MATH 220–03 (Kunkle), Final Exam — Name: ______________________

160 pts, 2 hours — Dec 7, 2022 —

No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points.

You are expected to know the values of all trig functions at multiples of $\pi/4$ and of $\pi/6$.

You may use, without proof, any of these **reduction formulas** that are relevant.

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$

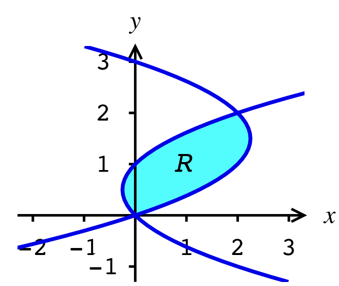
$$\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\,dx$$

$$\int \sec^n x\,dx = \frac{1}{n-1}\sec^{n-2} x\tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx \qquad (n \neq 1)$$

1(8 pts). A cable 20 ft long weighing 8 lbs hangs from the top of a tower 50 ft tall. Find the work to lift the entire cable to the top of the tower. Express your answer as a definite integral, but **do not evaluate**.

2(10 pts). Let L be the graph of $y = \ln x$ for $1 \leq x \leq e$. Find the surface area swept out when L is rotated about the x-axis. Express your answer as a definite integral, but **do not evaluate**.

3(15 pts). Let R be the region bounded by the curves $x = y^2 - y$ and $x = 3y - y^2$. (See figure.) Express the following as definite integrals but **do not evaluate**.



a. The volume swept out when R is rotated about $x = -1$.

b. The volume swept out when R is rotated about $y = 3$.

4(4 pts). Find real numbers x and y so that $e^{3\pi i/4} = x + iy$.

5(14 pts). Find the Maclaurin series for the given function.

a. $\sin(x^3)$

b. $\dfrac{x}{(2+x)^3}$

6(25 pts). Evaluate the indefinite integral.

a. $\displaystyle\int \cos^5 x \sin^3 x\,dx$

b. $\displaystyle\int \frac{x+5}{2x^2 - x - 1}\,dx$

7a(9 pts). Sketch the curve with the polar equation $r = 1 - 2\cos(\theta)$. List all points (r, θ) for $0 \le \theta \le 2\pi$ at which the curve intersects the x- or y-axis.

7b(16 pts). Find the area enclosed by the inner loop of the curve in 7a.

8(6 pts). Find the length of the curve with the polar equation $r = 1 - 2\cos(\theta)$. Express your answer as a definite integral, but **do not evaluate**.

9(15 pts). Evaluate the improper integral $\displaystyle\int_1^{\infty} xe^{-x}\,dx$ or show that it diverges.

10(20 pts). Determine the interval of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2 3^n}$.

11(8 pts). Evaluate the limit $\displaystyle\lim_{n\to\infty} \frac{(\ln n)^2}{n^2}$.

12a(5 pts). Eliminate the parameter to find a Cartesian equation of the curve given parametrically by

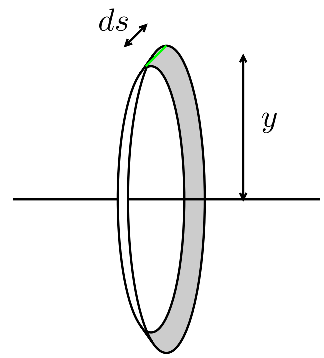
$$\begin{aligned} x &= \cos t \\ y &= \sin^2 t \end{aligned}$$

12b(5 pts). Sketch the curve in the xy-plane as it is drawn from $t = 0$ to $t = 2\pi$. Use arrows and label points to show clearly how the curve is traced as t increases over this interval.

1(8 pts).(Source: 6.4.13) The actual height of the tower is irrelevant if we measure altitude not from the ground, but from the initial position of the lower end of the cable; for instance, before any lifting, the end of the cable is at altitude 0, and the top of the tower is at altitude 20.

