No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points. You are expected to know the values of all trigonometric functions at multiples of $\pi/4$ and of $\pi/6$.

1(6 pts). Find the area swept out by the curve $y = \ln x$, $1 \le x \le e$ as it is rotated about the x-axis. Express your answer as a definite integral, but **do not evaluate**.

2. The series $s = \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n \ln n}$ converges.

a(4 pts). If we approximate s with the partial sum $s_{100} = \sum_{n=2}^{100} \frac{(-1)^{n+1}}{n \ln n}$, how large might the absolute error be? That is, find a number B so that $|s - s_{100}| \le B$.

b(4 pts). Is s_{100} an overestimate or an underestimate of s? Briefly explain.

3(18 pts). Evaluate the limit, if it exists. Show your work.

a. $\lim_{n\to\infty} \ln\left(\frac{n+1}{n}\right)$ b. $\lim_{n\to\infty} \frac{\sin(n^2)}{n^2}$ c. $\lim_{n\to\infty} \frac{3^n}{3^n + 2^n}$

4(30 pts). Determine whether the series converges or diverges. Justify your conclusion.

a. $\sum_{n=1}^{\infty} \left(\frac{1-2n}{3n-2}\right)$ b. $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$ c. $\sum_{n=1}^{\infty} \frac{e^{-1/n}}{n}$

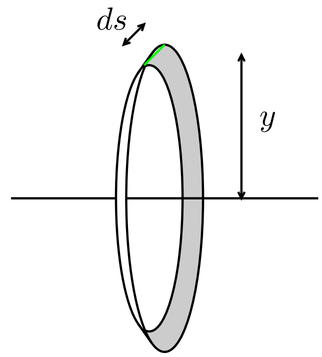
5(18 pts). Determine whether the series converges absolutely, converges conditionally, or diverges. Justify your conclusion.

a. $\sum_{n=1}^{\infty} \left(\frac{1-2n}{3n-2}\right)^n$ b. $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{n}{n^2-2}\right)$

6a(8 pts). Find the radius of convergence of the power series: $\sum_{n=0}^{\infty} \frac{(x-1)^n}{n!}$

6b(7 pts). Find the interval of convergence of the power series: $\sum_{n=1}^{\infty} (x+1)^n$

6c(5 pts). When it converges, what is the sum of the series in 6b?



1(6 pts).(Source: 8.1.9-20, 8.2.7-12, 6.5.1-8) Slice the curve into infinitesimal segments of length ds. Slope along this curve is $\frac{dy}{dx} = \frac{1}{x}$, so its length is

$$ds = \sqrt{1 + \frac{dy}{dx}^2}\,dx = \sqrt{1 + x^{-2}}\,dx$$

When the segment of curve at the point (x, y) is rotated about the x-axis, it generates a ribbon of radius y and area $dA = 2\pi y\,ds$. The total area is

$$\int_1^e 2\pi \ln x \sqrt{1 + x^{-2}}\,dx$$

2(Source: 11.5.more1a-f, part i,ii) . When its hypotheses are satisfied, the Alternating Series Test tells us that the sum of the series lies between any two consecutive partial sums. In this case, s is between s_{100} and $s_{101} = s_{100} + \frac{(-1)^{102}}{101\ln(101)}$.

a(4 pts). The distance between s and s_{100} is therefore less or equal the distance from s_{100} to s_{101}. That is, $|s - s_{100}| \le \left|\frac{(-1)^{102}}{101\ln(101)}\right| = \frac{1}{101\ln(101)}$.

b(4 pts). $\frac{(-1)^{102}}{101\ln(101)}$ is positive, and so $s_{100} < s_{101}$). Because s is between these, s must be greater than s_{100}. That is, s_{100} is an underestimate of s.

3a(4 pts).(Source: 11.1.31,32,42) By FLESK 5, $\lim_{n\to\infty}\left(\frac{n+1}{n}\right) = \lim_{n\to\infty}\left(\frac{n}{n}\right) = 1$, and since $\ln x$ is continuous, $\lim_{n\to\infty}\ln\left(\frac{n+1}{n}\right) = \ln\left(\lim_{n\to\infty}\frac{n+1}{n}\right) = \ln 1 = 0$.

