No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points.

You are expected to know the values of all trig functions at multiples of $\pi/4$ and of $\pi/6$.
You may use, without proof, any of these **reduction formulas** that are relevant.

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$

$$\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\,dx$$

$$\int \sec^n x\,dx = \frac{1}{n-1}\sec^{n-2} x\tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx \qquad (n \neq 1)$$

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1. Let $k(x) = x^2 - 4x$.

a(10 pts). Find the average value of $k(x)$ on the interval $[0, 3]$.
b(4 pts). Find all numbers c in $[0, 3]$ at which $k(c)$ equals its average value or explain why none exist.

Hooke's Law. *The magnitude of force necessary to hold a spring distance x beyond its natural length equals kx for some constant k. (That is, the force is proportional to x.)*

2(14 pts). It takes 2 ft-lbs of work to stretch a spring from its natural length of 3 ft to 4 ft. How much work is required to stretch the same spring from 4 ft to 4.5 ft?
Express your answer as a definite integral, but **do not evaluate**.
3(17 pts). Let R be the "triangular" region in the first quadrant bounded by $xy = 1$, $y = 4$, and $x = 1$. Express the following as definite integrals, but **do not evaluate**.

a. The area of R.

b. The volume swept out by R as it is rotated about $x = 2$.

c. The volume of the solid whose footprint in the xy-plane is R and whose cross-sections perpendicular to the x-axis are circles with diameter in R.
4a(4 pts). Find real numbers x and y so that $e^{-\pi i/3} = x + iy$.
4b(10 pts). Expand the binomial $(u + u^{-1})^6$.
4c(11 pts). Write $\sin 4x \cos 3x$ as a sum of sinusoidal functions.
5(30 pts). Evaluate the indefinite integral.

a. $\displaystyle\int \tan^4 x \sec^4 x\,dx$

b. $\displaystyle\int \tan^5 x\,dx$

c. $\displaystyle\int x \ln x\,dx$

d. $\displaystyle\int x \cosh x\,dx$

1.(Source: 6.5.9)

a(10 pts) The average value is $\frac{1}{3}\int_0^3 (x^2-4x)\,dx = \frac{1}{3}(\frac{1}{3}x^3 - 2x^2)\Big|_0^3 = -3$.

b(4 pts). Because $k(x)$ is continuous, the Mean Value Theorem for Integrals guarantees that $k(c)$ equals its average value at least once on the interval $[0,3]$.

$$k(c) = c^2 - 4c = -3 \quad\Longrightarrow\quad 0 = c^2 - 4c + 3 = (c-3)(c-1) \quad\Longrightarrow\quad c = 3 \text{ or } 1.$$

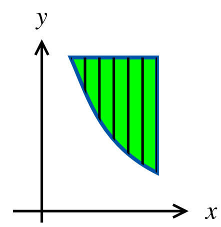
2(14 pts).(Source: 6.4.9a) Warning: don't confuse the **work** necessary to **stretch** the spring from 3 ft to 4 ft with the **force** necessary to **hold** the spring at 4 ft.

Let dw be the work to move the spring dx feet when it's extended x feet beyond its natural length 3. Then $dw = \text{force} \times \text{distance} = kx\,dx$. According to the first sentence, the work to stretch a spring from its natural length of 3 ft to 4 ft, that is, from $x=0$ to $x=1$, is

$$2 = \int_0^1 kx\,dx = \frac{1}{2}kx^2\Big|_0^1 = \frac{1}{2}k,$$

from which we learn that $k=4$. Therefore, the work to to stretch the same spring from 4 ft to 4.5 ft, that is, from $x=1$ to $x=1.5$, is $\int_1^{1.5} 4x\,dx$.

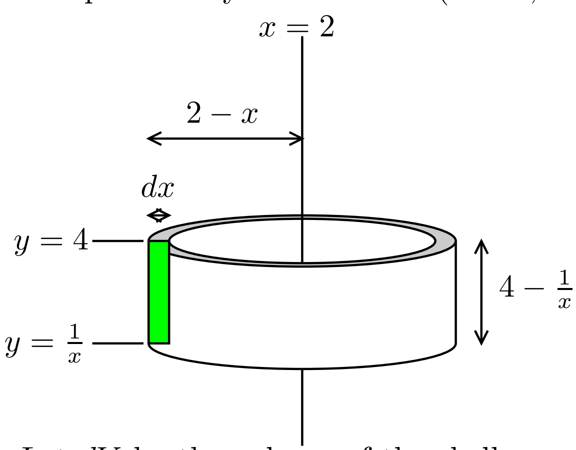
3. The curve $y = \frac{1}{x}$ decreases, hitting the horizontal line $y=4$ at $(\frac{1}{4},4)$ and the vertical line $x=1$ at $(1,1)$. Here's a graph of R, sliced vertically into rectangles (since that will be more useful in part c).

