No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points. You are expected to know the values of all trigonometric functions at multiples of $\pi/4$ and of $\pi/6$.

You may use any of these formulas that are relevant.

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x \cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$
$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$
$$\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\,dx$$
$$\int \sec^n x\,dx = \frac{1}{n-1}\sec^{n-2} x \tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx$$

$$|E_T| \le \frac{K(b-a)^3}{12n^2} \qquad |E_S| \le \frac{L(b-a)^5}{180n^4}$$

1(14 pts). Find the arclength of the curve $y = \frac{1}{4}x^2 - \frac{1}{2}\ln x$ over $1 \le x \le 3$.

2(12 pts). Evaluate the improper integral, if it converges: $\displaystyle\int_0^1 \frac{e^{-1/x}}{x^2}\,dx$

3(12 pts). Evaluate the limit, if it exists: $\lim_{x\to 1}\left(\frac{x}{x-1} - \frac{1}{\ln x}\right)$

4(10 pts). What trigonometric substitution would you make in the following integrals? Write your answer on the line, using the space below for any necessary work.
Do not carry out the substitution. Do not integrate.

a. $\int (x^2-4)^{-3/2}\,dx$ $x =$ ____________ $dx =$ ____________

b. $\int (x^2-2x+2)^{9/2}\,dx$ $x =$ ____________ $dx =$ ____________

5(14 pts). Evaluate the indefinite integral $\displaystyle\int \frac{dx}{(1-9x^2)^{3/2}}$

6(14 pts). Evaluate the indefinite integral $\displaystyle\int \frac{2x^2+2x+8}{x^3-4x^2}\,dx$

7(12 pts). Approximate the integral $\int_1^3 \sqrt{1+x^3}\,dx$ using the Trapezoid Rule with $n = 6$ subintervals. Repeat with Simpson's Rule. Label your answers so I can tell which is which. Leave unsimplified, unfinished arithmetic in your answers, but otherwise be completely explicit about how they are to be calculated. Your answer should not include "..." or anything else for me to fill in.

8(12 pts). Suppose $g(x)$ is a function for which

$$|g(x)| \leq 2 \qquad |g'(x)| \leq 3 \qquad |g''(x)| \leq 4 \qquad |g'''(x)| \leq 5 \qquad |g''''(x)| \leq 6$$

for all x in $[-1, 1]$, and that we wish to approximate $\int_{-1}^1 g(x)\,dx$ with the Trapezoid Rule. State your answers to the following questions as **inequalities**, rather than equations.

a. If we use $n = 10$ subintervals, how large might be the absolute error of our approximation?

b. If we need the absolute error to be at most 10^{-6}, how large must we take n?

1(14 pts).(Source: 8.1.12) $ds = \sqrt{\left(\frac{dy}{dx}\right)^2 + 1}\,dx =$

$$\sqrt{(\tfrac{1}{2}x - \tfrac{1}{2}x^{-1})^2 + 1}\,dx = \sqrt{\tfrac{1}{4}x^2 - \tfrac{1}{2} + \tfrac{1}{4}x^{-2} + 1}\,dx = \sqrt{\tfrac{1}{4}x^2 + \tfrac{1}{2} + \tfrac{1}{4}x^{-2}}\,dx$$
$$= \sqrt{(\tfrac{1}{2}x + \tfrac{1}{2}x^{-1})^2}\,dx = (\tfrac{1}{2}x + \tfrac{1}{2}x^{-1})\,dx$$

(since the latter is positive). Therefore the arclength is

$$s = \int ds = \int_1^3 \left(\tfrac{1}{2}x + \tfrac{1}{2}x^{-1}\right)\,dx = \left(\tfrac{1}{4}x^2 + \tfrac{1}{2}\ln x\right)\Big|_1^3 = 2 + \tfrac{1}{2}\ln 3.$$

2(12 pts).(Source: 7.8.14,39,40) The bad endpoint is $x = 0$, where the integrand doesn't exist. Rewrite the improper integral as a limit:

$$\int_0^1 \frac{e^{-1/x}}{x^2}\,dx = \lim_{h\to 0^+}\int_h^1 \frac{e^{-1/x}}{x^2}\,dx.$$

Substitute $u = -x^{-1}$, so that $du = x^{-2}\,dx$. The indefinite integral becomes $\int e^u\,du = e^u + C = e^{-x^{-1}} + C$, and then the improper integral is

$$\lim_{h\to 0^+} e^{-x^{-1}}\Big|_h^1 = \lim_{h\to 0^+}\left(e^{-1} - e^{-\frac{1}{h}}\right)$$

As $h \to 0^+$, $-\frac{1}{h} \to -\infty$, and $e^{-\frac{1}{h}} \to 0$. Therefore the limit is e^{-1}.

3(12 pts).(Source: 4.4.51) The limit looks like $\frac{1}{0} - \frac{1}{0}$, or $\infty - \infty$, which is indeterminate. Add the fractions (Corrections in Red)

$$\frac{x}{x-1} - \frac{1}{\ln x} = \frac{x\ln x - (x-1)}{(x-1)\ln x}$$

and try l'Hospital's Rule:

$$\lim_{x\to 1}\frac{x\ln x - (x-1)}{(x-1)\ln x} = \text{“}\frac{0}{0}\text{”} \overset{HR}{\longrightarrow} \lim_{x\to 1}\frac{\ln x + x\cdot\frac{1}{x} - 1}{\ln x + (x-1)x^{-1}}$$
$$= \lim_{x\to 1}\frac{\ln x}{\ln x + 1 - x^{-1}} = \text{“}\frac{0}{0}\text{”}$$
$$\overset{HR}{\longrightarrow} \lim_{x\to 1}\frac{x^{-1}}{x^{-1} + x^{-2}} = \frac{1}{2}$$

4a(4 pts).(Source: 7.3.3,7,8,13) In order for $x^2 - 4$ to equal $4\sec^2\theta - 4$, let $x = 2\sec\theta$, and then $dx = 2\sec\theta\tan\theta\,d\theta$.

