MATH 220–02 (Kunkle), Exam 4 — Name: ______________________

100 pts, 75 minutes — Nov 21, 2023 —

No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points. You are expected to know the values of all trigonometric functions at multiples of $\pi/4$ and of $\pi/6$.

1(4 pts). Evaluate the expression.

a. $\binom{-2/3}{3} =$ b. $\binom{\sqrt{2}}{0} =$ c. $\binom{4}{6} =$

Note: On later problems, you can use $\binom{k}{n}$ in your answers without having to explain what that symbol means.

2a(10 pts) Find $T_2(x)$, the second-degree Taylor polynomial for xe^x centered at $a = 2$.

2b(10 pts). Find an upper bound on the absolute error when xe^x is approximated by $T_2(x)$ (from part a), on the interval $[1.75, 2.25]$.

3(30 pts). Find the Maclaurin series for the given function.

a. $\dfrac{1}{(1+3x)^2}$ b. $\ln(3+x)$ c. x^2e^x

d. $\sinh x$ e. $\cos(x^3)$

4(10 pts). Find the solution to the initial value problem

$$x + x^2(y+y^3)\frac{dy}{dx} = 1 \qquad x=-1 \quad y=0$$

5a(6 pts). Eliminate the parameter to find a Cartesian equation for the curve parametrized by $x = 1+\sin t$, $y = -2+\cos t$.

5b(6 pts). Sketch the curve from 5a and indicate with an arrow the direction in which the curve is traced as the parameter t increases. Describe the curve you're trying to draw so that I can better understand your sketch.

6a(12 pts). Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ along the curve parametrized by $x = t-\sin t$, $y = t+\sin t$.

6b(3 pts). For what times t in $[0, 2\pi]$ is the line tangent to this curve horizontal?

6c(3 pts). For what times t in $[0, 2\pi]$ is the line tangent to this curve vertical?

6d(6 pts). Find the length of the curve for $0 \le t \le \pi$. Express your answer as a definite integral, but **do not evaluate**.

1. (Source: Students were told in class and on Oaks to expect this problem from 11.10.) See page 53 of the review notes https://kunklet.people.cofc.edu/MATH220/220review.pdf. The reason these don't include the formula $\binom{k}{n} = \frac{k!}{n!(k-n)!}$ is that it only applies when n, k and $k-n$ are nonnegative integers.

a(2 pts). $\binom{-2/3}{3} = \frac{(-\frac{2}{3})(-\frac{2}{3}-1)(-\frac{2}{3}-2)}{3\cdot 2\cdot 1} = -\frac{40}{81}$.

b(1 pts). $\binom{\sqrt{2}}{0} = 1$. In fact, $\binom{k}{0} = 1$ for all k.

c(1 pts). $\binom{4}{6} = \frac{4\cdot 3\cdot 2\cdot 1\cdot 0\cdot(-1)}{6!} = 0$.

2. Here are the first 3 derivatives and the first 3 coefficients of the Taylor series.

n	$f^{(n)}(x)$	$f^{(n)}(2)/n!$
0	xe^x	$2e^2$
1	$(x+1)e^x$	$3e^2$
2	$(x+2)e^x$	$4e^2/2$
3	$(x+3)e^x$	irrelevant

2a(10 pts).(Source: 11.11.13-21a) The second degree Taylor polynomial is $T_2(x) = 2e^2 + 3e^2(x-2) + 2e^2(x-2)^2$.

2b(10 pts).(Source: 11.11.13-21b) On $[1.75, 2.25]$, $|f^{(3)}(c)| = (c+3)e^c \le (5.25)e^{2.25}$, so, according to Taylor's Theorem,

$$|xe^2 - T_2(x)| = \frac{|f^{(3)}(c)||x-2|^3}{3!} \le \frac{(5.25)e^{2.25}}{3!}(0.25)^3$$

See https://www.desmos.com/calculator/n7esckeexs to compare this upper bound with the absolute error $|xe^x - T_2(x)|$ on $[1.75, 2.25]$.

3a(5 pts).(Source: 11.9.13, 11.10.33) **Solution one.** Differentiate $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ to obtain $\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}$ and then replace x by $-3x$:

$$\frac{1}{(1+3x)^2} = \sum_{n=1}^{\infty} n(-3x)^{n-1} = \sum_{n=1}^{\infty} n(-1)^{n-1}3^{n-1}x^{n-1}$$

Solution two. From the binomial series $(1+x)^{-2} = \sum_{n=0}^{\infty} \binom{-2}{n} x^n$ obtain

$$(1+3x)^{-2} = \sum_{n=0}^{\infty} \binom{-2}{n} 3^n x^n.$$

You can see that the two series are the same by comparing the coefficient of x^n in each:

$$\binom{-2}{n}3^n = \frac{\overbrace{-2(-3)(-4)\cdots(-n-1)}^{n \text{ factors}}}{n!}3^n = (-1)^n\frac{(n+1)!}{n!}3^n = (n+1)(-1)^n 3^n$$

