No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points.

You are expected to know the values of all trig functions at multiples of $\pi/4$ and of $\pi/6$.

1(5 pts). Find the center and radius of the sphere $x^2 - 2x + y^2 + z^2 + 4z = 4$.

2(12 pts). Sketch the graph of the given equation in each part. Label your axes x, y, z and use arrows to indicate the positive direction along each. Label your answers so I can tell which is which.

a. $4x^2 = z$ b. $4x^2 + y^2 = z$

3(14 pts). Let $\mathbf{u} = -2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = \mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$. Find each of the following.

a. The cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$.

b. The vector projection of $\mathbf{v}$ onto $\mathbf{u}$.

c. A nonzero vector perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

4(7 pts). Find an equation of the plane passing through the three points $(0, 1, 0)$, $(-2, 3, 1)$, $(1, 4, -2)$.

5(5 pts). Find an equation for the line passing through the two points $(4, 0, 1)$ and $(2, 2, 2)$.

6. Let $\mathbf{r}(t) = \langle 2t, t^2, \ln t\rangle$ be the position of a particle at time t.

a.(7 pts). Express the particle's velocity, acceleration, and speed as functions of t. Label your answers so I can tell which is which.

b.(8 pts). Find the distance traveled by the particle from time $t = 1$ to time $t = 2$.

c(12 pts). Find the curvature of the particle's path at the point $(2, 1, 0)$.

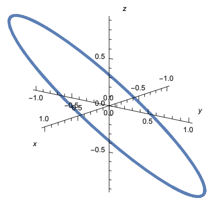
d(10 pts). Find the normal and tangential components of the particle's acceleration at the point $(2, 1, 0)$. Label your answers.

e.(9 pts). Find $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$ for the particle's path at the point $(2, 1, 0)$.

f.(5 pts). Find the osculating plane for the particle's path at the point $(2, 1, 0)$.

7(6 pts). Find the graph of given vector-valued function.

a. $\langle 2-3t, 2t+1, 4t\rangle$
b. $\langle \sin(10t), \cos(10t), t\rangle$
c. $\langle 2-t^2, t, t^2-1\rangle$
d. $\langle \sin t, \cos t, \frac{1}{2}\sin t - \frac{3}{4}\cos t\rangle$
e. $\langle \sqrt{t}\sin t, \sqrt{t}\cos t, \sqrt{t}\rangle$
f. $\langle \sin t, \cos t, \sin(4t)\rangle$



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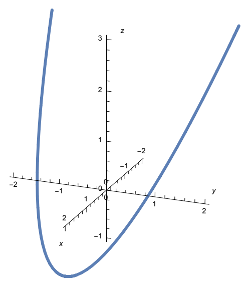

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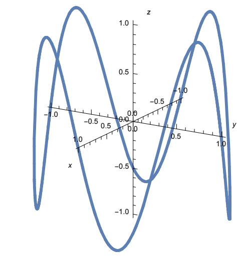

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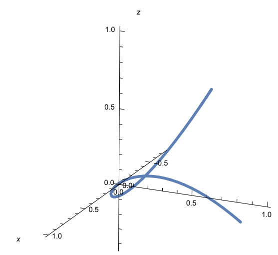

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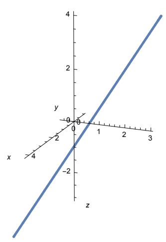

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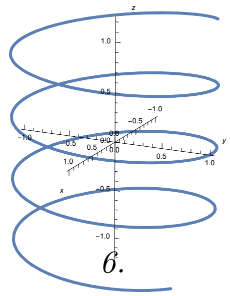

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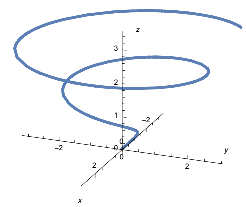

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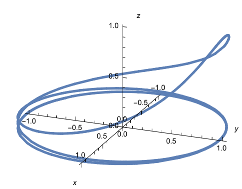


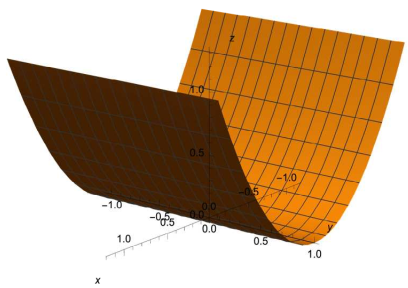
8.

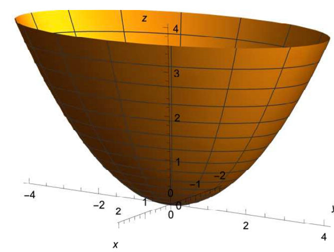
1.(Source: 12.1.18) Complete the square:

$$x^2 - 2x + y^2 + z^2 + 4z = 4$$
$$x^2 - 2x + 1 + y^2 + z^2 + 4z + 4 = 4 + 1 + 4$$
$$(x-1)^2 + y^2 + (z+2)^2 = 9$$

Center is $(1, 0, -2)$ and radius is 3.

2a.(6 pts).(Source: 12.6.5-6) $4x^2 = z$ is a cylinder with a parabolic cross section at every y-value. See below, left.





2b(6 pts).(Source: 12.6.11) $4x^2 + y^2 = z$ is an elliptical parabola. Its cross-sections at x- or y-values are parabolas, and at z-values are ellipses. See above, right.