3b(8 pts).(Source: 11.1.43)

$$-1 \le \sin(n^2) \le 1 \quad\Longrightarrow\quad -\frac{1}{n^2} \le \frac{\sin(n^2)}{n^2} \le \frac{1}{n^2}$$

Since $\lim_{n\to\infty} -\frac{1}{n^2} = \lim_{n\to\infty}\frac{1}{n^2} = 0$, the Squeeze Theorem implies that $\lim_{n\to\infty}\frac{\sin(n^2)}{n^2}$ also equals 0.

3c(6 pts).(Source: 11.1.30) As it is written, this limit is of the indeterminate form $\frac{\infty}{\infty}$, but l'Hospital's Rule doesn't produce a simpler limit, so, instead, rewrite the sequence by dividing top and bottom by 3^n:

$$(\star)\qquad \frac{3^n}{3^n + 2^n} = \frac{1}{1 + \frac{2^n}{3^n}} = \frac{1}{1 + (\frac{2}{3})^n}.$$

You could arrive at $(\star)$ by factoring out the dominant term 3^n from numerator and denominator and canceling. By FLESK 1, the limit of this is $\frac{1}{1+0} = 1$.

4a(6 pts).(Source: 11.2.33,36, 11.1.29) By FLESK 5, $\lim\limits_{n\to\infty}\left(\frac{1-2n}{3n-2}\right) = \lim\limits_{n\to\infty}\left(\frac{-2n}{3n}\right) = \frac{-2}{3}$.

Since this limit is not zero, $\sum_{n=1}^{\infty}\left(\frac{1-2n}{3n-2}\right)$ diverges by the nth Term Test.

4b(14 pts).(Source: 11.3.22, 11.4.40) Here are two solutions:
Solution one: The function $f(x) = \frac{\ln x}{x^3}$ is positive on $[2, \infty)$. To see if it's decreasing, examine its derivative:

$$f'(x) = \frac{x^{-1}x^3 - 3x^2 \ln x}{x^6} = \frac{x^2 - 3x^2 \ln x}{x^6} = \frac{1 - 3\ln x}{x^4}$$

x^4 is positive as long as $x \neq 0$, and $1 - 3\ln x$ must be < 0 on some interval $[K, \infty)$, since its limit is $-\infty$. (In fact, $1 - 3\ln x < 0$ when $x > e^{1/3}$.) Therefore, integral test says that

$$\sum_{n=1}^{\infty} f(n) \text{ and } \int_1^{\infty} f(x)\, dx$$

must both converge or both diverge.
Using integration by parts,

$$\begin{aligned} u &= \ln x & dv &= x^{-3}\, dx \\ du &= x^{-1}\, dx & v &= -\frac{1}{2}x^{-2} \end{aligned}$$

the indefinite integral

$$\begin{aligned} \int \frac{\ln x}{x^3}\, dx &= uv - \int v\, du \\ &= -\frac{1}{2}x^{-2}\ln x + \frac{1}{2}\int x^{-2}x^{-1}\, dx \\ &= -\frac{1}{2}x^{-2}\ln x + \frac{1}{2}\int x^{-3}\, dx = -\frac{1}{2}x^{-2}\ln x - \frac{1}{4}x^{-2} + C. \end{aligned}$$

To evaluate the improper integral, rewrite it as a limit:

$$\begin{aligned} \lim_{\beta\to\infty} \int_1^{\beta} \frac{\ln x}{x^3}\, dx &= \lim_{\beta\to\infty} \left(-\frac{1}{2}x^{-2}\ln x - \frac{1}{4}x^{-2} \right) \Big|_1^{\beta} \\ &= \lim_{\beta\to\infty} \left[\left(-\frac{1}{2}\beta^{-2}\ln\beta - \frac{1}{4}\beta^{-2} \right) - \left(-\frac{1}{4} \right) \right] \\ &= -\frac{1}{2} \lim_{\beta\to\infty} \left[\frac{\ln\beta}{\beta^2} \right] - \frac{1}{4}\cdot 0 + \frac{1}{4} \end{aligned}$$

Can use l'Hôpital's Rule on the remaining $\frac{\infty}{\infty}$ limit to obtain

$$\lim_{\beta\to\infty} \left[\frac{\beta^{-1}}{2\beta} \right] = \lim_{\beta\to\infty} \left[\frac{1}{2\beta^2} \right] = 0.$$

Therefore the improper integral converges to $\frac{1}{4}$. By the Integral Test, the series also converges.

