No notes, books, electronic devices, or outside materials of any kind.

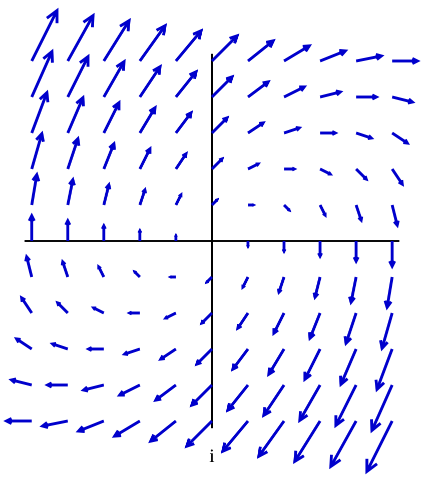
Read each problem carefully and simplify your answers.

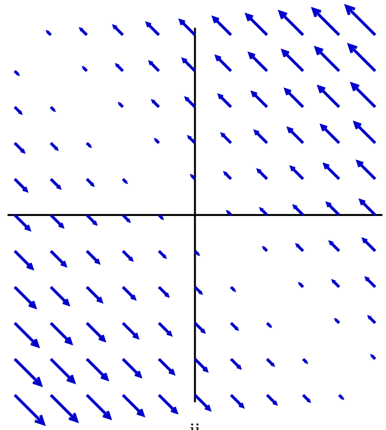
Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points.

You are expected to know the values of all trig functions at multiples of $\pi/4$ and of $\pi/6$.

1(4 pts). Find the graph of the given vector field.

a. $\langle -x-y,\, x+y\rangle$ b. $\langle y,\, x-y\rangle$ c. $\langle y,\, y-x\rangle$ d. $\langle x+y,\, x-y\rangle$


i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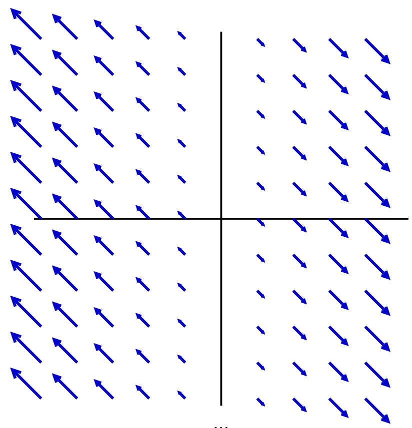
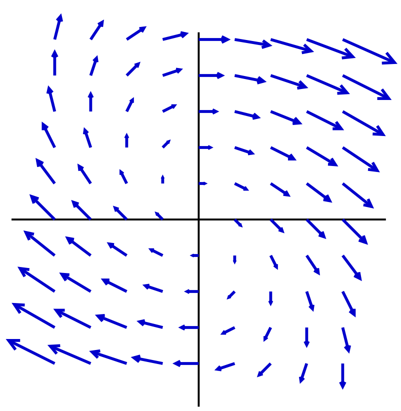
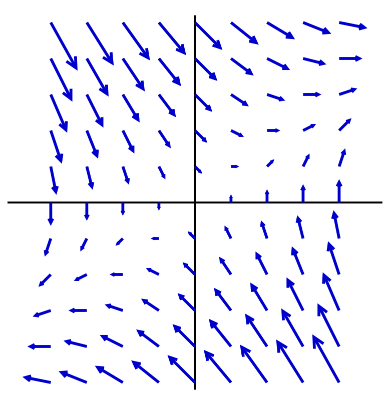

iii


iv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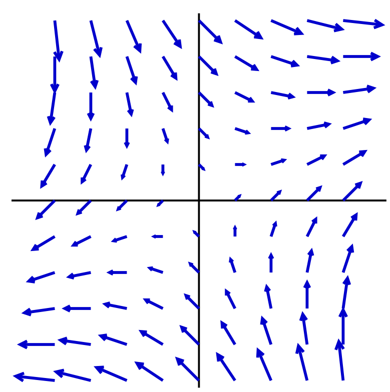
vi

2(6 pts). Suppose $\mathbf{F}$ and $\mathbf{G}$ are vector fields and u and v are scalar-valued functions. Indicate whether each of the following is a vector field (V), a scalar-valued function (S), or does not exist (dne).

a. $\operatorname{div}\mathbf{F}$ b. $\operatorname{grad}\mathbf{F}$ c. $\operatorname{grad}(uv)$
d. $\operatorname{grad}(\operatorname{div}(\mathbf{F}+\mathbf{G}))$ e. $\operatorname{curl}(\mathbf{F}\cdot\mathbf{G})$ f. $\operatorname{curl}(\operatorname{grad} u)$

2, continued(2 pts). Exactly one of a-i above is always zero (either the vector or the scalar) regardless of u, v, $\mathbf{F}$, and $\mathbf{G}$. Which is it?