Let dw be the work to lift the cable dy ft when the bottom of the cable is at altitude y. The cable weighs $\frac{8}{20} = \frac{2}{5}$ lb/ft, so when the lower end is at altitude y, the part of the chain still hanging from the tower weighs $(20 - y)$ ft $\times \frac{2}{5}$ lb/ft $= \frac{2}{5}(20 - y)$ lb. Lifting this dy ft takes $dw = \frac{2}{5}(20 - y)\,dy$ ft-lbs of work. To lift the entire chain takes dw ft-lbs of work for each y between 0 and 20, so the total work is $\int_0^{20} \frac{2}{5}(20 - y)\,dy$.

2(10 pts).(Source: 8.1.9-20, 8.2.7-12, 6.5.1-8) Slice the curve into infinitesimal segments of length ds. Slope along this curve is $\frac{dy}{dx} = \frac{1}{x}$, so its length is

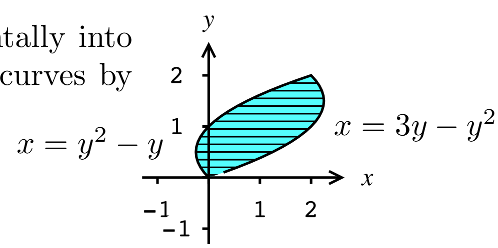


$$ds = \sqrt{1 + \frac{dy}{dx}^2}\,dx = \sqrt{1 + x^{-2}}\,dx$$

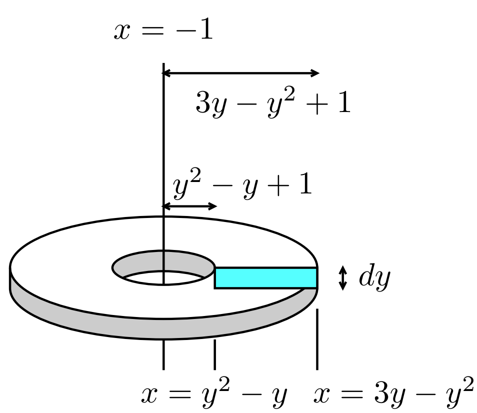
When the segment of curve at the point (x, y) is rotated about the x-axis, it generates a ribbon of radius y and area $dA = 2\pi y\,ds$. The total area is

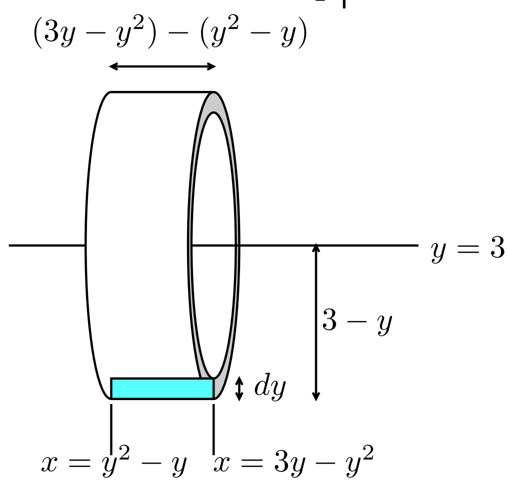
$$\int_1^e 2\pi \ln x \sqrt{1 + x^{-2}}\,dx$$

3(15 pts).(Source: 6.2.17,6.3.19,20) Here's R, sliced horizontally into rectangles. Solve for the intersection points of the two curves by setting $y^2 - y = 3y - y^2$ and find $y = 0$ and $y = 2$.



a. When each rectangle is rotated about the vertical line $x = -1$, it generates a washer, as shown below left.





The volume of this washer is

$$dV = \pi(3y - y^2 + 1)^2\,dy - \pi(y^2 - y + 1)^2\,dy,$$

and so the volume in question is

$$V = \int_0^2 \pi\big((3y - y^2 + 1)^2 - \pi(y^2 - y + 1)^2\big)\, dy$$

b. When each rectangle is rotated about the horizontal line $y = 3$, it generates a shell, as shown above right. The volume of this shell is

$$dV = 2\pi(3 - y)\big((3y - y^2) - (y^2 - y)\big)\, dy,$$

and the volume in question is

$$V = \int_0^2 2\pi(3 - y)(4y - 2y^2)\, dy$$

4(4 pts).(Source: Euler.1.gh) By Euler's formula, $e^{3\pi i/4} = \cos(\frac{3\pi}{4}) + i\sin(\frac{3\pi}{4}) = -\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$.