4b(6 pts).(Source: 7.3.23,27) Complete the square to eliminate the x^1 term:

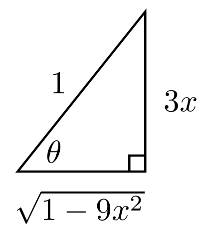
$$x^2 - 2x + 2 = x^2 - 2x + 1 + 1 = (x-1)^2 + 1.$$

So that this might equal $\tan^2\theta + 1$, we let $x - 1 = \tan\theta$, or $x = 1 + \tan\theta$, and $dx = \sec^2\theta\,d\theta$.

5(14 pts).(Source: 7.3.9,10) We wish $1 - 9x^2 = 1 - \sin^2\theta$, which equals $\cos^2\theta$, so substitute $3x = \sin\theta$ and $3\,dx = \cos\theta\,d\theta$. The integral becomes

$$\int \frac{\frac{1}{3}\cos\theta}{\cos^3\theta}\,d\theta = \frac{1}{3}\int \frac{1}{\cos^2\theta}\,d\theta = \frac{1}{3}\int \sec^2\theta\,d\theta = \frac{1}{3}\tan\theta + C.$$

If you didn't recognize it as the derivative of $\tan\theta$, you still could have integrated $\sec^2\theta$ using the reduction formula. (And now, learn the derivatives of the six trig functions.) To convert back to x, draw a right triangle that contains θ. Label two sides using $3x = \sin\theta$ and find the third by Pythagorus. Conclude that



$$\tan\theta = \frac{3x}{\sqrt{1-9x^2}}$$

and so the integral equals

$$\frac{x}{\sqrt{1-9x^2}} + C.$$

6(14 pts).(Source: 7.4.16,19) The degree of the numerator is strictly less than that of the denominator, so long division is not necessary. Factor the denominator and then look for constants A, B, and C for which

$$\frac{2x^2+2x+8}{(x-4)x^2} = \frac{A}{x-4} + \frac{B}{x} + \frac{C}{x^2}.$$

Multiply both sides by $(x-4)x^2$:

$$2x^2 + 2x + 8 = Ax^2 + B(x-4)x + C(x-4).$$

Now do the same things to both sides to obtain three equations in the three unknowns A, B, and C. The easiest thing to do in this case is to evaluate at $x = 0$ and $x = 4$. After that, I chose to equate the x^2 coefficients:

$$\begin{aligned} x = 0: &\quad 8 = -4C \\ x = 4: &\quad 48 = 16A \qquad \Longrightarrow \quad A = 3, \quad B = -1, \quad C = -2. \\ x^2\text{-coefficient}: &\quad 2 = A + B \end{aligned}$$

Now integrate:

$$\int \frac{2x^2+2x+8}{x^3-4x^2}\,dx = \int \left(\frac{3}{x-4} - \frac{1}{x} - \frac{2}{x^2}\right)dx = 3\ln|x-4| - \ln|x| + 2x^{-1} + C.$$

7(12 pts).(Source: 7.7.7) The subintervals each have length $\Delta x = (3-1)/6 = 1/3$, so their seven endpoints are 1, 4/3, 5/3, 6, 7/3, 8/3, and 3.

$$T_6 = \frac{1/3}{2}\Big(\sqrt{1+2^3} + 2\sqrt{1+(\tfrac{4}{3})^3} + 2\sqrt{1+(\tfrac{5}{3})^3} \\ +2\sqrt{1+2^3} + 2\sqrt{1+(\tfrac{7}{3})^3} + 2\sqrt{1+(\tfrac{8}{3})^3} + \sqrt{1+3^3}\Big)$$

$$S_6 = \frac{1/3}{3}\Big(\sqrt{1+2^3} + 4\sqrt{1+(\tfrac{4}{3})^3} + 2\sqrt{1+(\tfrac{5}{3})^3} \\ +4\sqrt{1+2^3} + 2\sqrt{1+(\tfrac{7}{3})^3} + 4\sqrt{1+(\tfrac{8}{3})^3} + \sqrt{1+3^3}\Big)$$

8a(6 pts).(Source: 7.7.more.1,2) Using $K = 4$, and $b - a = 2$

$$|E_T| \le \frac{4 \cdot 2^3}{12 \cdot 10^2}.$$

8b(6 pts).(Source: 7.7.more.1,2) To make $|E_T| \le 10^{-6}$, find n that will make $\frac{4 \cdot 2^3}{12 \cdot n^2} \le 10^{-6}$. Solve this inequality for n:

$$\frac{4 \cdot 2^3}{12} 10^6 \le n^2 \quad \Longrightarrow \quad \sqrt{\frac{4 \cdot 2^3}{12} 10^6} \le n,$$

(or, $10^3\sqrt{\frac{8}{3}} \le n$.) **(done)**

A common mistake is to conclude either $n = 10^3\sqrt{\frac{8}{3}}$ or $n \le 10^3\sqrt{\frac{8}{3}}$. The first isn't possible, since $10^3\sqrt{\frac{8}{3}}$ isn't an integer. The second is also incorrect, since it actually guarantees *nothing* about the size of the error. It also suggests that trapezoid rule is less accurate for larger n than for smaller n, which is counterintuitive.