3(8 pts). Determine whether the vector field $\mathbf{F} = \langle -ye^{-x} + \ln y, e^{-x} + \frac{x}{y} \rangle$ is conservative, and, if it is, find a potential function for $\mathbf{F}$.

4(14 pts). Evaluate the line integral.

a. $\int_D (-ye^{-x} + \ln y)\,dx + (e^{-x} + \frac{x}{y})\,dy$, where D is the hyperbola $y = \sqrt{7 + x^2}$ from the point $(-3, 4)$ to the point $(3, 4)$.

b. $\int_W (\cos(\sqrt{x}) - \frac{1}{2}y)\,dx + (\frac{1}{2}x + e^{y^2})\,dy$, where W is the left half of $x^2 + y^2 = 1$ in the positive direction from $(0, -1)$ to $(0, 1)$ followed by the line segment from $(0, 1)$ to $(0, -1)$.

5(9 pts). Let C be the path parametrized by $\langle t, -t, t^2 \rangle$ from $(-1, 1, 1)$ to $(1, -1, 1)$. Write the line integral $\int_C z^2\ dx + x^2\,dy +\ dz$ as a definite integral in t, but **do not evaluate**.

6(14 pts). Let T be the interior of the parallelogram in the xy-plane with vertices $(0, 0)$, $(1, 3)$, $(2, 1)$ and $(3, 4)$. Rewrite the double integral $\iint_T (4x - 3y)\,dA$ as an iterated integral in the variables $u = x - 2y$ and $v = 3x - y$, but **do not evaluate**.

7(20 pts). Let H be the part of the cone $x^2 + y^2 = z^2$ between $z = 2$ and $z = 3$.

a. Find a parametrization of H.

b. Find the area of H. Express your answer as an iterated double integral in your parameters, but **do not evaluate**.

8(24 pts). Let P be the part of the plane $x + y + 3z = 3$ in the first octant.

a. Express the surface integral $\iint_P xy\,dS$ as an iterated double integral, but **do not evaluate**.

b. Find the flux of $\langle y, 3z, x \rangle$ across P when P is oriented upward. Express your answer as an iterated double integral in your parameters, but **do not evaluate**.

This exam contained **1** bonus points.

1(4 pts).(Source: 16.1.11-13) a = ii. b = v. c = i. d = vi.

2(6 pts).(Source: 16.5.12) a S. b DNE. c V. d V. e DNE. f V.

f: $\operatorname{curl}(\operatorname{grad} u) = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle u$ is always $\mathbf{0}$.

grad $= \nabla$	div $= \nabla \cdot$	curl $= \nabla \times$
grad scalar = vector	div scalar DNE	curl scalar DNE
grad vector DNE	div vector = scalar	curl vector = vector

3(8 pts).(Source: 16.3.7) Integrate $f_y = e^{-x} + \frac{x}{y}$ to obtain $f = ye^{-x} + x \ln y + C(x)$. Differentiate this w.r.t. x to obtain $f_x = -ye^{-x} + \ln y + C'(x)$. Compare with $-ye^{-x} + \ln y$ and conclude $C'(x) = 0$, so C is any constant. $f(x, y) = ye^{-x} + x \ln y$ is a potential function.

4a(6 pts).(Source: 16.3.12-14) Since $\langle -ye^{-x} + \ln y, e^{-x} + \frac{x}{y} \rangle$ is conservative, we can evaluate the integral by using the Fundamental Theorem for Line Integrals [2] p. 1087:

$$\int_D (-ye^{-x} + \ln y)\, dx + (e^{-x} + \frac{x}{y})\, dy = (ye^{-x} + x \ln y)\Big|_{(-3,4)}^{(3,4)} = (4e^{-3} + 3\ln 4) - (4e^{3} - 3\ln 4),$$

or $4e^{-3} - 4e^3 + 6 \ln 4$.

