No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points.

1(16 pts). Find curvature κ of the curve parametrized by $\mathbf{r}(t) = \langle \ln t, 4\sqrt{t}, 2t \rangle$ at the point $(0, 4, 2)$.

2.(14 pts) Find the length of the curve parametrized by $\mathbf{r}(t) = \langle \ln t, 4\sqrt{t}, 2t \rangle$ for $1 \leq t \leq 3$.

3(23 pts). Let $g(x, y) = 3 + \ln(5x - 2y)$, and find the following.

a. g_x, g_y, g_{xx}, g_{xy}, g_{yy}. Label your answers so I can tell which is which.

b. The gradient of $g(x, y)$ at the point $(1, 2)$.

c. The linearization of $g(x, y)$ at the point $(1, 2)$.

d. The maximum directional derivative of $g(x, y)$ at the point $(1, 2)$ and the direction in which it occurs.

4(12 pts). Find equation(s) the line described in each part.

a. The line through the two points $(1, 0, -1)$ and $(3, 1, 2)$.

b. The line tangent to the curve parametrized by $\mathbf{r}(t) = \langle \ln t, 4\sqrt{t}, 2t \rangle$ at the point $(0, 4, 2)$.

5(16 pts). Find an equation of the plane described in each part.

a. The plane through the three points $(1, 0, -1)$, $(3, 1, 2)$, $(0, 1, 0)$.

b. The plane tangent to the surface $z = 3 + \ln(5x - 2y)$ at the point $(1, 2, 3)$.

6(16 pts). Let R be the solid in the first octant bounded by the plane $x + y = 3$ and the cylinder $z = 9 - x^2$. Express the volume of R as either a double or triple integral, but **do not evaluate**.

7(13 pts). Let $f(x, y, z) = xy - xz + yz$. Evaluate the line integral $\int_C \nabla f \cdot d\mathbf{r}$, where C is the curve parametrized by $\mathbf{r}(t) = \langle \cos \pi t, t^3 - 2t, t^3 + 2t \rangle$ for $0 \leq t \leq 1$.

8(16 pts). Find the dimensions of the rectangular box of maximum volume such that the sum of the length of its 12 edges is 60.

9(8 pts). Sketch the surface $x^2 + z^2 = y^2 - 1$.
Label your x-, y- and z-axes and use arrows to indicate the positive direction on each. It may help you to find the intercepts of this surface, although you are not required to do so. If you think your drawing needs some explaining, describe what you're trying to draw.

10(25 pts). Let T be the parallelogram parametrized by $\mathbf{r}(u, v) = \langle u + v, u - v, 1 + 2u \rangle$ for $0 \leq u \leq 2$ and $0 \leq v \leq 1$.

a. Find the flux of $\mathbf{F} = \langle e^x, -e^x, e^{-y} \rangle$ across T (oriented upward).

b. Find the area of T.

11(18 pts). Let E be the the solid given by $x^2 + z^2 \leq 1$ and $0 \leq y \leq 2$ and let $\mathbf{F}$ be the vector field $\langle xz^2, yx^2, xy^2 \rangle$. Find the flux $\iint_{\partial E} \mathbf{F} \cdot \mathbf{n}\, dS$ of $\mathbf{F}$ across the surface of E (oriented outward) by first writing the flux as a triple integral.

12(23 pts). Let $\mathbf{G} = \langle ye^z, 2z - x, e^{x-z} \rangle$.

a. Find $\operatorname{curl} \mathbf{G}$.

b. Let the surface P be the part of the ellipsoid $x^2 + y^2 + \frac{1}{2}z^2 = 1$ on which $z \geq 0$, oriented away from the origin. Find the flux $\iint_P \operatorname{curl} \mathbf{G} \cdot \mathbf{n}\, dS$ of $\operatorname{curl} \mathbf{G}$ across P by first writing it as a line integral.

1.(Source: 13.3.18) $\langle \ln t, 4\sqrt{t}, 2t\rangle = (0,4,2)$ at $t=1$. At this time, $\mathbf{v} = \frac{d\mathbf{r}}{dt} = \langle t^{-1}, 2t^{-1/2}, 2\rangle$ $= \langle 1,2,2\rangle$ and $\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = \langle -t^{-2}, -t^{-3/2}, 0\rangle = \langle -1,-1,0\rangle$. The cross product of these is

$$\mathbf{v}\times\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 2 \\ -1 & -1 & 0 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 2 & 2 \\ -1 & 0 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 & 2 \\ -1 & 0 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & 2 \\ -1 & -1 \end{vmatrix} = \langle 2,-2,1\rangle$$

and $\kappa = |\mathbf{v}\times\mathbf{a}|/|\mathbf{v}|^3 = 3/3^3 = 1/9$.

