MATH 221–02 (Kunkle), Exam 4 | Name: ________________________

100 pts, 75 minutes | April 9, 2019 | 

No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points.

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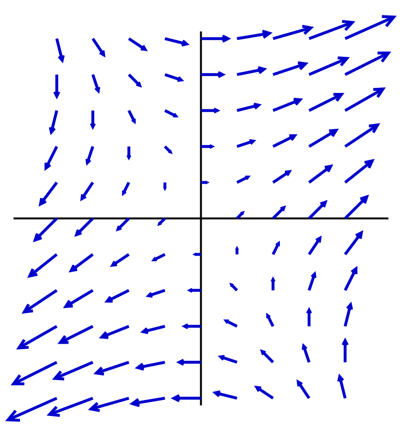
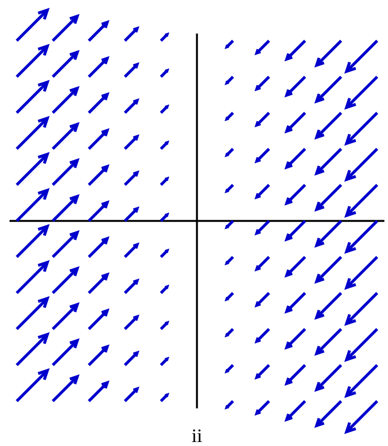
1(4 pts). Find the graph of the given vector field.

$\mathbf{F}(x,y) = \langle 1, 1 \rangle$ $\quad$ $\mathbf{G}(x,y) = \langle y, -x \rangle$ $\quad$ $\mathbf{H}(x,y) = \langle x+y, x \rangle$ $\quad$ $\mathbf{K}(x,y) = \langle y, y \rangle$


i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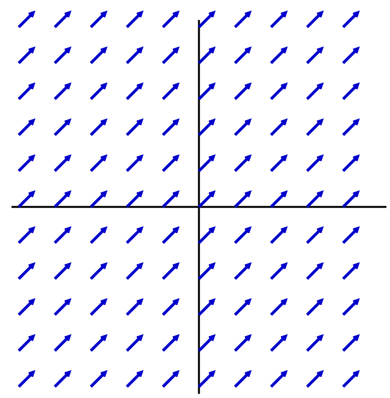
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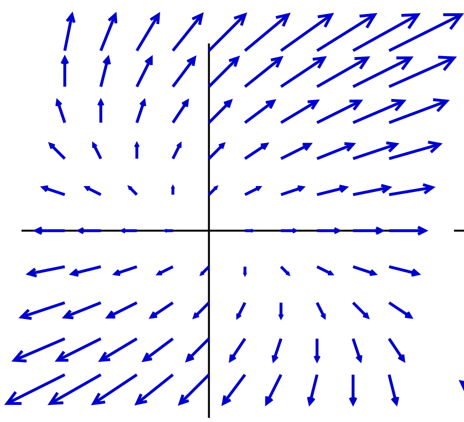
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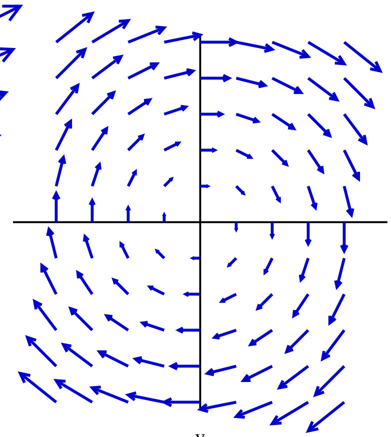
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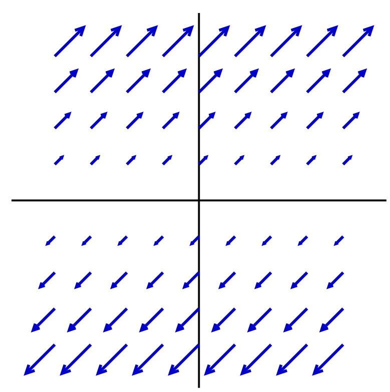
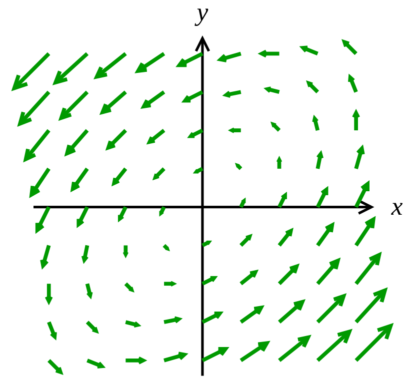

vi

2(17 pts). Evaluate the triple integral $\iiint_E \sqrt{4 - x^2 - y^2}\, dV$, where $E$ is the region above $z = x^2 + y^2$ and below $z = 4$.

3(17 pts). Evaluate the triple integral $\iiint_R z\, dV$, where $R$ is the region above the cone $z = \sqrt{x^2 + y^2}$ and between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$.

4(3 pts). Explain why the vector field graphed at right is not conservative.