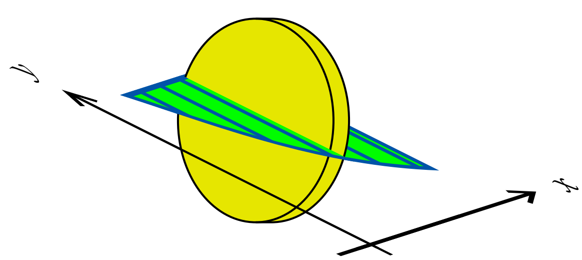


3a(5 pts).(Source: 6.1.9) Let dA be the area of the rectangle located at position x. Then $dA = \text{height} \cdot \text{base} = (4 - \frac{1}{x})\,dx$ and $A = \int dA = \int_{x=1/4}^{x=1} (4-\frac{1}{x})\,dx$.

(You could also slice R horizontally, in which case the left and right ends of the rectangle at altitude y are $x = \frac{1}{y}$ and $x=1$, respectively, and the total area is $A = \int dA = \int_{y=1}^{y=4}(1-\frac{1}{y})\,dy$.)

3b(7 pts).(Source: 6.2.15,6.3.15) When R is rotated about $x=2$, each vertical rectangle sweeps out a cylindrical shell (below, left)





Let dV be the volume of the shell generated by the rectangle at position x. Its height is $4-\frac{1}{x}$, its radius is $2-x$, and its thickness is dx, so $V = \int dV = \int_{1/4}^1 2\pi(2-x)(4-\frac{1}{x})\,dx$.

(If you slice R horizontally, rotating each rectangle generates a washer with inner radius 2 and outer radius $2-\frac{1}{y}$, and so $V=\int_{y=1}^{y=4}\pi\big((2-\frac{1}{y})^2-1^2\big)\,dy$.)

3c(5 pts).(Source: 6.2.more.1p) Slice the solid into infinitely many slices with knife cuts perpendicular to the x-axis. (One slice is shown in yellow above, right.) Let dV be the volume of the (cylindrical) slice at position x. Its circular base has diameter $4-\frac{1}{x}$ and its height is dx, and so $V=\int dV=\int_{1/4}^{1}\pi\big(\frac{1}{2}(4-\frac{1}{x})\big)^2\,dx$, or $\int_{1/4}^{1}\frac{\pi}{4}(4-\frac{1}{x})^2\,dx$

4a(4 pts).(Source: Euler.1.gh) By Euler's formula, $e^{-\pi i/3}=\cos(\frac{-\pi}{3})+i\sin(\frac{-\pi}{3})=\frac{1}{2}-i\frac{\sqrt{3}}{2}$.

4b(10 pts).(Source: Euler.8.a,e) The sixth row of Pascal's triangle is 1 6 15 20 15 6 1 and so

$$
\begin{aligned}
&(u+u^{-1})^6\\
&\quad=u^6+6u^5(u^{-1})^1+15u^4(u^{-1})^2+20u^3(u^{-1})^3+15u^2(u^{-1})^4+6u^1(u^{-1})^5+(u^{-1})^6\\
&\quad=u^6+6u^4+15u^2+20+15u^{-2}+6u^{-4}+u^{-6}
\end{aligned}
$$

4c(11 pts).(Source: Euler.9.a)

$$
\begin{aligned}
\sin 4x\cos 3x &= \left(\frac{e^{i4x}-e^{-i4x}}{2i}\right)\left(\frac{e^{i3x}+e^{-i3x}}{2}\right)=\frac{e^{i7x}-e^{-i7x}+e^{ix}-e^{-ix}}{4i}\\
&=\frac{1}{2}\left(\frac{e^{i7x}-e^{-i7x}}{2i}+\frac{e^{ix}-e^{-ix}}{2i}\right)=\frac{1}{2}(\sin 7x+\sin x).
\end{aligned}
$$

