MATH 221–01 (Kunkle), Final Exam Name: ______________________

160 pts, 2 hours April 30, 2022

No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points.

You are expected to know the values of all trig functions at multiples of $\pi/4$ and of $\pi/6$.

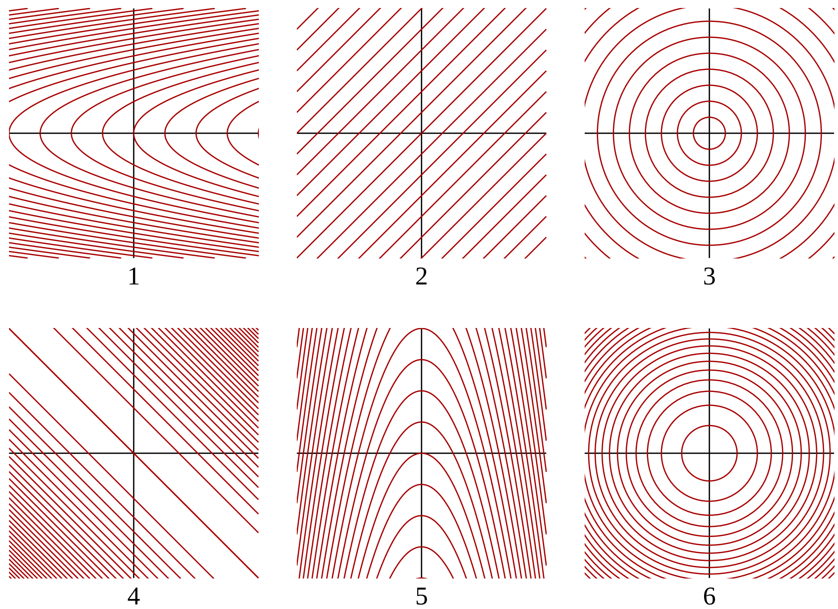
1(8 pts). Find the contour plot of each of the four functions below.

a. $(x+y)^2$

b. $\sqrt{x^2+y^2}$

c. $1-x^2-y^2$

d. $x-y^2$



2a(5 pts). Find the vector projection (also known as the orthogonal projection) of $\langle 1,2,1\rangle$ onto $\langle -1,-1,1\rangle$. 2b(1 pts). Which of these does the correct answer to 2a tell us? (circle one)

i. The point on the line $\langle -t,-t,t\rangle$ closest to the point $(1,2,1)$.

ii. The point on the line $\langle t,2t,t\rangle$ closest to the point $(-1,-1,1)$.

iii. Neither of these.

3(29 pts). Let $\mathbf{r}(t) = \langle e^{1-t}, t^{-1}, \frac{1}{2}t^2+1\rangle$ be the position of a particle at time t.

a. Find a parametric equation of the line tangent to the particle's path at the point corresponding to $t=1$.

b. Find the particle's velocity, acceleration, and speed at $t=1$.

c. Find a_T and a_N, the tangential and normal components of the particle's acceleration at $t=1$.

d. Is the particle is speeding up or slowing down at $t=1$? Explain briefly.

4a(5 pts). Find the area of the parallelogram determined by the vectors $\langle 1,2,1\rangle$ and $\langle -1,-1,1\rangle$. (That is, the one with vertices $(0,0,0)$, $(1,2,1)$, $(-1,-1,1)$, $(0,1,2)$.)

4b(5 pts). Find the volume of the parallelepiped determined by the vectors $\langle 1,2,1\rangle$, $\langle -1,-1,1\rangle$, and $\langle 0,4,3\rangle$.

5. Find an equation—either implicit or parametric—of the plane described in each part.
a(5 pts). Passing through the points $(0,0,0)$, $(1,2,1)$, and $(-1,-1,1)$.
b(10 pts). Tangent to the surface $xe^{y-z} + yz\ln x = 1$ at the point $(1,1,1)$.
c(13 pts). Tangent to the surface parametrized by $\mathbf{r}(u,v) = \langle v^2, uv, \frac{1}{2}u^2\rangle$ at the point in 3-space corresponding to $(u,v) = (1,1)$.
6(4 pts). Two students are working on the problem of maximizing and minimizing a function $q(x,y)$ on the disk $D = \{(x,y) \mid x^2 + y^2 \leq 1\}$. They observe that $q(x,y)$ is continuous and differentiable everywhere in the xy plane.
The first student has determined that the maximum and minimum values of q on the circle $x^2 + y^2 = 1$ are 3 and -2.
The second student has determined that the only solutions to $q_x(x,y) = q_y(x,y) = 0$ are $(x,y) = (0, 1 \pm \frac{1}{3})$, and that $q(0, \frac{2}{3}) = 7$ and $q(0, \frac{4}{3}) = -4$.
Assuming their work is correct, what, if anything, can you say about the maximum and minimum values of $q(x,y)$ on D?
7(19 pts). Determine the (x,y) coordinates of all local maxima, local minima, and saddle points of $\phi(x,y) = \frac{1}{3}x^3 - 4x + xy^2$, and state which is which.
8(4 pts). A friend needs your help. He needs to calculate the integral

