MATH 220–02 (Kunkle), Exam 1 Name: ______________________________
100 pts, 75 minutes Sept 12, 2023

No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points.

You are expected to know the values of all trigonometric functions at multiples of $\pi/4$ and of $\pi/6$.
You may use, without proof, any of these **reduction formulas** that are relevant.

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$
$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$
$$\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\,dx$$
$$\int \sec^n x\,dx = \frac{1}{n-1}\sec^{n-2} x\tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx$$

1(10 pts). How much work is required to lift 60 lbs of coal from the bottom of a 200-foot mine shaft with a (200-foot) cable weighing 20 lbs?
Express your answer as a definite integral, but **do not evaluate**.

2(23 pts). Let R be the region in the xy-plane bounded by $y = \sin x$ and $y = \cos x$ between $x = 0$ and $x = \frac{\pi}{4}$. Find the following. Work in the space provided and label your answers. Express your answers as definite integrals, but **do not evaluate**.

a. The area of R.

b. The volume of the solid whose base is R and whose cross-sections perpendicular to the x-axis are squares with one side in the xy-plane.

c. The volume swept out by R as it rotated about the y-axis.

d. The volume swept out by R as it rotated about the line $y = 1$.

3(7 pts). Find the average value of $\tan x$ on the interval $\left[0, \frac{\pi}{4}\right]$.

4(12 pts). Express $\cos^6 x$ as a sum of sinusoidal functions. You are not required to integrate this sum.

5(48 pts). Evaluate the indefinite integral.

a. $\int \sin^5 x \cos^3 x\,dx$

b. $\int \tan^2 x \sec^3 x\,dx$

c. $\int \sin(5x)\cos(6x)\,dx$

d. $\int x^2 \sinh x\,dx$

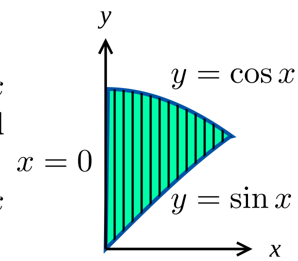
1(10 pts).(Source: 6.4.15) The weight-density of the cable is $\frac{20}{200}$lb/ft $= \frac{1}{10}$lb/ft. Let dw be the work it takes to lift the coal and remaining cable dy feet when the coal is at altitude y. At that moment, the remaining $200 - y$ feet of cable weighs $\frac{1}{10}(200 - y)$ lb, so

$$dw = (\tfrac{1}{10}(200 - y) + 60)\, dy.$$

The total work is

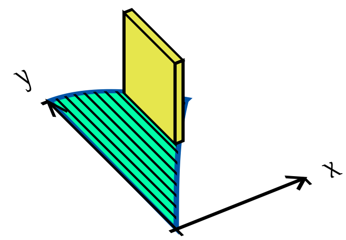
$$w = \int dw = \int_0^{200} (\tfrac{1}{10}(200 - y) + 60)\, dy$$

(or $\int_0^{200}(80 - \frac{1}{10}y)\, dy$).



2(23 pts). To get a decent working sketch of R, remember that $y = \cos x$ is above $y = \sin x$ between $x = 0$ and $x = \frac{\pi}{4}$. See sketch of R (sliced vertically) at the right.

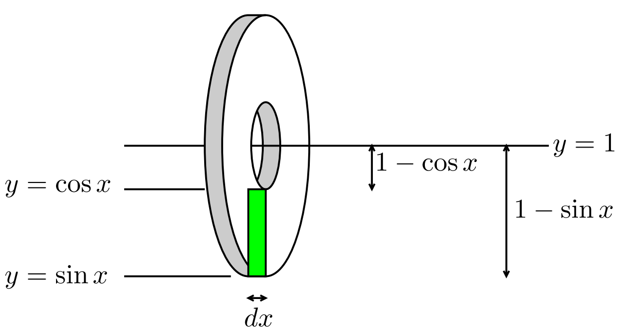
a.(Source: 6.1.15,16) The rectangle at position x has height $\cos x - \sin x$ and width dx, so the area of R equals $\int dA = \int_0^{\pi/4}(\cos x - \sin x)\, dx$.