2.(Source: 13.3.2,6) $s = \int_1^3 \frac{ds}{dt}\,dt = \int_1^3 |\mathbf{v}|\,dt = \int_1^3 \sqrt{t^{-2}+4t^{-1}+4}\,dt = \int_1^3 \sqrt{(t^{-1}+2)^2}\,dt =$ $\int_1^3 (t^{-1}+2)\,dt = (2t+\ln t)\Big|_1^3 = 4+\ln 3$

3a.(Source: 14.3.19,54)

$$g_x = \frac{5}{5x-2y} \quad g_y = \frac{-2}{5x-2y} \quad g_{xx} = \frac{-25}{(5x-2y)^2} \quad g_{xy} = \frac{10}{(5x-2y)^2} \quad g_{yy} = \frac{-4}{(5x-2y)^2}$$

3b.(Source: 14.6.8) $\nabla g(1,2) = \langle g_x(1,2), g_y(1,2)\rangle = \langle 5,-2\rangle$.

3c.(Source: 14.4.11) $L(x,y) = g(1,2)+g_x(1,2)(x-1)+g_y(1,2)(y-2) = 3+5(x-1)-2(y-2)$.

3d.(Source: 14.6.23) At $(1,2)$, the greatest directional derivative is $|\nabla g| = \sqrt{29}$, which occurs in the direction of ∇g, that is, $\langle 5,-2\rangle/\sqrt{29}$.

4a.(Source: 12.5.7) The line is parallel $\langle 3,1,2\rangle - \langle 1,0,-1\rangle = \langle 2,1,3\rangle$. It is given by the parametrically by $\mathbf{r} = \langle 1,0,-1\rangle + t\langle 2,1,3\rangle$, or, if you prefer,

$$x = 1+2t \qquad y = t \qquad z = -1+3t.$$

4b.(Source: 13.2.23) $\langle \ln t, 4\sqrt{t}, 2t\rangle$ passes through $(0,4,2)$ at $t=1$. The line is parallel to $\mathbf{v}(1) = \langle 1,2,2\rangle$ and is given parametrically by the vector-valued function (which cannot be named $\mathbf{r}$) $\mathbf{u} = \langle 0,4,2\rangle + t\langle 1,2,2\rangle$

5a.(Source: 12.5.31) The plane is parallel the vectors $\langle 3,1,2\rangle - \langle 1,0,-1\rangle = \langle 2,1,3\rangle$ and $\langle 0,1,0\rangle - \langle 1,0,-1\rangle = \langle -1,1,1\rangle$ and is therefore perpendicular to their cross product

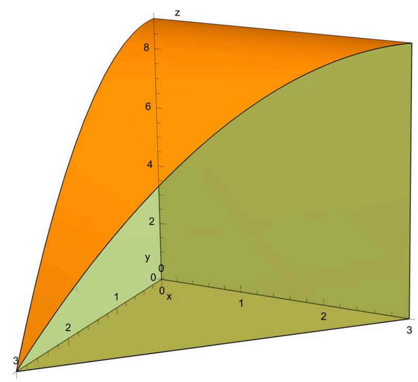
$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & 3 \\ -1 & 1 & 1 \end{vmatrix} = \langle -2,-5,3\rangle.$$

Its equation is $-2(x-1)-5y+3(z+1) = 0$, or $-2x-5y+3z = -5$.

5b.(Source: 14.4.6) The plane tangent to the graph of a function is the graph of its linearization. From problem 3c, this is $z = 3+5(x-1)-2(y-2)$.

6.(Source: 15.2.26,15.6.21,34) To write the limits for R, its best to choose the inside variable to be either y or z. Then volume can be written in any of these eight ways:

$$\begin{aligned}
\int_0^3\int_0^{9-x^2}\int_0^{3-x} dy\,dz\,dx &= \int_0^3\int_0^{9-x^2}(3-x)\,dz\,dx \\
\int_0^9\int_0^{\sqrt{9-z}}\int_0^{3-x} dy\,dx\,dz &= \int_0^9\int_0^{\sqrt{9-z}}(3-x)\,dx\,dz \\
\int_0^3\int_0^{3-x}\int_0^{9-x^2} dz\,dy\,dx &= \int_0^3\int_0^{3-x}(9-x^2)\,dy\,dx \\
\int_0^3\int_0^{3-y}\int_0^{9-x^2} dz\,dx\,dy &= \int_0^3\int_0^{3-y}(9-x^2)\,dx\,dy
\end{aligned}$$