5a(? pts).(Source: 11.10.35,36, and 11.9 generally) $\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$, so

$$\sin(x^3) = \sum_{n=0}^{\infty}(-1)^n \frac{(x^3)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{6n+3}}{(2n+1)!}$$

5b(? pts).(Source: 11.9.13,17,11.10.33,41) Solution one. Use the binomial series $(1+x)^k = \sum_{n=0}^{\infty}\binom{k}{n}x^n$ to rewrite $x(2+x)^{-3} = 2^{-3}x(1+\frac{x}{2})^{-3} = 2^{-3}x(1+\frac{x}{2})^{-3}$ as

$$2^{-3}x\sum_{n=0}^{\infty}\binom{-3}{n}\left(\frac{x}{2}\right)^n = \sum_{n=0}^{\infty}\binom{-3}{n}\frac{x^{n+1}}{2^{n+3}}$$

5b. Solution two. $\frac{2}{(1-x)^3} = \frac{d^2}{dx^2}\left(\frac{1}{1-x}\right)$, so

$$\frac{1}{(1-x)^3} = \frac{1}{2}\cdot\frac{d^2}{dx^2}\left(\frac{1}{1-x}\right) = \frac{1}{2}\cdot\frac{d^2}{dx^2}\sum_{n=0}^{\infty}x^n = \sum_{n=2}^{\infty}\frac{n(n-1)}{2}x^{n-2}.$$

Replace x with $-x/2$:

$$\frac{1}{(1+\frac{x}{2})^3} = \frac{1}{2}\sum_{n=2}^{\infty}n(n-1)\left(\frac{-x}{2}\right)^{n-2} = \sum_{n=2}^{\infty}\frac{n(n-1)}{2}\frac{(-1)^{n-2}x^{n-2}}{2^{n-2}}$$

Multiply the above by $\frac{1}{2^3}x$:

$$\frac{x}{(2+x)^3} = \frac{1}{2^3}x\cdot\sum_{n=2}^{\infty}n(n-1)\frac{(-1)^{n-2}x^{n-2}}{2^{n-1}} = \sum_{n=2}^{\infty}n(n-1)\frac{(-1)^{n-2}x^{n-1}}{2^{n+2}}.$$

You could rewrite $(-1)^{n-2}$ as $(-1)^n$, but no penalty if you did not.

6a(? pts).(Source: 7.2.13) Since the exponents of sine and cosine are both odd, you can't use a reduction formula, but you can substitute either $u = \sin x$ or, more simply, $u = \cos x$. Using this, the integral becomes

$$\begin{aligned}\int \cos^5 x \sin^2 x \sin x\, dx &= \int u^5(1-u^2)(-du) = \int (u^7 - u^5)\, du \\ &= \frac{1}{8}u^8 - \frac{1}{6}u^6 + C = \frac{1}{8}\cos^8 x - \frac{1}{6}\cos^6 x + C.\end{aligned}$$

6b.(Source: 7.4.29) Find the partial fraction decomposition of the integrand. The degree of the numerator is strictly less than that of the denominator, so long division is not necessary. The denominator factors as $(2x+1)(x-1)$, so we look for constants A and B for which

$$\frac{x+5}{(2x+1)(x-1)} = \frac{A}{2x+1} + \frac{B}{x-1}.$$

Multiply both sides by $(2x+1)(x-1)$:

$$x + 5 = A(x-1) + B(2x+1).$$

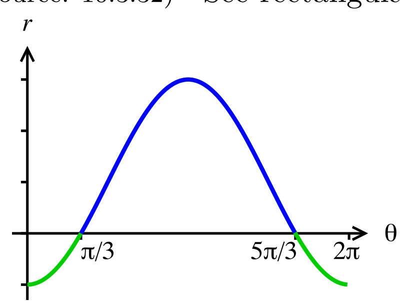
Now do the same things to both sides to obtain two equations in the two unknowns A and B. The easiest thing to do in this case is to evaluate at $x = 1$ and $x = -1/2$.