3a(5 pts).(Source: 12.3.19) $\cos\theta = \dfrac{\mathbf{u}\cdot\mathbf{v}}{|\mathbf{u}||\mathbf{v}|} = \dfrac{-2+6-2}{\sqrt{9}\sqrt{14}} = \dfrac{2}{3\sqrt{14}}.$

3b(4 pts).(Source: 12.3.43) $\operatorname{proj}_{\mathbf{u}} \mathbf{v} = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u} = \dfrac{2}{9}\langle -2, 2, 1\rangle.$

3c(5 pts).(Source: 12.4.5) $\mathbf{u}\times\mathbf{v} =$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 2 & 1 \\ 1 & 3 & -2 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix} - \mathbf{j}\begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix} + \mathbf{k}\begin{vmatrix} -2 & 2 \\ 1 & 3 \end{vmatrix} = \langle -7, -3, -8\rangle$$

4(7 pts).(Source: 12.5.32) The plane is parallel the vectors $\langle -2, 3, 1\rangle - \langle 0, 1, 0\rangle = \langle -2, 2, 1\rangle$ and $\langle 1, 4, -2\rangle - \langle 0, 1, 0\rangle = \langle 1, 3, -2\rangle$. The cross product of these, found in Problem 3, is normal to the plane, so the plane is given by

$$-7x - 3(y-1) - 8z = 0,$$

or, $7x + 3y + 8z = 3$.

5(5 pts).(Source: 12.5.7-9) The line is parallel the vector $\langle 4, 0, 1\rangle - \langle 2, 2, 2\rangle = \langle 2, -2, -1\rangle$ and is parametrized by by

$$\mathbf{l}(t) = \langle 4, 0, 1\rangle + t\langle 2, -2, -1\rangle.$$

6a.(7 pts).(Source: 13.4.12) $\mathbf{v} = \dfrac{d\mathbf{r}}{dt} = \langle 2, 2t, t^{-1}\rangle$. $\mathbf{a} = \dfrac{d^2\mathbf{r}}{dt^2} = \langle 0, 2, -t^{-2}\rangle$.

$$\text{Speed} = |\mathbf{v}| = \sqrt{4 + 4t^2 + t^{-2}} = \sqrt{(2t + t^{-1})^2} = 2t + t^{-1}.$$

6b.(8 pts).(Source: 13.3.2,3) The length of the curve is

$$s = \int_1^2 \frac{ds}{dt}\,dt = \int_1^2 (2t + t^{-1})\,dt = (t^2 + \ln t)\Big|_1^2 = 4 + \ln 2 - 1 - \ln 1 = 3 + \ln 2.$$

6c(12 pts).(Source: 13.3.24,25) $\mathbf{r} = \langle 2, 1, 0\rangle$ at $t = 1$, when $\mathbf{v} = \langle 2, 2, 1\rangle$ and $\mathbf{a} = \langle 0, 2, -1\rangle$. Calculate their cross product:

$$\mathbf{v} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 1 \\ 0 & 2 & -1 \end{vmatrix} = \mathbf{i}\begin{vmatrix} 2 & 1 \\ 2 & -1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 2 & 1 \\ 0 & -1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 2 & 2 \\ 0 & 2 \end{vmatrix} = \langle -4, 2, 4\rangle$$

and then

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \frac{2|\langle -2, 1, 2\rangle|}{|\langle 2, 2, 1\rangle|^3} = \frac{2}{9}.$$

6d(10 pts).(Source: 13.4.41)

Solution one: $a_T = \frac{d^2s}{dt^2} = (2t + t^{-1})' = 2 - t^2$, which equals 1 at $t = 1$.

$a_N = \kappa(\frac{ds}{dt})^2 = \frac{2}{9}|\langle 2, 2, 1\rangle|^2 = 2.$

Solution two: $a_T = \frac{\mathbf{v}\cdot\mathbf{a}}{|\mathbf{v}|} = \frac{3}{3} = 1$, and $a_N = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|} = \frac{|\langle -4,2,4\rangle|}{|\langle 2,2,1\rangle|} = \frac{6}{3} = 2.$

6e.(9 pts).(Source: 13.3.47,48) $\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{1}{3}\langle 2, 2, 1\rangle$. $\mathbf{B} = \frac{\mathbf{v}\times\mathbf{a}}{|\mathbf{v}\times\mathbf{a}|} = \frac{1}{6}\langle -4, 2, 4\rangle = \frac{1}{3}\langle -2, 1, 2\rangle$.

$$\begin{aligned} \mathbf{N} = \mathbf{B} \times \mathbf{T} &= \frac{1}{3}\langle -2, 1, 2\rangle \times \frac{1}{3}\langle 2, 2, 1\rangle \\ &= \frac{1}{9}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix} = \frac{1}{9}\left(\mathbf{i}\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} -2 & 1 \\ 2 & 2 \end{vmatrix}\right) \\ &= \frac{1}{9}\langle -3, 6, -6\rangle = \frac{1}{3}\langle -1, 2, -2\rangle. \end{aligned}$$

If you chose to normalize $\frac{d\mathbf{T}}{dt}$ to find $\mathbf{N}$, you'd find

$$\begin{aligned} \mathbf{T} &= (2t + t^{-1})^{-1}\langle 2, 2t, t^{-1}\rangle \\ \frac{d\mathbf{T}}{dt} &= -(2t + t^{-1})^{-2}(2 - t^{-2})\langle 2, 2t, t^{-1}\rangle + (2t + t^{-1})^{-1}\langle 0, 2, -t^{-2}\rangle \end{aligned}$$

which equals $\frac{1}{9}\langle -2, 4, -4\rangle$. To find $\mathbf{N}$, normalize any positive scalar multiple of this, e.g. $\langle -1, 2, -2\rangle$.

f.(5 pts).(Source: 13.3.50) The osculating plane is normal to $\mathbf{v}\times\mathbf{a} = \langle -4, 2, 4\rangle$, so its equation is $-4(x - 2) + 2(y - 1) + 4z = 0$, or $-2x + y + 2z = -3$.

7(6 pts).(Source: 13.1.21-26) a5, b6, c2, d1, e7, f3.