7.(Source: 16.4.12b,18b) The curve C begins at $\mathbf{r}(0) = (1,0,0)$ and ends at $\mathbf{r}(1) = (-1,-1,3)$. According the Fundamental Theorem of Calculus for Line Integrals,

$$\int_C \nabla f\cdot d\mathbf{r} = f(x,y,z)\Big|_{(1,0,0)}^{(-1,-1,3)} = f(-1,-1,3) - f(1,0,0) = 1-(-3)+(-3)-0 = 1.$$

8.(Source: 14.7.51) Call the dimensions of the box x, y and z. We wish to maximize $V = xyz$ over positive x, y, and z subject to the constraint that $4x+4y+4z = 60$, or $x+y+z = 15$. We can solve with either with or without the method of Lagrange Multipliers.

With Lagrange Multipliers: Let $g = x+y+z$. The maximum of V must at occur at a point where $g = 15$ and

$$\nabla g\times\nabla V = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ yz & xz & xy \end{vmatrix} = \langle xy-xz, yz-xy, xz-yz\rangle = \langle 0,0,0\rangle, \text{ or}$$

$$x(y-z)=0 \qquad y(z-x)=0 \qquad z(x-y)=0$$

We safely ignore points where x, y, or z equals 0, since $V = 0$ at any such points. Therefore,

$$y-z=0 \qquad z-x=0 \qquad x-y=0 \qquad \Longrightarrow \qquad x=y=z,$$

which, combined with $x+y+z = 15$, implies $x = y = z = 5$. Since this is the only critical point, these must be the dimensions of the box of maximum volume.

Without Lagrange Multipliers: Solve for $z = 15-x-y$ in the constraint and substitute this into $V = xy(15-x-y)$. The maximum volume must occur at a critical point of this function of x and y. Set its first partial derivatives equal 0:

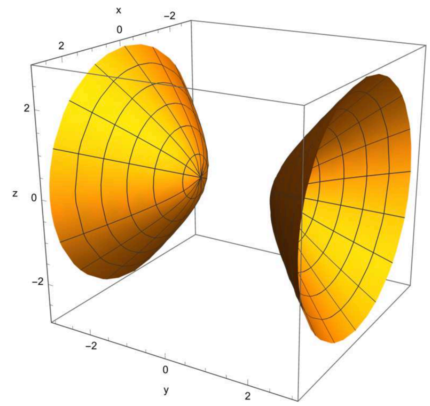
$$\begin{aligned}
V_x &= 15y-2xy-y^2 &= y(15-2x-y) &= 0 \\
V_y &= 15x-x^2-2xy &= x(15-x-2y) &= 0
\end{aligned}$$

We can safely ignore critical points where x or y equals 0. Solving

$$15 - 2x - y = 15 - x - 2y = 0$$

gives $x = y = 5$. Since this is the only critical point, V must be maximized there, and the dimensions of the box are $x = y = 5$ and $z = 15 - x - y = 5$.

9.(Source: 12.6.24) Some clues: the y-intercepts are $y = \pm 1$, found when $x = z = 0$. There are no other intercepts, since $x = y = 0$ and $y = z = 0$ both lead to no solutions. Cross-sections when x or z are constants are hyperbolas, and cross-sections when y is a constant (of absolute value greater than one) are circles. The surface is a **hyperboloid of two sheets**.
In the graph, which has been rotated for better viewing, the positive z direction is up, the positive y-direction is down and to the right, and the positive x direction is down and to the left.



10a.(Source: 16.7.21) $\mathbf{n}\,dS = \pm(\mathbf{r}_u \times \mathbf{r}_v)\,du\,dv = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 2 \\ 1 & -1 & 0 \end{vmatrix} = \pm\langle 2, 2, -2\rangle\,du\,dv$. Choose $-$ so that $\mathbf{n}\,dS$ points upward. Then $\mathbf{F}\cdot\mathbf{n}\,dS = \langle e^x, -e^x, e^{-y}\rangle \cdot \langle -2, -2, 2\rangle\,du\,dv = 2e^{-y}\,du\,dv$, and the flux of $\mathbf{F}$ across T is the integral of this, or

$$\int_0^1 \int_0^2 2e^{-(u-v)}\,du\,dv = 2\int_0^2 e^{-u}\,du \int_0^1 e^v\,dv = 2(1 - e^{-2})(e - 1)$$

