No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points.

You are expected to know the values of all trig functions at multiples of $\pi/4$ and of $\pi/6$.

1(5 pts). Sketch the set of points in the xy-plane where the function $w(x, y) = \sqrt{x-1} + \sqrt{x^2 + y^2 - 4}$ is continuous.
Give a list of algebraic conditions that (x, y) must satisfy to belong to this set.

2(12 pts). Suppose T is a function of u and v and that $\frac{\partial T}{\partial u} = \frac{2v}{u^2 - v^2}$, and $\frac{\partial T}{\partial v} = \frac{-2u}{u^2 - v^2}$.
If $u = x - y$ and $v = xy^{-1}$, find $\frac{\partial T}{\partial y}$ at the point $x = 1$, $y = 2$.

3. Let $p(x, y) = x^2 - 3xy + \sin(2x - y)$.

a(14 pts). Find the linearization of $p(x, y)$ at the point $(x, y) = (\frac{\pi}{2}, \frac{\pi}{2})$.

b(6 pts). Find all the second partial derivatives of $p(x, y)$.

4. Let $g(x, y, z) = xy - yz + xz$.

a(8 pts). Find the derivative of g in the direction $\frac{1}{\sqrt{2}}\langle 1, -1, 0\rangle$ at the point $(1, 0, 2)$.

b(2 pts). What is the greatest directional derivate of g at the point $(1, 0, 2)$?

c(2 pts). In what direction does the greatest derivative of g at $(1, 0, 2)$ occur?

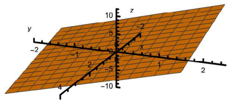
5(23 pts). Find the locations (x, y) of all local extrema and saddle points of $h(x, y) = x^3 - y^3 - 6xy + 6$ and state which is which.

6(21 pts). Find the absolute extrema of $r(x, y, z) = 2x - 2y + z$ subject to the constraint $x^2 + 2y^2 + z^2 = 28$.

7(3 pts). Let D be the closed triangular region with vertices $(2, 2)$, $(-2, -2)$, $(-2, 2)$ (that is, the triangle and its interior). Explain why the absolute extrema of $f(x, y) = 2x + 5y$ on D must occur somewhere on the boundary of D (that is, the three edges of the triangle). You are not required to find these extrema.

8(4 pts). Find the graph of given function.

a. $\sqrt{4-x^2-y^2}$ b. $\sqrt{x^2+y^2}$ c. $\frac{1}{1+x^2+y^2}$ d. $1-2x+3y$



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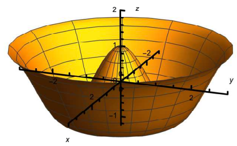

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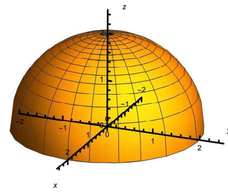

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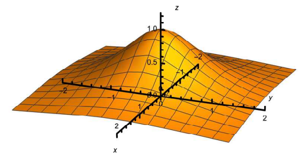

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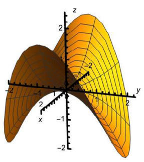

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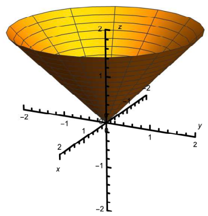

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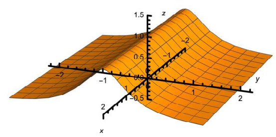

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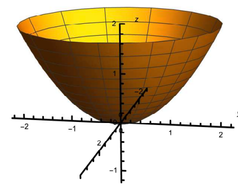

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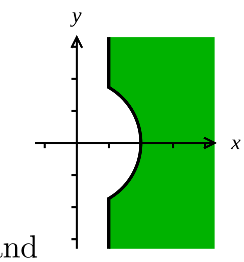

1(5 pts).(Source: 14.1.16,14.2.33) As a combination of polynomials and radicals, $w(x, y)$ is continuous everywhere it is defined. For both square roots to be defined requires $x \geq 1$ *and* $x^2 + y^2 \geq 4$. Geometrically, (x, y) must be outside the circle of radius 2 centered at the origin and to the right of the vertical line $x = 1$.

2(12 pts).(Source: 14.5.22) At the point $(x, y) = (1, 2)$, $(u, v) = (-1, \frac{1}{2})$, and

$$T_u = \frac{2v}{u^2 - v^2} = \frac{1}{1 - \frac{1}{4}} = \frac{4}{3} \qquad u_y = -1$$
$$T_v = \frac{-2u}{u^2 - v^2} = \frac{-2}{1 - \frac{1}{4}} = -\frac{8}{3} \qquad v_y = -xy^{-2} = -\frac{1}{4}$$

By the Chain Rule

$$T_y = T_u u_y + T_v v_y = \frac{4}{3}(-1) + (-\frac{8}{3})(-\frac{1}{4}) = -\frac{6}{3} = -2.$$

