No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points.

1(18 pts). Let the $\mathbf{r}(t) = \langle t, t^2, \frac{2}{3}t^3 \rangle$ be the position of a particle at time t.

a. Express the particle's velocity, acceleration, and speed as functions of t.

b. Find the tangential and normal components of acceleration at time t.

c. At what point(s) (x, y, z) is the particle's normal plane parallel to $x + 4y + 8z = 3$?

2(6 pts). Evaluate $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4 - 5y^4}{x^2 + y^4}$ or explain why the limit does not exist.

3(18 pts). Let $p(x, y, z) = xe^{y-z}$.

a. Find the gradient of p.

b. Find the derivative of p at the point $(2, 1, 0)$ in the direction $\langle -\frac{2}{3}, \frac{2}{3}, \frac{1}{3} \rangle$.

c. Find the greatest directional derivative $D_{\mathbf{u}}p$ of p at $(2, 1, 0)$ and the unit vector $\mathbf{u}$ at which it occurs.

d. Find an equation of the plane tangent to the surface $xe^{y-z} = 2e$ at the point $(2, 1, 0)$.

4(12 pts). Express $\frac{\partial y}{\partial s}$ and $\frac{\partial y}{\partial t}$ in terms of u, v, w, s, and t if

$$y = \sqrt{u^2 + v^2 - w^2} \qquad u = s^2 t \qquad v = st^{-1} \qquad w = \ln(s + t)$$

5(14 pts). Let $g(x, y) = \dfrac{x}{2x - y}$.

a. Find g_x, g_y, g_{xx}, g_{xy}, and g_{yy}.

b. Find the linearization of g at the point $(x, y) = (3, 4)$.

6(16 pts). Find the points (x, y) where the function $k(x, y) = 4x^2 - \frac{1}{3}y^3 + 2y^2 - 4xy$ has a local maximum, a local minimum, or a saddle point and clearly identify which is which.

7(16 pts). Find the absolute maximum and minimum of $f(x, y) = (x - 1)^2 - y^2$ on the closed triangular region in the xy-plane with vertices $(-1, -1)$, $(1, -1)$, and $(1, 1)$.

1a.(Source: 13.4.9) $\mathbf{v} = \frac{d\mathbf{r}}{dt} = \langle 1, 2t, 2t^2\rangle$. $\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = \langle 0, 2, 4t\rangle$. speed $= \frac{ds}{dt} = |\mathbf{v}| = \sqrt{1+(2t)^2+(2t^2)^2} = \sqrt{1+4t^2+4t^4} = \sqrt{(1+2t^2)^2} = 1+2t^2$.

1b.(Source: 13.4.40) $a_T = \frac{d^2s}{dt^2} = 4t$. $a_N = \sqrt{|\mathbf{a}|^2 - a_t^2} = \sqrt{2^2+(4t)^2-(4t)^2} = 2$.

Alternate solution: $a_T = \frac{\mathbf{v}\cdot\mathbf{a}}{|\mathbf{v}|} = \frac{4t+8t^3}{1+2t^2} = \frac{4t(1+2t^2)}{1+2t^2} = 4t$.

$\mathbf{v}\times\mathbf{a} = \langle 1, 2t, 2t^2\rangle \times \langle 0, 2, 4t\rangle = 2\langle 2t^2, -2t, 1\rangle$. Then $a_N = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|} = \frac{2|\langle 2t^2,-2t,1\rangle|}{|\langle 1,2t,2t^2\rangle|} = 2$.

1c.(Source: 13.3.53) Seek times t at which $\mathbf{v}$ is parallel the normal vector of the given plane:

$$\langle 1, 2t, 2t^2\rangle = \alpha\langle 1, 4, 8\rangle \Rightarrow \alpha = 1,\, t = 2,$$

at which time $(x, y, z) = \mathbf{r}(2) = (2, 4, \frac{16}{3})$.

2.(Source: 14.2.9) Evaluate $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4-5y^4}{x^2+y^4}$ or explain why the limit does not exist.

If (x, y) approaches the origin along the line $y = 0$, then $\frac{x^4-5y^4}{x^2+y^4} = \frac{x^4}{x^2} = x^2$, which goes to zero as $x \to 0$. But if (x, y) approaches the origin along the line $x = 0$, then $\frac{x^4-5y^4}{x^2+y^4} = \frac{-5y^4}{y^4} = -5$. Since the limit cannot simultaneously equal 0 and -5, it does not exist.

3a.(Source: 14.6.10) $\nabla p = \langle p_x, p_y, p_z\rangle = \langle e^{y-z}, xe^{y-z}, -xe^{y-z}\rangle$.

3b.(Source: 14.6.10) If $\mathbf{u} = \langle -\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\rangle$, then $D_{\mathbf{u}}p(2,1,0) = \nabla p(2,1,0)\cdot\mathbf{u}$ $= \langle e, 2e, -2e\rangle \cdot \langle -\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\rangle = \frac{e}{3}\langle 1, 2, -2\rangle\cdot\langle -2, 2, 1\rangle = 0$.