$$\oint (x-y)\,dx + (x+y),dy$$

over a certain closed path, but he's forgotten the path. All he remembers is that the it begins and ends at $(1,0)$. Evaluate the line integral for your friend or (briefly) explain to him why it's not possible.
9(16 pts). Find the volume of the solid above the cone $z = \sqrt{3}\sqrt{x^2+y^2}$ and below the sphere $x^2 + y^2 + z^2 = 16$.
10. A fluid flowing through 3-space has velocity $\langle x^2 - y + z, x + y^2 - z, -x + y + z^2\rangle$ at the point (x,y,z).
a(14 pts). Find the net rate of flow (a.k.a. "flux") of the fluid out of the rectangular parallelogram with vertices

$$\begin{array}{cccc} (0,0,0) & (0,2,0) & (0,0,3) & (0,2,3) \\ (1,0,0) & (1,2,0) & (1,0,3) & (1,2,3) \end{array}$$

by first writing the flux as a triple integral.
b(22 pts). Find the net rate of flow (a.k.a. "circulation") of the fluid around the triangle

$$\begin{array}{ccccc} & & (0,0,2) & & \\ & \swarrow & & \nwarrow & \\ (1,0,0) & & \longrightarrow & & (0,1,0) \end{array}$$

by first writing the circulation as a surface integral.

1(8 pts).(Source: 14.1.61-66) A contour map of $f(x, y)$ consists of curves $f(x, y) = k$ for some evenly spaced constants k.
a. level curves are $\{(x+y)^2 = k\}$, or $\{x + y = \pm\sqrt{k}\}$, lines of slope -1. graph 4.
b. $\{\sqrt{x^2 + y^2} = k\}$ are circles of radius k. These are evenly spaced when k is. graph 3.
c. $\{1 - x^2 - y^2 = k\}$, or $\{x^2 + y^2 = 1 - k\}$, are circles of radius $\sqrt{1-k}$. When k is evenly spaced, these are *not*. graph 6.
d. $\{x = y^2 + k\}$ are parabolas opening to the right. graph 1.

2a(5 pts).(Source: 12.3.41)

$$\text{proj}_{\langle -1,-1,1\rangle}\langle 1, 2, 1\rangle = \frac{\langle 1, 2, 1\rangle \cdot \langle -1, -1, 1\rangle}{\langle -1, -1, 1\rangle \cdot \langle -1, -1, 1\rangle}\langle -1, -1, 1\rangle = \frac{-2}{3}\langle -1, -1, 1\rangle,$$

or $\langle \frac{2}{3}, \frac{2}{3}, -\frac{2}{3}\rangle$.
2b(1 pts). Correct response is **i**.

3a&b(16 pts). velocity $\mathbf{v} = \frac{d\mathbf{r}}{dt} = \langle -e^{1-t}, -t^{-2}, t\rangle$. acceleration $\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = \langle e^{1-t}, 2t^{-3}, 1\rangle$. At $t = 1$, $\mathbf{r} = \langle 1, 1, \frac{3}{2}\rangle$, $\mathbf{v} = \langle -1, -1, 1\rangle$, $\mathbf{a} = \langle 1, 2, 1\rangle$, and speed $= |\mathbf{v}| = \sqrt{3}$. The tangent line passes through the point $(1, 1, \frac{3}{2})$ and is parallel $\langle -1, -1, 1\rangle$. The line is parametrized by the vector-valued function

$$\langle 1, 1, \frac{3}{2}\rangle + t\langle -1, -1, 1\rangle,$$

or $\langle 1 - t, 1 - t, \frac{3}{2} + t\rangle$.

