MATH 221–02 (Kunkle), Exam 1 Name: ______________________

100 pts, 75 minutes January 29, 2019

No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points.

1(9 pts). Determine whether the lines L_1 and L_2 are intersecting, parallel, or skew. If they intersect, find their point (x, y, z) of intersection.

$$\begin{array}{llll} L_1: & x = t - 1 & y = 10 - 4t & z = -7 + 2t \\ L_2: & x = -2 - 3s & y = 3 + s & z = 3 + 2s \end{array}$$

2(22 pts). Express $\mathbf{T}$, $\mathbf{N}$, and the curvature κ in terms of t along the curve parametrized by $\mathbf{r}(t) = \langle t\cos t - \sin t,\ t\sin t + \cos t,\ \frac{1}{2}t^2\rangle$ $(t > 0)$. Label your answers so I can tell which is which.

3(18 pts). Let $\mathbf{u} = \langle 1, -1, -2\rangle$ and $\mathbf{v} = \langle 3, 1, 0\rangle$ and find the following.

a. $2\mathbf{u} + 3\mathbf{v}$

b. The length of $\mathbf{u}$.

c. The cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$. (You are not require to state the angle.)

d. The vector projection of $\mathbf{v}$ onto $\mathbf{u}$.

e. A nonzero vector orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

f. The area of the parallelogram with sides $\mathbf{u}$ and $\mathbf{v}$.

4(16 pts). Let $\mathbf{r}(t) = e^t\mathbf{i} + (t-1)\sqrt{2}\mathbf{j} + e^{-t}\mathbf{k}$ and find the following.

a. The domain of $\mathbf{r}(t)$

b. $\dfrac{d\mathbf{r}}{dt}$

c. The line tangent to the curve parametrized by $\mathbf{r}(t)$ at the point $(e, 0, e^{-1})$.

d. The length of the curve parametrized by $\mathbf{r}(t)$ for $0 \le t \le 1$.

5(23 pts). Find the equation(s) of the line or plane described in each part.

a. The line passing through the points $(1, -1, -2)$ and $(3, 1, 0)$.

b. The plane containing the point $(0, 1, -1)$ and the line $\frac{x-1}{2} = y + 2 = \frac{z-3}{4}$.

c. The plane passing through the point $(0, 1, -1)$ parallel to $x - y - 2z = -1$.

6(12 pts). Sketch the surface. Label your x-, y-, and z-axes and use arrows to indicate the positive direction on each. If you think your drawing needs some explaining, describe what you're trying to draw.

a. $x^2 + \dfrac{y^2}{4} = 1$

b. $x^2 + \dfrac{y^2}{4} + z^2 = 1$

1(1 pts).(Source: 12.5.19) The lines are not parallel, since L_1 is parallel the vector $\langle 1, -4, 2\rangle$ and L_2 is parallel $\langle -3, 1, 2\rangle$, and neither of these is a scalar multiple of the other.
To see if the lines intersect, set the coordinates equal and solve for s and t:

$$\begin{aligned} t-1 &= -2-3s & \quad & t+3s=-1 \\ 10-4t &= 3+s & \Rightarrow \quad & 4t+s=7 \\ -7+2t &= 3+2s & \quad & 2t-2s=10 \end{aligned} \tag{1}$$

Solving the 1st and 2nd equations yields $t = 2$, $s = -1$, and but when we check, these values cause the third equation to read "$-3 = 1$," and so (1) has no solution and lines don't intersect. Correct response is **skew**.

2(1 pts).(Source: 13.3.19) $\frac{d\mathbf{r}}{dt} = \langle -t\sin t,\ t\cos t,\ t\rangle$. $\frac{ds}{dt} = \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{t^2 + t^2\sin^2 t + t^2\cos^2 t} = t\sqrt{2}$. Normalize $\mathbf{r}'(t)$ to obtain $\mathbf{T} = \frac{1}{\sqrt{2}}\langle -\sin t,\ \cos t,\ 1\rangle$.
$\frac{d\mathbf{T}}{dt} = \frac{1}{\sqrt{2}}\langle -\cos t,\ -\sin t,\ 0\rangle$, which we normalize to find $\mathbf{N} = \langle -\cos t,\ -\sin t,\ 0\rangle$.
To find curvature, use its definition:

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \left|\frac{d\mathbf{T}}{dt}\right| \div \frac{ds}{dt} = \frac{1}{\sqrt{2}} \cdot \frac{1}{t\sqrt{2}} = \frac{1}{2t}.$$

3a.(Source: 12.2.22) $2\langle 1,-1,-2\rangle + 3\langle 3,1,0\rangle = \langle 2,-2,-4\rangle + \langle 9,3,0\rangle = \langle 11,1,-4\rangle$
3b.(Source: 12.2.22) $|\mathbf{u}| = \sqrt{1^2 + (-1)^2 + (-2)^2} = \sqrt{6}$.
3c.(Source: 12.3.17) The dot product $\mathbf{u}\cdot\mathbf{v}$ is the scalar $(1)(3) + (-1)(1) + (-2)(0) = 2$, which equals $|\mathbf{u}||\mathbf{v}|\cos\theta$, therefore $\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \frac{2}{\sqrt{6}\sqrt{10}}$, or $\frac{1}{\sqrt{15}}$.
3d.(Source: 12.3.41) $\text{proj}_{\mathbf{u}}\,\mathbf{v} = \frac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} = \frac{2}{6}\langle 1,-1,-2\rangle = \frac{1}{3}\langle 1,-1,-2\rangle$, or $\langle \frac{1}{3}, -\frac{1}{3}, -\frac{2}{3}\rangle$.
3e.(Source: 12.4.1,19)

$$\mathbf{u}\times\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -2 \\ 3 & 1 & 0 \end{vmatrix} = \mathbf{i}\begin{vmatrix} -1 & -2 \\ 1 & 0 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 & -2 \\ 3 & 0 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & -1 \\ 3 & 1 \end{vmatrix}$$

$$= (0 - 1(-2))\mathbf{i} - (0 - 3(-2))\mathbf{j} + (1\cdot 1 - 3(-1))\mathbf{k} = \langle 2,-6,4\rangle$$