10b.(Source: 16.6.40,16.7.5,12.4.27-31) The area of T is

$$\iint_T dS = \int_0^1 \int_0^2 |\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv = \int_0^1 \int_0^2 2\sqrt{3}\,du\,dv = 4\sqrt{3}.$$

You could also find the area of the parallelogram T using a method from Chapter 12: the sides of T are $\mathbf{r}(1,0) - \mathbf{r}(0,0) = \langle 1, 1, 2\rangle$ and $\mathbf{r}(0,2) - \mathbf{r}(0,0) = \langle 2, -2, 0\rangle$, and its area is $|\langle 1, 1, 2\rangle \times \langle 2, -2, 0\rangle| = |\langle 4, 4, -4\rangle| = 4\sqrt{3}$.
(Note that T is a rectangle, since $\langle 1, 1, 2\rangle \perp \langle 2, -2, 0\rangle$.)

11.(Source: 16.9.7) $\operatorname{div}\mathbf{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\rangle \cdot \langle xz^2, yx^2, xy^2\rangle = (xz^2)_x + (yx^2)_y + (xy^2)_z = z^2 + x^2$
By the Divergence Theorem, the flux of $\mathbf{F}$ across the surface of E is the triple integral over E of $\operatorname{div}\mathbf{F}$, which we integrate by using polar coordinates in the xz plane:

$$\iiint_E (z^2 + x^2)\,dV = \int_0^2 \int_0^{2\pi} \int_0^1 r^2 \cdot r\,dr\,d\theta\,dy = 2 \cdot 2\pi \cdot \frac{1}{4} = \pi.$$

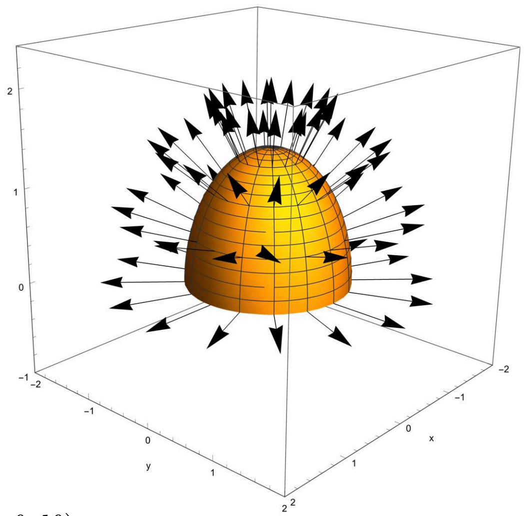
12a.(Source: 16.5.1-8) curl $\mathbf{G} = \nabla \times \mathbf{G}$ is the vector

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ye^z & 2z - x & e^{x-z} \end{vmatrix} = \mathbf{i}\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2z - x & e^{x-z} \end{vmatrix} - \mathbf{j}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ ye^z & e^{x-z} \end{vmatrix} + \mathbf{k}\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ ye^z & 2z - x \end{vmatrix}$$

$$= \langle -2, -e^{x-z} + ye^z, -1 - e^z \rangle^*$$

*corrected Dec 4, 2019, by JL and TU.

12b.(Source: 16.8.6) By Stokes's Theorem, $\iint_P \operatorname{curl} \mathbf{G} \cdot \mathbf{n}\, dS = \int_{\partial P} \mathbf{G} \cdot d\mathbf{r}$ where ∂P is P's boundary, the unit circle in the xy-plane, traversed in the positive direction. Since the normals of P point away from the origin, the positive direction around the circle is the counterclockwise direction when viewed from above. Since $z = 0$ along this curve,



$$\int_{\partial P} \mathbf{G} \cdot d\mathbf{r} = \int_{\partial P} ye^z\, dx + 2z - x\, dy + e^{x-z}\, dz = \int_{\partial P} y\, dx - x\, dy$$

You can calculate this line integral directly by parametrizing the unit circle, using $x = \cos\theta$ and $y = \sin\theta$:

$$\int_{\partial P} y\, dx - x\, dy = \int_0^{2\pi} \sin\theta(-\sin\theta\, d\theta) - \cos\theta(\cos\theta\, d\theta)$$

$$= \int_0^{2\pi} (-\sin^2\theta - \cos^2\theta)\, d\theta = \int_0^{2\pi} (-1) d\theta = -2\pi.$$

Or you could use Green's Theorem, which says that the line integral around the circle equals a double integral over the interior U of the circle:

$$\int_{\partial P} y\, dx - x\, dy = \iint_U \left((-x)_x - (y)_y\right) dA = \iint_U (-2)\, dA = -2\,\text{area}(U) = -2\pi.$$

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