3a(14 pts).(Source: 14.4.11-16)

$$p(x, y) = x^2 - 3xy + \sin(2x - y) \qquad p\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = \frac{\pi^2}{4} - 3\frac{\pi^2}{4} + \sin\left(\frac{\pi}{2}\right) = -\frac{\pi^2}{2} + 1$$
$$p_x(x, y) = 2x - 3y + 2\cos(2x - y) \qquad p_x\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = 2\frac{\pi}{2} - 3\frac{\pi}{2} + 2\cos\left(\frac{\pi}{2}\right) = -\frac{\pi}{2}$$
$$p_y(x, y) = -3x - \cos(2x - y) \qquad p_y\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = -3\frac{\pi}{2} - \cos\left(\frac{\pi}{2}\right) = -\frac{3\pi}{2}$$

The linearization of p is

$$L(x, y) = p\left(\frac{\pi}{2}, \frac{\pi}{2}\right) + p_x\left(\frac{\pi}{2}, \frac{\pi}{2}\right)\left(x - \frac{\pi}{2}\right) + p_y\left(\frac{\pi}{2}, \frac{\pi}{2}\right)\left(y - \frac{\pi}{2}\right)$$
$$= -\frac{\pi^2}{2} + 1 - \frac{\pi}{2}\left(x - \frac{\pi}{2}\right) - \frac{3\pi}{2}\left(y - \frac{\pi}{2}\right)$$

3b(6 pts).(Source: 14.3.53,54)

$$p_{xx} = \big(2x - 3y + 2\cos(2x - y)\big)_x = 2 - 4\sin(2x - y)$$
$$p_{xy} = \big(2x - 3y + 2\cos(2x - y)\big)_y = -3 + 2\sin(2x - y)$$
$$p_{yy} = \big(-3x - \cos(2x - y)\big)_y = -\sin(2x - y)$$

4a(8 pts).(Source: 14.6.15) $\nabla g = \langle y + z, x - z, x - y\rangle$, which, at the point $(1, 0, 2)$ equals $\langle 2, -1, 1\rangle$. The directional derivative in the direction $\frac{1}{\sqrt{2}}\langle 1, -1, 0\rangle$ is

$$\frac{1}{\sqrt{2}}\langle 1, -1, 0\rangle \cdot \nabla g = \frac{1}{\sqrt{2}}\langle 1, -1, 0\rangle \cdot \langle 2, -1, 1\rangle = \frac{3}{\sqrt{2}}.$$

4b(2 pts).(Source: 14.6.21) At $(1, 0, 2)$ the greatest derivative of g is $|\nabla g| = |\langle 2, -1, 1\rangle| = \sqrt{6}$.

4c(2 pts).(Source: 14.6.21) The greatest directional derivative of g at $(1, 0, 2)$ occurs in the direction of the gradient: $\frac{1}{\sqrt{6}}\langle 2, -1, 1\rangle$.

5(23 pts).(Source: 14.7.more.11) Search for critical points:

$$h_x(x, y) = 3x^2 - 6y = 0 \Rightarrow y = \tfrac{1}{2}x^2.$$
$$h_y(x, y) - 3y^2 - 6x = 0 \Rightarrow x = -\tfrac{1}{2}y^2 = -\tfrac{1}{2}(\tfrac{1}{2}x^2)^2 = -\tfrac{1}{8}x^4$$

Factoring $0 = x^4 + 8x = x(x^3 + 8)$ yields $x = 0$ and $x = -2$ Use $y = \frac{1}{2}x^2$ to find the accompanying y-values 0 and 2

Now use the Second Derivative Test at the critical points.

$$D = \begin{vmatrix} h_{xx} & h_{xy} \\ h_{xy} & h_{yy} \end{vmatrix} = \begin{vmatrix} 6x & -6 \\ -6 & -6y \end{vmatrix} = 6 \cdot 6 \begin{vmatrix} x & -1 \\ -1 & -y \end{vmatrix} = 36(-xy - 1)$$

critical point	D	h_{xx}	conclusion
$(-2, 2)$	$36 \cdot 3$	-6	local maximum
$(0, 0)$	-36	irrelevant	saddle point

6(21 pts).(Source: 14.8.7,8) Use Lagrange multipliers with $r(x, y, z) = 2x - 2y + z$ and $g = x^2 + 2y^2 + z^2$. Find the critical points, i.e., the points on $g = 28$ at which $\nabla r \times \nabla g = \mathbf{0}$:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & 1 \\ 2x & 4y & 2z \end{vmatrix} = 2\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & 1 \\ x & 2y & z \end{vmatrix} = 2\langle -2z - 2y, x - 2z, 2x + 4y\rangle = \mathbf{0}$$

Express x and y in terms of z:

$$y = -z \qquad x = 2z$$

and substitute these into the constraint:

$$4z^2 + 2z^2 + z^2 = 28 \Rightarrow z^2 = 4 \Rightarrow z = \pm 2.$$

Find the corresponding values of x and y and evaluate r at the two critical points:

critical point	$2x - 2y + z$	conclusion
$(4, -2, 2)$	14	absolute maximum
$(-4, 2, -2)$	-14	absolute minimum

7(3 pts).(Source: 14.7.31) f can take its absolute extrema only at critical points in D's interior or on the boundary of D. Since $f_x = 2$ is ever zero, f has no interior critical points.

8(4 pts).(Source: 14.1.32) a3, b6, c4, d1.