4b(8 pts).(Source: 16.4.7) Since the curve is closed, we can use Green's Theorem p. 1096 to rewrite the line integral as a double integral over the interior D of the path W, multiplied by -1 since path around the half-circle is traveled in the negative direction.

$$-\iint_D \left((\frac{1}{2}x + e^{y^2})_x - (\cos(\sqrt{x}) - \frac{1}{2}y)_y \right) dA = -\iint_D 1\, dA,$$

which equals negative one times area of the half-circle, or $-\frac{1}{2}\pi$.

5(9 pts).(Source: 16.2.13,14) Using

$$\begin{array}{ccc} x = t & y = -t & z = t^2 \\ dx = dt & dy = -\,dt & dz = 2t\, dt \end{array}$$

the line integral becomes

$$\int_{-1}^{1} (t^4\, dt + t^2\,(-dt) + 2t\, dt) = \int_{-1}^{1} (t^4 - t^2 + 2t)\, dt.$$

6(14 pts).(Source: 15.9.15) Equations of the edges of the parallelogram are

$$\begin{array}{cc} x - 2y = 0 & 3x - y = 0 \\ x - 2y = -5 & 3x - y = 5 \end{array}$$

Solve for x and y to find

$$x = -\frac{1}{5}u - \frac{2}{5}v \qquad y = -\frac{3}{5}u + \frac{1}{5}v$$

and therefore

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -\frac{1}{5} & -\frac{2}{5} \\ -\frac{3}{5} & \frac{1}{5} \end{vmatrix} = \frac{1}{5}.$$

The integrand $4x - 3y = u + v$. In these new variables, the double integral is

$$\int_{-5}^{0} \int_{0}^{5} (u+v)\frac{1}{5}\, dv\, du.$$

You could have avoided solving for x and y had you noticed that $4x-3y = x-2y+3x-y = u+v$ and used

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{\begin{vmatrix} 1 & -2 \\ 3 & -1 \end{vmatrix}} = \frac{1}{5}.$$

7(20 pts). a.(Source: 16.6.25) You could parametrize by $\mathbf{r} = \langle x, y, \sqrt{x^2+y^2}\rangle$, but

$$\mathbf{r} = \langle r\cos\theta, r\sin\theta, r\rangle \tag{0.1}$$

will be easier to use in the next part, since the limits will be $0 \le \theta \le 2\pi$ and $2 \le r \le 3$.
b.(Source: 16.6.25,42) Using (0.1), find the surface area by integrating

$$dS = |\mathbf{r}_\theta \times \mathbf{r}_r|\, dr\, d\theta = \left\| \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 1 \end{matrix} \right\| dr\, d\theta = |\langle r\cos\theta, r\sin\theta, -r\rangle|\, dr\, d\theta.$$

Surface area is

$$\int_0^{2\pi} \int_2^3 \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + r^2}\, dr\, d\theta,$$

or $\int_0^{2\pi}\int_2^3 r\sqrt{2}\, dr\, d\theta$.

8(24 pts). a.(Source: 16.7.10) Can parametrize P with $\mathbf{r} = \langle x, y, 1 - \frac{1}{3}x - \frac{1}{3}y\rangle$. Then

$$|\mathbf{r}_x \times \mathbf{r}_y| = \left\| \begin{matrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -\frac{1}{3} \\ 0 & 1 & -\frac{1}{3} \end{matrix} \right\| = |\langle \tfrac{1}{3}, \tfrac{1}{3}, 1\rangle| = \sqrt{\tfrac{11}{9}}.$$

Integrate over the region in the xy-plane bounded by the x and y-axes and the line $x+y=3$ (where P intersects $z=0$):

$$\iint_P xy\, dS = \sqrt{\tfrac{11}{9}} \int_0^3 \int_0^{3-y} xy\, dx\, dy.$$

b.(Source: 16.7.24) $\mathbf{n}\,dS = \pm(\mathbf{r}_x \times \mathbf{r}_y)\,dx\,dy = \pm\langle \frac{1}{3}, \frac{1}{3}, 1\rangle\,dx\,dy$. Upward orientation requires $\mathbf{n}$ to have a positive $\mathbf{k}$-coefficient, so choose $+$. The flux is

$$\int_0^3 \int_0^{3-y} \langle y, 3z, -x\rangle \cdot \langle \tfrac{1}{3}, \tfrac{1}{3}, 1\rangle\,dx\,dy = \int_0^3 \int_0^{3-y} (\tfrac{1}{3}y + 1 - \tfrac{1}{3}x - \tfrac{1}{3}y - x)\,dx\,dy$$
$$= \int_0^3 \int_0^{3-y} (1 + \tfrac{2}{3}x)\,dx\,dy.$$

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