5(18 pts). Evaluate the double integral $\iint_P (x-y)\sin(x^2-y^2)\,dA$, where $P$ is the rectangle bounded by the lines $x-y=-1$, $x-y=2$, $x+y=0$, and $x+y=\pi$.

6(12 pts). Evaluate the line integral $\int_C xe^y\,ds$ where $C$ is the curve parametrized by $x=\sin t$, $y=\cos t$, $z=2t$, $0\le t\le \frac{\pi}{2}$.

7(9 pts). Evaluate the line integral $\int_D (y+e^{x^2})\,dx+(-x+\sin\sqrt{y})\,dy$, where $D$ is the triangle with the vertices $(0,0)$, $(2,0)$, $(2,1)$, traversed in the positive direction.

8. Define the vector field $\mathbf{G}(x,y,z)=ze^{xz}\mathbf{i}-z\mathbf{j}+(-y+xe^{xz})\mathbf{k}$.

a(4 pts). Compute $\operatorname{div}\mathbf{G}$.

b(4 pts). Compute $\operatorname{curl}\mathbf{G}$.

c(6 pts). Is $\mathbf{G}$ conservative? If so, find a function $g$ for which $\nabla g=\mathbf{G}$.

d(6 pts). Evaluate the line integral $\int_C \mathbf{G}\cdot d\mathbf{r}$ where $C$ is the curve parametrized by $\mathbf{r}(t)=(t^2+1)\mathbf{i}+(1-t^2)\mathbf{j}+(2t+t^2)\mathbf{k}$, $-1\le t\le 1$.

1.(Source: 16.1.29-32) a. iii. b. v. c. i. d. vi.
Some observations: $\mathbf{F}$ is constant. $\mathbf{G}$ is orthogonal to the prosition vector $\langle x, y\rangle$. In c, when $x = 0$, $\mathbf{H}$ is a multiple of $\mathbf{i}$, and when $y = 0$, $\mathbf{H}$ is a multiple of $\mathbf{i} + \mathbf{j}$. Above [below] the $x$-axis, $\mathbf{K}$ is a positive [negative] multiple of $\langle 1, 1\rangle$.

2.(Source: 15.7.17,18) See the graph of the domain of integration below. The limits of this integral aren't hard to write in rectangular coordinates, but it's difficult to integrate. Convert the integral to cylindrical coordinates:

$$\int_0^{2\pi}\int_0^2\int_{r^2}^4 \sqrt{4-r^2}\,dz\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^2 (4-r^2)\sqrt{4-r^2}\,r\,dr\,d\theta$$
$$= \int_0^{2\pi}\int_0^2 (4-r^2)^{3/2} r\,dr\,d\theta = \int_0^{2\pi} d\theta \cdot \frac{-1}{5}(4-r^2)^{5/2}\Big|_0^2 = 2\pi\cdot\frac{1}{5}\cdot 4^{5/2}$$

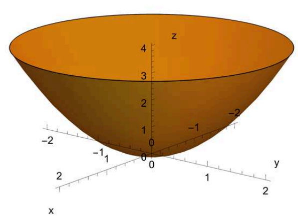


Figure for Problem 2.

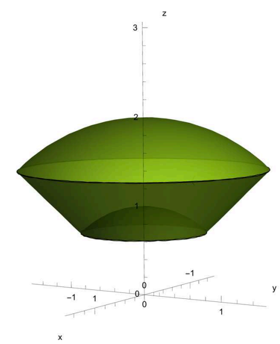
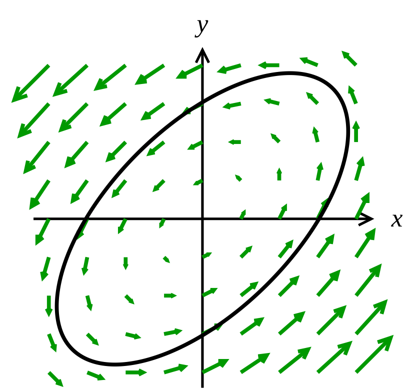


Figure for Problem 3.

3.(Source: 15.8.22,23) See the graph of the domain of integration above. It's not too difficult to perform the integration in rectangular coordinates, but the limits are hard to write. Convert the integral to spherical coordinates:

$$\int_0^{2\pi}\int_0^{\pi/4}\int_1^2 \rho\cos\theta\rho^2\sin\theta\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi/4}\sin\theta\cos\theta\int_1^2 \rho^3\,d\rho\,d\phi\,d\theta$$
$$= \int_0^{2\pi}\int_0^{\pi/4}\sin\theta\cos\theta\frac{15}{4}\,d\phi\,d\theta = 2\pi\cdot\frac{15}{4}\cdot\frac{1}{2}\sin^2\theta\Big|_0^{\pi/4} = 2\pi\cdot\frac{15}{4}\cdot\frac{1}{4}$$

4.(Source: 16.3.25,26) Call the vector field $\mathbf{F}$,and let $C$ be the curve in the figure, taken in the positive (counterclockwise) direction. Along this curve, $\mathbf{F}\cdot d\mathbf{r} > 0$, since $\mathbf{F}$ and the tangent vector $d\mathbf{r}$ always meet at an acute angle. Therefore $\int_C \mathbf{F}\cdot d\mathbf{r} > 0$. If $\mathbf{F}$ were conservative, $\int \mathbf{F}\cdot d\mathbf{r}$ would be zero around every closed curve.



