Due by end-of-class, 10/10/2014, either in class, in my mailbox (339 RS Small), or in my emailbox (kunklet@cofc.edu).
Use any outside materials that you want (other than a tutor), work with your classmates or not, but do **not** simply submit a solution that you got from someone else. Your proofs should demonstrate not just the correctness of the statements below but also your complete understanding of their meaning. Pay particular attention to instructions I gave you in "How to write subspace proofs," available at http://kunklet.people.cofc.edu/syll203_2015-10.html. I will penalize you for sloppy, ambiguous, ungrammatical, or poor wording, including no wording.
Write your solution on whatever paper you prefer, using as many pages as necessary. Please be neat. You needn't turn in this sheet.

(It's often convenient to refer to a function f without attaching the words "of x," for instance, when no formula for calculating f is required. For example, one can write the familiar sum formula for derivatives as simply $(f+g)' = f' + g'$. Think of f as the name of a function, and $f(x)$ as its output at some real number x.)

1 (8 pts). Define $H = \{f \in C^2(-\infty,\infty) \mid f'' - 2f' + f = \mathbf{0}\}$. That is, H consists of all twice-continuously differentiable functions f on $\mathbb{R}$ for which $f''(x) - 2f'(x) + f(x) = 0$ for all real numbers x.

For example, the function $f(x) = xe^x$ is in H, since

$$f(x) = xe^x \qquad f'(x) = xe^x + e^x \qquad f''(x) = xe^x + 2e^x$$

and so

$$f''(x) - 2f'(x) + f(x) = xe^x + 2e^x - 2(xe^x + e^x) + xe^x = xe^x - 2xe^x - 2e^x + xe^x = 0$$

Prove that H is a subspace of $C^2(-\infty,\infty)$.

2 (2 pts). Define $K = \{f \in C^2(-\infty,\infty) \mid f''(x) - 2f'(x) + f(x) = \sin x \text{ for all real } x\}$. Prove that K is **not** a subspace of $C^2(-\infty,\infty)$.

3 (2 extra credit pts). Find a nonzero function in H (other than a scalar multiple of xe^x) and another in K. Hint: try exponentials and trig functions. "Nonzero function" means a function other than $\mathbf{0}$.

Solutions

1. In the vector space $C^2(-\infty,\infty)$, the zero vector $\mathbf{0}$ is the constant function whose output is the number 0 for every input: $x \in \mathbb{R} \to \mathbf{0}(x) = 0$. Because the derivative of a constant is $\mathbf{0}$, both $\mathbf{0}'$ and $\mathbf{0}''$ equal $\mathbf{0}$. Therefore $\mathbf{0}'' - 2\mathbf{0}' + \mathbf{0} = \mathbf{0} + \mathbf{0} + \mathbf{0} = \mathbf{0}$, so $\mathbf{0}$ is in H.

Suppose that u and v are in H, so that both $u'' - 2u' + u = \mathbf{0}$ and $v'' - 2v' + v = \mathbf{0}$. Then $u + v$, the function defined by the rule $(u+v)(x) = u(x) + v(x)$, is in H because

$$\begin{aligned}
&\big(u+v\big)'' - 2\big(u+v\big)' + \big(u+v\big) \\
&= u'' + v'' - 2u' - 2v' + u + v \\
&= u'' - 2u' + u + v'' - 2v' + v \\
&= \mathbf{0} + \mathbf{0} = \mathbf{0}.
\end{aligned}$$

Thus H is closed under vector addition.

Suppose that u is in H, so that $u'' - 2u' + u = 0$ and that c is a scalar. Then cu, the function defined by the rule $(cu)(x) = c \cdot u(x)$, is in H because $(cu)' = cu'$ and $(cu)'' = cu''$, and therefore

$$\begin{aligned} &(cu)'' - 2(cu)' + (cu) \\ &= cu'' - 2cu' + cu \\ &= c\big(u'' - 2u' + u\big) \\ &= c \cdot \mathbf{0} = \mathbf{0}. \end{aligned}$$

Thus H is closed under scalar multiplication.

Since H is nonempty and closed under vector addition and scalar multiplication, H is a subspace of $C^2(-\infty, \infty)$. QED

2. As noted above, $\mathbf{0}'' - 2\mathbf{0}' + \mathbf{0} = \mathbf{0}$, not $\sin x$, so $\mathbf{0}$ isn't in K. Therefore K is not a vector space. QED

3. The function e^x is in H because $(e^x)'' - 2(e^x)' + e^x = e^x - 2e^x + e^x = 0$ for all real numbers x. (In fact, H is the span of the functions e^x and xe^x.)

The function $\frac{1}{2}\cos x$ is in K, since

$$(\tfrac{1}{2}\cos x)' = -\tfrac{1}{2}\sin x$$

and

$$(\tfrac{1}{2}\cos x)'' = (-\tfrac{1}{2}\sin x)' = -\tfrac{1}{2}\cos x$$

and therefore for all real numbers x,

$$\begin{aligned} &(\tfrac{1}{2}\cos x)' - 2(\tfrac{1}{2}\cos x)' + \tfrac{1}{2}\cos x \\ =\ & -\tfrac{1}{2}\cos x - 2(-\tfrac{1}{2}\sin x) + \tfrac{1}{2}\cos x \\ =\ & \sin x. \end{aligned}$$

(In fact, K equals $\{\frac{1}{2}\cos x + c_1 e^x + c_2 x e^x \mid c_1, c_2 \in \mathbb{R}\}$.) QED