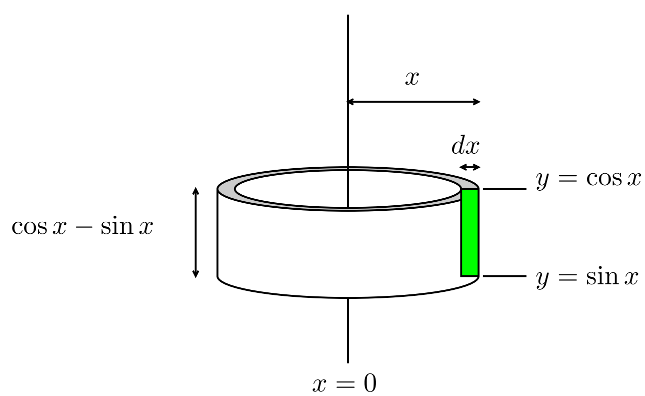


b.(Source: 6.2.58,6.2.more.1.c,f,h,. . .) Slice the solid with knife cuts perpendicular the the x-axis and let dV be the volume of the slice at position x, shown at the right.
The dimensions of this slice are $(\cos x - \sin x) \times (\cos x - \sin x) \times dx$, so $V = \int dV = \int_0^{\pi/4}(\cos x - \sin x)^2\, dx$.

c.(Source: 6.3.6,7) When R is rotated about the y-axis, the result is a shell (below right) and

$$V = \int dV = \int_0^{\pi/4} 2\pi x(\cos x - \sin x)\, dx$$





d.(Source: 6.2.14) When R is rotated about $y = 1$, the result is a washer (above left) and

$$V = \int dV = \int_0^{\pi/4} \pi\big((1 - \sin x)^2 - (1 - \cos x)^2\big)\, dx$$

3(7 pts).(Source: 6.5.1-8, 7.2.more.2b, 7.2.23) The average value equals

$$\frac{1}{\frac{\pi}{4}}\int_0^{\pi/4}\tan x\,dx = \frac{4}{\pi}\ln|\sec x|\Big|_0^{\pi/4} = \frac{4}{\pi}(\ln|\sec\frac{\pi}{4}| - \ln|\sec 0|) = \frac{4}{\pi}(\ln\sqrt{2} - \ln 1) = \frac{4}{\pi}\ln\sqrt{2}.$$

4(12 pts).(Source: Euler.9e) A sinusoidal function is a function of the form $A\sin(Bx+C)+D$ or $A\cos(Bx+C)+D$.
Use Euler's formula, and the 6th row of Pascal's Triangle, 1 6 15 20 15 6 1:

$$\begin{aligned}\cos^6 x = \left(\frac{e^{ix}+e^{-ix}}{2}\right)^6 &= \frac{1}{64}\left(e^{i6x}+6e^{i4x}+15e^{i2x}+20+15e^{-i2x}+6e^{-i4x}+e^{-i6x}\right)\\ &= \frac{1}{32}\left(\frac{e^{i6x}+e^{-i6x}}{2}+6\frac{e^{i4x}+e^{-i4x}}{2}+15\frac{e^{i2x}+e^{-i2x}}{2}+10\right)\\ &= \frac{1}{32}(\cos 6x+6\cos 4x+15\cos 2x+10)\end{aligned}$$

5a(12 pts).(Source: 7.2.2,3) Since the exponents of sine and cosine are both odd, you can't use a reduction formula, but you can substitute either $u=\cos x$ or, more simply, $u=\sin x$. Using this, $du=\cos x\,dx$, and the integral becomes

$$\begin{aligned}\int \sin^5 x\cos^2 x\cos x\,dx &= \int u^5(1-u^2)\,du = \int (u^5-u^7)\,du\\ &= \frac{1}{6}u^6-\frac{1}{8}u^8+C = \frac{1}{6}\sin^6 x-\frac{1}{8}\sin^8 x+C.\end{aligned}$$