Any nonzero scalar multiple of this vector is also a legitimate answer.
3f.(Source: 12.4.28) The area of the parallelogram $= |\mathbf{u}\times\mathbf{v}| = 2|\langle 1,-3,2\rangle| = 2\sqrt{14}$.
4a.(Source: 13.1.1,2) The domain of $\mathbf{r}(t)$ is $(-\infty,\infty)$, since each component of $\mathbf{r}$ is defined for all real numbers.
4b.(Source: 13.2.3,5) $\dfrac{d\mathbf{r}}{dt} = \langle e^t, \sqrt{2}, -e^{-t}\rangle$.
4c.(Source: 13.2.23) The curve passes through $(e, 0, e^{-1})$ at $t = 1$. Evaluate $\dfrac{d\mathbf{r}}{dt}$ at this time to find the tangent vector $\langle e, \sqrt{2}, -e^{-1}\rangle$. The tangent line is parametrized by

$$x = e + et \qquad y = t\sqrt{2} \qquad z = e^{-1} - te^{-1}$$

4d.(Source: 13.3.3)

$$s = \int_0^1 \frac{ds}{dt}\,dt = \int_0^1 \sqrt{e^{2t} + 2 + e^{-2t}}\,dt = \int_0^1 \sqrt{(e^t + e^{-t})^2}\,dt$$

$$= \int_0^1 (e^t + e^{-t})\,dt = (e^t - e^{-t})\Big|_0^1 = e - e^{-1}$$

5a.(Source: 12.5.7,8) The line is parallel the vector $\langle 3,1,0\rangle - \langle 1,-1,-2\rangle = \langle 2,2,2\rangle$, or any vector parallel this, say $\langle 1,1,1\rangle$. The line is

$$x = 1+t \qquad y = -1+t \qquad z = -2+t$$

5b.(Source: 12.5.36) Find any two points on the line, for instance by setting

$$\frac{x-1}{2} = y+2 = \frac{z-3}{4} = 0 \qquad \Rightarrow \qquad (x,y,z) = (1,-2,3)$$

and

$$\frac{x-1}{2} = y+2 = \frac{z-3}{4} = 1 \qquad \Rightarrow \qquad (x,y,z) = (3,-1,7)$$

Now find two vectors between these and the given point $(0,1,-1)$:

$$\langle 0,1,-1\rangle - \langle 1,-2,3\rangle = \langle -1,3,-4\rangle$$
$$\langle 3,-1,7\rangle - \langle 1,-2,3\rangle = \langle 2,1,4\rangle$$

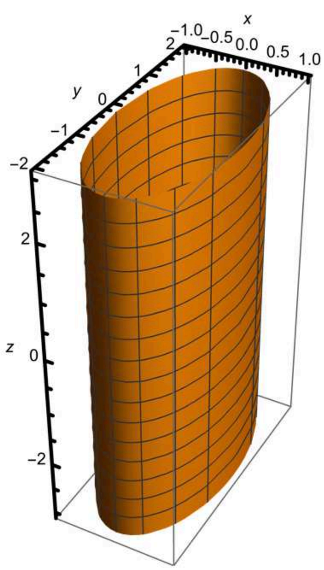
For the normal vector, use the cross product of these:

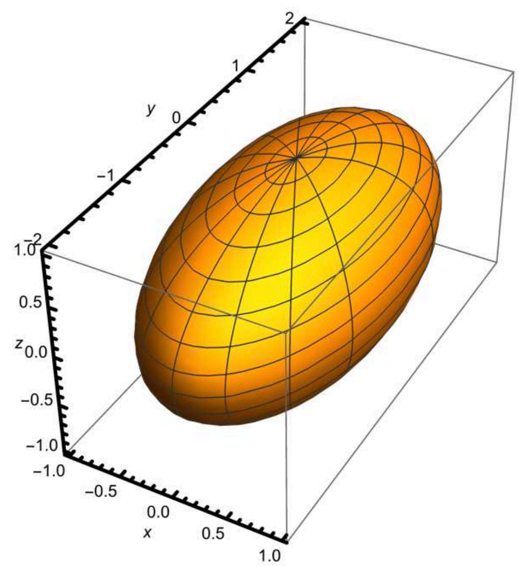
$$\langle -1,3,-4\rangle \times \langle 2,1,4\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 3 & -4 \\ 2 & 1 & 4 \end{vmatrix} = \langle 16,-4,-7\rangle$$

The plane is $16x - 4(y-1) - 7(z+1) = 0$, or $16x - 4y - 7z = 3$

5c.(Source: 12.5.27) To be parallel $x - y - 2z = -1$, the plane must have the same normal vector, $\langle 1,-1,-2\rangle$. Therefore the desired plane is $x-(y-1)-2(z+1) = 0$, or $x-y-2z = 1$.

6a.(Source: 12.6.4) The graph is an elliptic cylinder obtained by graphing the ellipse $x^2 + \frac{y^2}{4} = 1$ in the xy-plane and then dragging that curve in the direction of the z-axis. See graph below left.





6b.(Source: 12.6.17) The graph is an ellipsoid. Its cross sections perpendicular to each of the coordinate axes are ellipses. See graph above right.