$$\begin{aligned} x = -1/2: &\qquad \tfrac{9}{2} = A \cdot \left(-\tfrac{3}{2}\right) &\implies&\quad A = -3 \\ x = 1: &\qquad 6 = B \cdot 3 &\implies&\quad B = 2. \end{aligned}$$

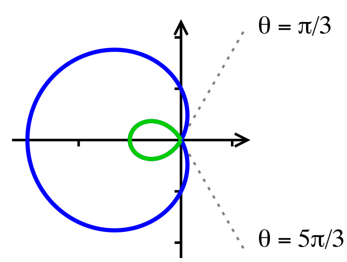
Now that we have the PDF, it's easy to integrate:

$$\int \frac{x+5}{(2x+1)(x-1)}\, dx = \int \left(-\frac{3}{2x+1} + \frac{2}{x-1}\right) dx = -\frac{3}{2}\ln|2x+1| + 2\ln|x-1| + C.$$

7a(9 pts).(Source: 10.3.32) See rectangular and polar graphs of $r = 1 - 2\cos\theta$ below.



$r = 1 - 2\cos\theta$ (rectangular)



$r = 1 - 2\cos\theta$ (polar)

The polar graph is a limaçon. Since $-1 \le r \le 3$, the curve contains a loop-inside-a-loop. The small loop is drawn with negative r's when $-\pi/3 \le \theta \le \pi/3$. Intercepts:

θ	0	$\pi/3$	$\pi/2$	π	$3\pi/2$	$5\pi/3$	2π
r	-1	0	1	3	1	0	-1

7b(16 pts).(Source: 10.4.21) The area is $\int_{-\pi/3}^{\pi/3} \frac{1}{2} r^2 \, d\theta$, and by symmetry, this is twice the area from $\theta = 0$ to $\theta = \pi/3$:

$$2 \int_0^{\pi/3} \frac{1}{2}(1 - 2\cos\theta)^2 \, d\theta = \int_0^{\pi/3} (1 - 4\cos\theta + 4\cos^2\theta) \, d\theta.$$

Rewrite $\cos^2\theta$ either by Euler's formula:

$$\cos^2\theta = \left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)^2 = \frac{1}{4}(e^{i2\theta} + e^{-i2\theta} + 2) = \frac{1}{2}(\cos(2\theta) + 1)$$

or by remembering this formula from trigonometry. The area equals

$$\begin{aligned}\int_0^{\pi/3} (1 - 4\cos\theta + 2(\cos(2\theta) + 1)) \, d\theta &= \int_0^{\pi/3} (3 - 4\cos\theta + 2\cos(2\theta)) \, d\theta \\ &= \big(3\theta - 4\sin\theta + \sin(2\theta)\big)\Big|_0^{\pi/3} \\ &= \pi - 4\sin(\pi/3) + \sin(2\pi/3) = \pi - \frac{3\sqrt{3}}{2}\end{aligned}$$

8(6 pts).(Source: 10.4.45-48) Using $ds = \sqrt{r^2 + (\frac{dr}{d\theta})^2} \, d\theta$, total length is

$$\int_0^{2\pi} \sqrt{(1 - 2\cos(\theta))^2 + (-2\sin\theta)^2} \, d\theta$$

9(15 pts).(Source: 7.8.19,20) First find the indefinite integral by parts:

$$\begin{aligned} u &= x \quad & dv &= e^{-x} \, dx \\ du &= dx \quad & v &= -e^{-x} \end{aligned}$$

The integral becomes

$$\begin{aligned}\int xe^{-x} \, dx &= \int u \, dv = uv - \int v \, du \\ &= -xe^{-x} + \int e^{-x} \, dx = -xe^{-x} - e^{-x} + C = -(x+1)e^{-x} + C\end{aligned}$$

The improper integral is the limit

$$\lim_{G \to \infty} \int_1^G xe^{-x} \, dx = \lim_{G \to \infty} -(x+1)e^{-x}\Big|_1^G = \lim_{G \to \infty} (-\frac{G+1}{e^G} + 2e^{-1})$$

By l'Hospital's Rule,

$$\lim_{G\to\infty} -\frac{G+1}{e^G} = \text{“}\frac{\infty}{\infty}\text{”} \overset{HR}{\hookrightarrow} \lim_{G\to\infty} -\frac{1}{e^G} = 0$$

and so the improper integral equals $2e^{-1}$.