If we had substituted $w=\cos x$ instead, then $dw=-\sin x\,dx$, and the integral is

$$\begin{aligned}\int \sin^4 x\cos^3 x\sin x\,dx &= \int (1-w^2)^2w^3\,(-dw) = -\int (1-2w^2+w^4)w^3\,dw\\ &= -\int (w^3-2w^5+w^7)\,dw = -\frac{1}{4}w^4+\frac{2}{6}w^6-\frac{1}{8}w^8+C\\ &= -\frac{1}{4}\cos^4 x+\frac{1}{3}\cos^6 x-\frac{1}{8}\cos^8 x+C.\end{aligned}$$

Finally, you could use Euler to rewrite the integrand (as in Example 10 of Euler) and then integrate the result. It's a fun exercise and too much effort for an exam but the result is that

$$\sin^5 x\cos^3 x = \frac{1}{2^7}\big(\sin(8x)-2\sin(6x)-2\sin(4x)+6\sin(2x)\big)$$

so that the integral is

$$\frac{1}{2^7}\left(-\frac{1}{8}\cos(8x)+\frac{1}{3}\cos(6x)+\frac{1}{2}\cos(4x)-3\cos(2x)\right)+C.$$

See these three antiderivatives at https://www.desmos.com/calculator/5sofhlu062

5b(12 pts).(Source: 7.2.more.2r) When $\tan x$ appears to an even power and $\sec x$ to an odd power, substituting $u = \tan x$ or $u = \sec x$ won't work. Instead, rewrite the integral as

$$\int \tan^2 x \sec^3 x\, dx = \int (\sec^2 x - 1) \sec^3 x\, dx = \int \sec^5 x\, dx - \int \sec^3 x\, dx$$

and use the reduction formula:

$$\begin{aligned}
&= \tfrac{1}{4} \sec^3 x \tan x + \tfrac{3}{4} \int \sec^3 x\, dx - \int \sec^3 x\, dx \\
&= \tfrac{1}{4} \sec^3 x \tan x - \tfrac{1}{4} \int \sec^3 x\, dx \\
&= \tfrac{1}{4} \sec^3 x \tan x - \tfrac{1}{4} \left(\tfrac{1}{2} \sec x \tan x - \tfrac{1}{2} \int \sec x\, dx \right) \\
&= \tfrac{1}{4} \sec^3 x \tan x - \tfrac{1}{8} \sec x \tan x + \tfrac{1}{8} \ln |\sec x + \tan x| + C
\end{aligned}$$

5c(12 pts).(Source: 7.2.41, Euler.10c) Rewrite the integrand using Euler's formula:

$$\begin{aligned}
\sin 5x \cos 6x &= \left(\frac{e^{i5x} - e^{-i5x}}{2i} \right) \left(\frac{e^{i6x} + e^{-i6x}}{2} \right) \\
&= \frac{e^{i11x} - e^{-i11x} - e^{ix} + e^{-ix}}{4i} \\
&= \frac{1}{2} \left(\frac{e^{i11x} - e^{-i11x}}{2i} - \frac{e^{ix} - e^{-ix}}{2i} \right) \\
&= \frac{1}{2} (\sin 11x - \sin x).
\end{aligned}$$

Now integration is straightforward:

$$\begin{aligned}
\int \sin 5x \cos 6x\, dx &= \frac{1}{2} \int (\sin 11x - \sin x)\, dx \\
&= \frac{-1}{22} \cos 11x + \frac{1}{2} \cos x + C
\end{aligned}$$

5d(12 pts).(Source: 7.1.25,28) Integrate by parts twice:

$$\begin{aligned}
u &= x^2 & dv &= \sinh x\, dx \\
du &= 2x\, dx & v &= \cosh x
\end{aligned}$$

$$\int x^2 \sinh x\, dx = uv - \int v\, du = x^2 \cosh x - \int 2x \cosh x\, dx$$

$$\begin{aligned}
U &= 2x & dV &= \cosh x\, dx \\
dU &= 2\, dx & V &= \sinh x
\end{aligned}$$

$$\begin{aligned}
&= x^2 \cosh x - \left[2x \sinh x - \int 2 \sinh x\, dx \right] \\
&= x^2 \cosh x - 2x \sinh x + 2 \cosh x + C
\end{aligned}$$