Solution two: As in an example seen in class Monday, we can try to limit-compare this series to $\sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$ with the help of l'Hospital's Rule:

$$\lim_{n\to\infty} \frac{\frac{\ln n}{n^3}}{\frac{1}{n^{5/2}}} = \lim_{n\to\infty} \frac{\ln n}{n^{1/2}} = \text{“}\frac{\infty}{\infty}\text{”} \overset{HR}{\hookrightarrow} \lim_{n\to\infty} \frac{n^{-1}}{\frac{1}{2}n^{-1/2}} = \lim_{n\to\infty} \frac{1}{\frac{1}{2}n^{1/2}} = 0.$$

Since the limit is zero, the Limit Comparison Test is inconclusive. But, we can now try the comparison test. Because $\frac{\frac{\ln n}{n^3}}{\frac{1}{n^{5/2}}} \to 0$, the numerator must be less than the denominator on some interval $[K, \infty)$:

$$0 < \frac{\ln n}{n^3} \leq \frac{1}{n^{5/2}}$$

(See **Limit Comparison Test, continued**, review notes, page 45.) It is important to note that original series is positive, so that we can apply the Comparison Test. Since $\sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$ is a convergent p-series ($p = \frac{5}{2} > 1$), $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$ converges by the Comparison Test.

4c(10 pts).(Source: 11.4.28) Limit-compare with harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$:

$$\lim_{n\to\infty} \frac{e^{-1/n}}{n} \div \frac{1}{n} = \lim_{n\to\infty} e^{-1/n} = e^{\lim_{n\to\infty} -1/n} = e^0 = 1$$

(using the continuity of e^x). Since this limit is positive and finite, the Limit Comparison Test says that $\sum_{n=1}^{\infty} \frac{e^{-1/n}}{n}$ and the harmonic series must both converge or both diverge. Since the harmonic series diverges, so must $\sum_{n=1}^{\infty} \frac{e^{-1/n}}{n}$.

5a(7 pts).(Source: 11.6.32) Root Test:

$$\lim_{n\to\infty} \left(\left| \frac{1-2n}{3n-2} \right|^n \right)^{1/n} = \lim_{n\to\infty} \left| \frac{1-2n}{3n-2} \right| = \left| \lim_{n\to\infty} \left(\frac{1-2n}{3n-2} \right) \right| = \left| -\frac{2}{3} \right| = \frac{2}{3}$$

(as seen in problem 4a). Since this limit is less than 1, the series converges absolutely.

5b(11 pts).(Source: 11.6.6) Test first for absolute convergence. The series

$$\sum_{n=1}^{\infty} \left| (-1)^{n+1} \left(\frac{n}{n^2-2} \right) \right| = \sum_{n=1}^{\infty} \left(\frac{n}{n^2-2} \right)$$

can be compared to the harmonic series because

$$0 \leq \frac{1}{n} = \frac{n}{n^2} \leq \frac{n}{n^2-2}$$

Since the harmonic series diverges, so does $\sum_{n=1}^{\infty} \frac{n}{n^2-2}$, and so the original series fails to converge absolutely.

Now check for conditional convergence. $b_n = \frac{n}{n^2-2}$ is positive for $n \geq 2$ and decreasing since its derivative

$$\frac{1(n^2-2) - n \cdot 2n}{(n^2-2)^2} = \frac{-n^2-2}{(n^2-2)^2}$$

is always negative. Therefore, the Alternating Series Test tells us that the original series converges. Since it does not converge absolutely, it converges conditionally.

6a(8 pts).(Source: 11.8.7,19) At $x = 1$ the power series is $1 + 0 + 0 + \cdots$, which converges. (In fact, every power series converges at its center.) If $x \neq 1$, then we can take the limit

$$\lim_{n\to\infty} \frac{\left|\frac{(x-1)^{n+1}}{(n+1)!}\right|}{\left|\frac{(x-1)^n}{n!}\right|} = \lim_{n\to\infty} \frac{|x-1|^{n+1}}{|x-1|^n} \cdot \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{|x-1|}{(n+1)} = 0,$$

and since this is less than 1, the series converges absolutely. Because the power series converges for all x, its radius of convergence is ∞.

6b(7 pts).(Source: 11.2.58, also 11.8.3-20) The series is geometric with $r = x + 1$. It converges if and only if $-1 < x + 1 < 1$, so its interval of convergence is $(-2, 0)$.

6c(5 pts).(Source: 11.2.58) When it converges, the sum of the geometric series is

$$\frac{\text{first term}}{1-r} = \frac{x+1}{1-(x+1)} = -\frac{x+1}{x}$$