5a(9 pts).(Source: 7.2.22,26) Since the exponent of $\sec x$ is even, we can substitute $t=\tan x$ so that $dt=\sec^2 x\,dx$. The integral becomes

$$
\begin{aligned}
&\int\tan^4 x\sec^4 x\,dx=\int\tan^4 x\sec^2\sec^2 x\,dx=\int\tan^4 x(\tan^2 x+1)\sec^2 x\,dx\\
&=\int t^4(t^2+1)\,dt=\int(t^6+t^4)\,dt=\frac{1}{7}t^7+\frac{1}{5}t^5+C=\frac{1}{7}\tan^7 x+\frac{1}{5}\tan^5 x+C
\end{aligned}
$$

5a. Alternate solution.

$$
\begin{aligned}
&\int\tan^4 x\sec^4 x\,dx=\int\tan^4 x(\sec^2)^2\,dx=\int\tan^4 x(\tan^2 x+1)^2\,dx\\
&=\int\tan^4 x(\tan^4 x+2\tan^2 x+1)\,dx=\int(\tan^8 x+2\tan^6 x+\tan^4 x)\,dx\\
&=\int\tan^8 x\,dx+2\int\tan^6 x\,dx+\int\tan^4 x\,dx\\
&=\frac{1}{7}\tan^7 x-\int\tan^6 x\,dx+2\int\tan^6 x\,dx+\int\tan^4 x\,dx\\
&=\frac{1}{7}\tan^7 x+\int\tan^6 x\,dx+\int\tan^4 x\,dx\\
&=\frac{1}{7}\tan^7 x+\frac{1}{5}\tan^5 x-\int\tan^4 x\,dx+\int\tan^4 x\,dx=\frac{1}{7}\tan^7 x+\frac{1}{5}\tan^5 x+C
\end{aligned}
$$

You could similarly rewrite the integrand entirely in terms of $\sec x$ and use another reduction formula.

5b(5 pts).(Source: 7.2.more.2e) Using the reduction formula for $\int \tan^n x\,dx$ on page 1,

$$\begin{aligned}\int \tan^5 x\,dx &= \frac{1}{4}\tan^4 x - \int \tan^3 x\,dx = \frac{1}{4}\tan^4 x - \left[\frac{1}{2}\tan^2 x - \int \tan x\,dx\right]\\ &= \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x + \ln|\sec x| + C\end{aligned}$$

5b. Alternate solution. Rewrite the integrand in terms of $\sin x$ and $\cos x$.

$$\int \frac{\sin^5 x}{\cos^5 x}\,dx = \int \frac{(1-\cos^2 x)^2}{\cos^5 x}\sin x\,dx = \int \frac{1 - 2\cos^2 x + \cos^4 x}{\cos^5 x}\sin x\,dx.$$

Now substitute $\xi = \cos x$, which means $\sin x\,dx = -d\xi$, and the integral becomes

$$\begin{aligned}&\int \frac{1-2\xi^2+\xi^4}{\xi^5}(-d\xi) = \int (-\xi^{-5} + 2\xi^{-3} - \xi^{-1})\,d\xi\\ &= \frac{1}{4}\xi^{-4} - \xi^{-2} - \ln|\xi| + C = \frac{1}{4}\sec^4 x - \sec^2 x + \ln|\sec x| + C\end{aligned}$$

5c(10 pts).(Source: 7.1.11,26,27) When using integration by parts, choose u and dv so that their product $u\,dv$ exactly equals what follows the integral sign in your problem.

$$\begin{aligned} u &= \ln x & dv &= x\,dx\\ du &= x^{-1}\,dx & v &= \tfrac{1}{2}x^2,\end{aligned}$$

The integral becomes

$$\tfrac{1}{2}x^2\ln x - \int \tfrac{1}{2}x\,dx = \tfrac{1}{2}x^2\ln x - \tfrac{1}{4}x^2 + C$$

5d(6 pts).(Source: 7.1.25, 3.11) Integrate by parts:

$$\begin{aligned} u &= x & dv &= \cosh x\,dx\\ du &= dx & v &= \sinh x\end{aligned}$$

and the integral becomes

$$x\sinh x - \int \sinh x\,dx = x\sinh x - \cosh x + C$$