10(20 pts).(Source: 11.8.12) Begin with either Root or Ratio test. Here's Ratio:

$$\lim_{n\to\infty}\left|\frac{\frac{(x-2)^{n+1}}{(n+1)^2 3^{n+1}}}{\frac{(x-2)^n}{n^2 3^n}}\right| = \lim_{n\to\infty}\frac{n^2}{(n+1)^2}\frac{|x-2|}{3}.$$

By l'Hôpital's Rule, or by Limit 6 of https://kunklet.people.cofc.edu/MATH220/flesk.pdf, $\frac{n^2}{(n+1)^2}$ goes to 1, and so the limit is $\frac{|x-2|}{3}$.

$$\frac{|x-2|}{3} < 1 \iff |x-2| < 3 \iff -3 < x-2 < 3 \iff -1 < x < 5.$$

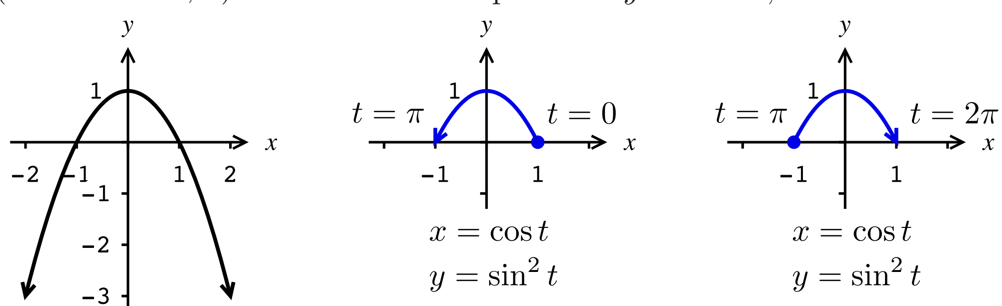
Now test for convergence at the endpoints. At $x = 5$, power series is $\sum_{n=1}^{\infty}\frac{3^n}{n^2 3^n} = \sum_{n=1}^{\infty}\frac{1}{n^2}$. The p-series $\sum\frac{1}{n^p}$ converges iff $p > 1$, so the power series converges at $x = 5$. At $x = -1$, the power series is $\sum_{n=1}^{\infty}\frac{(-3)^n}{n^2 3^n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$, which converges absolutely, since $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges.

Therefore the interval of convergence is $[-1, 5]$.

11(8 pts).(Source: 4.4.32,11.1.50) The limit is $\frac{\infty}{\infty}$, and we can apply l'Hospital's Rule to this quotient as long it remains indeterminate:

$$\lim_{n\to\infty}\frac{(\ln n)^2}{n^2} \overset{HR}{\hookrightarrow} \lim_{n\to\infty}\frac{2(\ln n)\frac{1}{n}}{2n} = \lim_{n\to\infty}\frac{(\ln n)}{n^2} = \text{“}\frac{\infty}{\infty}\text{”} \overset{HR}{\hookrightarrow} \lim_{n\to\infty}\frac{\frac{1}{n}}{2n} = \lim_{n\to\infty}\frac{1}{2n^2} = 0.$$

12a(5 pts).(Source: 10.1.18,22) The curve is the parabola $y = 1 - x^2$, since $\sin^2 t = 1 - \cos^2 t$.



$y = 1 - x^2$

$x = \cos t$
$y = \sin^2 t$
$0 \le t \le \pi$

$x = \cos t$
$y = \sin^2 t$
$\pi \le t \le 2\pi$

12b(5 pts). The parametrization covers the part of the parabola in which $-1 \le x \le 1$. As t goes from 0 to π, as $x = \cos t$ decreases from 1 to -1, the parametrization draws this arc once from right to left. Then, as t goes from π to 2π, as x increases from -1 to 1, the parametrization redraws the arc from left to right. See figure.