrix} c = x$, so $c = \begin{pmatrix} 4 & -1 & 0 \\ 0 & 1/3 & -1 \\ 0 & 0 & 1 \end{pmatrix}^{-1} x = \begin{pmatrix} 1/4 & 3/4 & 3/4 \\ 0 & 3 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} 1/4 \\ 0 \\ 2 \end{pmatrix}$. Alt. coord $= \begin{pmatrix} 1/4 & 3/4 & 3/4 \\ 0 & 3 & 3 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} = \begin{pmatrix} 1/4 \\ 0 \\ -2 \end{pmatrix}$.

6. rank C + dim Nul C = # columns C; rank = dim Row C = 7

7 + dim Nul C = 20 $\Rightarrow$ dim Nul C = 13.

Alt. rank = 8 $\Rightarrow$ dim Nul C = 12.

7. Every spanning set includes a basis, so dim $U \leq 4$.

8. S is a basis for W. (see Basis Thm, p. 259)

(would also accept S spans W.)