3c.(Source: 14.6.24) At $(2,1,0)$ the greatest value of $D_{\mathbf{v}}p$ is $|\nabla p| = |\langle e, 2e, -2e\rangle| = e|\langle 1, 2, -2\rangle| = 3e$. It occurs when $\mathbf{v}$ is in the same direction as ∇p, that is, when $\mathbf{v} = \frac{1}{3e}\langle e, 2e, -2e\rangle = \frac{1}{3}\langle 1, 2, -2\rangle$.

d.(Source: 14.6.43) $\nabla p(2,1,0)$ is normal to the tangent plane, so the plane's equation is $e(x-2)+2e(y-1)-2ez = 0$, or, more simply, $x+2y-2z = 4$

4.(Source: 14.5.7,8)

$$\begin{aligned}
\frac{\partial y}{\partial s} &= \frac{\partial y}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial s} + \frac{\partial y}{\partial w}\frac{\partial w}{\partial s} \\
&= \frac{u}{\sqrt{u^2+v^2-w^2}}2st + \frac{v}{\sqrt{u^2+v^2-w^2}}t^{-1} - \frac{w}{\sqrt{u^2+v^2-w^2}}\frac{1}{s+t} \\
&= \frac{1}{\sqrt{u^2+v^2-w^2}}\left(2ust + vt^{-1} - \frac{w}{s+t}\right) \\
\frac{\partial y}{\partial t} &= \frac{\partial y}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial y}{\partial v}\frac{\partial v}{\partial t} + \frac{\partial y}{\partial w}\frac{\partial w}{\partial t} \\
&= \frac{1}{\sqrt{u^2+v^2-w^2}}\left(us^2 - vst^{-2} - \frac{w}{s+t}\right)
\end{aligned}$$

5.a.(Source: 14.3.22,23,55) Here a solution using the product and chain rules. It pays to factor out the lowest power of $2x - y$ throughout.

$$\begin{aligned} g &= x(2x-y)^{-1} \\ g_x &= (2x-y)^{-1} + x(-1)(2x-y)^{-2}2 \\ &= (2x-y-2x)(2x-y)^{-2} \\ &= -y(2x-y)^{-2} \\ g_y &= x(2x-y)^{-2} \end{aligned}$$

$$\begin{aligned} g_{xx} &= 4y(2x-y)^{-3} \\ g_{xy} &= -(2x-y)^{-2} - y(-2)(2x-y)^{-3}(-1) \\ &= -(2x-y)^{-3}(2x-y+2y) \\ &= -(2x-y)^{-3}(2x+y) \\ g_{yy} &= 2x(2x-y)^{-3} \end{aligned}$$

5b.(Source: 14.4.14) $L(x,y) = g(3,4)+g_x(3,4)(x-3)+g_y(3,4)(y-4) = \frac{3}{2}-(x-3)+\frac{3}{4}(y-4)$.

6.(Source: 14.7.more.1j) Search for critical points:

$$k_x(x,y) = 0 \Rightarrow 8x - 4y = 0 \Rightarrow 2x = y.$$
$$k_y(x,y) = 0 \Rightarrow y^2 + 4y - 4x = 0 \Rightarrow y^2 + 2y = 0 \Rightarrow y(y+2) = 0 \Rightarrow y = 0 \text{ or } y = -2.$$

Therefore, the critical points are $(0,0)$ and $(-1,-2)$.
Now use the Second Derivative Test.

$$D = \begin{vmatrix} k_{xx} & k_{xy} \\ k_{xy} & k_{yy} \end{vmatrix} = \begin{vmatrix} 8 & -4 \\ -4 & 2y+4 \end{vmatrix} = 16(y+1)$$

critical point	D	k_{xx}	conclusion
$(0,0)$	16	8	Local Minimum
$(-1,-2)$	-16	irrelevant	Saddle Point

7.(Source: 14.7.31) Setting $f_x = 2(x-1) = 0$ and $f_y = -2y = 0$, we learn that the only critical point $(1,0)$ is not interior to the region. Consequently, the absolute extrema must occur on the boundary, which we examine one line segment at a time.
(Generally, it is insufficient at this point to look for points along the boundary where $f_x = f_y = 0$. Instead, we write f along each segment of the boundary as a function of one variable, which we then maximize with Calc 1 techniques. In effect, we're looking for points along these segments at the derivative of f *in the direction of the boundary* is zero.)
Along the horizontal line $y = -1$, $-1 \le x \le 1$, $f = (x-1)^2 - 1$. $f' = 2(x-1) = 0$ only at $x = 1$. The extrema of f along this line must occur at its endpoints, $(-1,-1)$ and $(1,-1)$.
Along the vertical line $x = 1$, $-1 \le y \le 1$, $f = -y^2$. $f' = -2y = 0$ only at $y = 0$. The extrema of f along this line must occur either at the critical point $(1,0)$ or at its endpoints, $(1,-1)$ and $(1,1)$.
Along the line $y = x$, $-1 \le x \le 1$, $f = (x-1)^2 - x^2 = 2x - 1$. $f' = 2 \ne 0$. The extrema of f along this line must occur at its endpoints, $(-1,-1)$ and $(1,1)$.
So, we'll find the absolute max and min of f if we evaluate f at these four points:

(x,y)	$(-1,-1)$	$(1,-1)$	$(1,1)$	(1,0)
f	3	-1	-1	0