3b(7 pts).(Source: 11.9.15, 11.10.40) **Solution one.** First find the Maclaurin series for $\dfrac{1}{3+x} = \dfrac{1}{3}\cdot\dfrac{1}{1-\left(\frac{-x}{3}\right)} = \dfrac{1}{3}\sum_{n=0}^{\infty}\left(\dfrac{-x}{3}\right)^n = \sum_{n=0}^{\infty}(-1)^n\dfrac{x^n}{3^{n+1}}$. Now integrate:

$$\ln(3+x) = \int \frac{1}{3+x}\,dx = \sum_{n=0}^{\infty}(-1)^n\int\frac{x^n}{3^{n+1}}\,dx = C + \sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{(n+1)3^{n+1}}.$$

Evaluate at $x=0$ to find C:

$$\ln 3 = C + \sum_{n=0}^{\infty} 0 \quad\Longrightarrow\quad \ln(3+x) = \ln 3 + \sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{(n+1)3^{n+1}}.$$

Solution two. $\ln(1+x) = \sum_{n=0}^{\infty}(-1)^n\dfrac{x^{n+1}}{n+1}$, and so

$$\ln(3+x) = \ln\left(3\left(1+\frac{x}{3}\right)\right) = \ln 3 + \ln\left(1+\frac{x}{3}\right) = \ln 3 + \sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{(n+1)3^{n+1}}$$

3c(6 pts).(Source: 11.10.47) $x^2e^x = x^2\sum_{n=0}^{\infty}\dfrac{1}{n!}x^n = \sum_{n=0}^{\infty}\dfrac{1}{n!}x^{n+2}$

3d(6 pts).(Source: 11.10.17) **Solution one.** Calculate the first few terms of the Maclaurin series:

n	$f^{(n)}(x)$	$f^{(n)}(0)/n!$
0	$\sinh x$	0
1	$\cosh x$	1
2	$\sinh x$	0
3	$\cosh x$	$1/3!$
4	$\sinh x$	0
5	$\cosh x$	$1/5!$
6	$\sinh x$	0

The Maclaurin series is $x + \dfrac{1}{3!}x^3 + \dfrac{1}{5!}x^5 + \cdots = \sum_{n=0}^{\infty}\dfrac{x^{2n+1}}{(2n+1)!}$

Solution two. Use the Maclaurin series for e^x and e^{-x} and the definition of $\sinh x$:

$$\begin{aligned}\sinh x &= \frac{1}{2}(e^x - e^{-x})\\ &= \frac{1}{2}(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \cdots)\\ &\quad -(1 - x + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \cdots)\\ &= \frac{1}{2}(\quad 2x \quad + 2\frac{1}{2!}x^3 \quad + 2\frac{1}{5!}x^5 + \cdots)\\ &= x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}\end{aligned}$$

3e(6 pts).(Source: 11.10.39) Since $\cos x = \sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n)!}x^{2n}$,

$$\cos(x^3) = \sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n)!}(x^3)^{2n} = \sum_{n=0}^{\infty}(-1)^n \frac{1}{(2n)!}x^{6n}$$

4(10 pts).(Source: 9.3.14) Separate variables and integrate:

$$\begin{aligned}\int (y + y^3)\, dy &= \int \left(\frac{1-x}{x^2}\right) dx = \int (x^{-2} - x^{-1})\, dx\\ \frac{1}{2}y^2 + \frac{1}{4}y^4 &= -x^{-1} - \ln|x| + C\end{aligned}$$

Evaluate at the initial data $x = -1$, $y = 0$ to find $0 = -1 - 0 + C$, so the solution is

$$\frac{1}{2}y^2 + \frac{1}{4}y^4 = -x^{-1} - \ln|x| + 1$$

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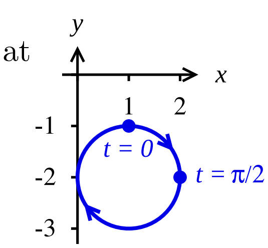
5a(6 pts).(Source: 10.1.8,20) Observe $x - 1 = \sin t$ and $y + 2 = \cos t$, so $(x-1)^2 + (y+2)^2 = 1$.

5b(6 pts). These equations parametrize the circle of radius 1 centered at the point $(1, -2)$. By calculating x and y at $t = 0$ and $t = \pi/2$:



t	x	y
0	1	-1
$\pi/2$	2	-2

we see that the circle is traced in the clockwise direction. Not convinced? See the animated graph at https://www.desmos.com/calculator/naxejqnesx.

6a(12 pts).(Source: 10.2.15) $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{1+\cos t}{1-\cos t}$.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{-\sin t(1-\cos t)-(1+\cos t)\sin t}{(1-\cos t)^2}}{1-\cos t} = \frac{-2\sin t}{(1-\cos t)^3}$$

6b(3 pts).(Source: 10.2.19-20) $\frac{dy}{dx}$ is zero when $1+\cos t = 0$, at $t = \pi$.

6c(3 pts).(Source: 10.2.19-20) $\frac{dy}{dx} \to \pm\infty$ when $1-\cos t \to 0$, at $t = 0$ and $t = 2\pi$.

6d(6 pts).(Source: 10.2.41-44) $ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$, so the length is

$$\int_0^{\pi} \sqrt{(1-\cos t)^2 + (1+\cos t)^2}\, dt$$

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