5.(Source: 15.9.24) Let $u = x - y$ and $v = x + y$. Solve for $x$ and $y$ to obtain $x = \frac{1}{2}u + \frac{1}{2}v$ and $y = \frac{1}{2}v - \frac{1}{2}u$. Calculate the Jacobian:

$$\frac{\partial(x,y)}{\partial(u,v)} = \left\| \begin{matrix} x_u & x_v \\ y_u & y_v \end{matrix} \right\| = \left\| \begin{matrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{matrix} \right\| = |1/4 - (-1/4)| = 1/2$$

The integral is

$$\int_{-1}^{2} \int_{0}^{\pi} u \sin(uv) \frac{1}{2}\, dv\, du = \frac{1}{2} \int_{-1}^{2} (-\cos(uv)) \Big|_0^{\pi} du$$
$$= \frac{1}{2} \int_{-1}^{2} (-\cos(\pi u) + 1)\, du = \frac{1}{2} \left( -\frac{1}{\pi} \sin(\pi u) + u \right) \Big|_{-1}^{2} = \frac{3}{2}.$$

(To integrate first with respect to $u$ instead of $v$ would require integration by parts.) Since $\frac{\partial(x,y)}{\partial(u,v)}$ is a constant, it could also be calculated as the reciprocal of $\frac{\partial(u,v)}{\partial(x,y)}$, which avoids the work of solving for $x$ and $y$ in terms of $u$ and $v$.

6.(Source: 16.2.4,9) $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt = \sqrt{\cos^2 t + \sin^2 t + 2^2}\, dt = \sqrt{5}\, dt.$

The line integral is $\int_0^{\pi/2} (\sin t) e^{\cos t} \sqrt{5}\, dt = -\sqrt{5} e^{\cos t} \Big|_0^{\pi/2} = \sqrt{5}(e-1).$

7.(Source: 16.4.6,7) By Green's Theorem, the integral around this closed path equals a double integral over the interior $T$ of the triangle:

$$\iint_T \left( (-x + \sin\sqrt{y})_x - (y + e^{x^2})_y \right) dA = \iint_T (-2)\, dA = -2 \cdot \text{area}(T) = -2.$$

8a.(Source: 16.5.4-8) $\operatorname{div} \mathbf{G}$ is a scalar field (that is, a scalar-valued function of $(x, y, z)$): $\operatorname{div} \mathbf{G} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle ze^{xz}, -z, -y + xe^{xz} \rangle = z^2 e^{xz} + 0 + x^2 e^{xz} = (x^2 + z^2) e^{xz}.$

8b.(Source: 16.5.4-8) $\operatorname{curl} \mathbf{G}$ is a vector field (that is, a vector-valued function of $(x, y, z)$):

$$\operatorname{curl} \mathbf{G} = \langle \tfrac{\partial}{\partial x}, \tfrac{\partial}{\partial y}, \tfrac{\partial}{\partial z} \rangle \times \langle ze^{xz}, -z, -y + xe^{xz} \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ze^{xz} & -z & -y + xe^{xz} \end{vmatrix} = \langle 0, 0, 0 \rangle.$$

8c.(Source: 16.5.17, 16.3.17) $\mathbf{G}$ is conservative, since it is defined on all of $\mathbb{R}^3$ and its curl is zero. To find a potential function $g$, suppose $\langle ze^{xz}, -z, -y + xe^{xz} \rangle = \nabla g$. Then, starting with the simplest derivative, we integrate with respect to one variable, differentiate with respect to another, and compare the results with the entries of $\mathbf{G}$:

$$\begin{aligned} g_y &= -z & g &= -yz + e^{xz} + b(z) \\ g &= -yz + c(x,z) & g_z &= -y + xe^{xz} + b'(z) = -y + xe^{xz} \\ g_x &= c_x = ze^{xz} & b'(z) &= 0 \Rightarrow b = \text{const} \\ c &= e^{xz} + b(z) & g &= -yz + e^{xz} + \text{const} \end{aligned}$$

8d.(Source: 16.3.15,17) The curve $C$ begins at the point $(2, 0, -1)$ and ends at $(2, 0, 3)$. Since $\mathbf{G}$ is conservative, the easiest way to calculate $\int_C \mathbf{G} \cdot d\mathbf{r}$ is by the Fundamental Theorem of Calculus for line integrals:

$$\int_C (\nabla g) \cdot d\mathbf{r} = g \Big|_{(2,0,-1)}^{(2,0,3)} = g(2,0,3) - g(2,0,-1) = e^6 - e^{-2}.$$