3c(10 pts).(Source: 13.4.41-42)

$$\mathbf{v} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -1 & 1 \\ 1 & 2 & 1 \end{vmatrix} = \mathbf{i}\begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} -1 & -1 \\ 1 & 2 \end{vmatrix} = \langle -3, 2, -1\rangle$$

Recall that if θ is the angle between $\mathbf{v}$ and $\mathbf{a}$, then

$$a_T = |\mathbf{a}|\cos\theta = \frac{\mathbf{v}\cdot\mathbf{a}}{|\mathbf{v}|} = \frac{-2}{\sqrt{3}}, \quad \text{and} \quad a_N = |\mathbf{a}|\sin\theta = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|} = \frac{|\langle -3, 2, -1\rangle|}{\sqrt{3}} = \sqrt{\frac{14}{3}}.$$

3d(3 pts).(Source: 13.4. 7) $\frac{d^2s}{dt^2} = a_T < 0$. The particle is slowing down since its speed is decreasing.

4a(5 pts).(Source: 12.4.28) See top of page 818.
Area $= |\langle -1, -1, 1\rangle \times \langle 1, 2, 1\rangle| = |\langle -3, 2, -1\rangle| = \sqrt{14}$.

4b(5 pts).(Source: 12.4.33) By 14 , page 820,
Volume $= |\langle 0, 4, 3\rangle \cdot \langle -1, -1, 1\rangle \times \langle 1, 2, 1\rangle| = |\langle 0, 4, 3\rangle \cdot \langle -3, 2, -1\rangle| = 5$.

5. See 7 , p. 827. To find the equation of a plane, we need a point on the plane and a vector normal to the plane.

5a(5 pts).(Source: 12.5.31-33) This plane is parallel the vectors $\langle -1,-1,1\rangle$ and $\langle 1,2,1\rangle$, so their cross product $\langle -3,2,-1\rangle$ is normal to the plane. Since it passes through $(0,0,0)$, the plane is given implicitly by the equation $-3x+2y-z=0$.

5b(10 pts).(Source: 14.6.41-46) The gradient of a function is perpendicular to its level curves/surfaces. $\nabla(xe^{y-z}+yz\ln x) = \langle e^{y-z}+yz\frac{1}{x}, xe^{y-z}+z\ln x, -xe^{y-z}+y\ln x\rangle$, which, at $(1,1,1)$, equals $\langle 2,1,-1\rangle$. The plane is given by the equation $2(x-1)+(y-1)-(z-1)=0$, or $2x+y-z=2$.

5c(13 pts).(Source: 16.6.33-36) The point corresponding to $(u,v)=(1,1)$ is given by $\mathbf{r}(1,1) = \langle 1, 1\frac{1}{2}\rangle$. The vector $\mathbf{r}_u \times \mathbf{r}_v =$

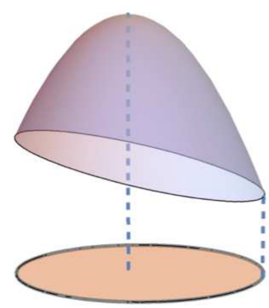
$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & v & u \\ 2v & u & 0 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 1 \\ 2 & 1 & 0 \end{vmatrix} = \langle -1,2,-2\rangle$$

is normal to the plane, which is given implicitly by $-(x-1)+2(y-1)-2(z-\frac{1}{2})=0$, or $-x+2y-2z=0$.

See review notes example 16.6.re.5. Since it is parallel to both $\mathbf{r}_u(1,1)=\langle 0,1,1\rangle$ and $\mathbf{r}_v=\langle 2,1,0\rangle$ and passes through $(1,1\frac{1}{2})$, the plane can be expressed parametrically by

$$\rho(s,t) = \langle 1, 1\tfrac{1}{2}\rangle + s\langle 0,1,1\rangle + t\langle 2,1,0\rangle,$$

or $\langle 1+2t, 1+s+t, \frac{1}{2}+s\rangle$.



6(4 pts).(Source: 14.8.31-42) The absolute extrema of a function on a region must occur at a boundary point or at a critical point interior to the region. Since the max and min along the boundary of D are 3 and -2 and q equals 7 at the only critical point interior to D, the maximum and minimum of q on D are 7 and -2, respectively.

7(19 pts).(Source: 14.7.5-13) First, find ϕ's critical points: $\phi_y(x,y)=2xy=0$ implies either $x=0$ or $y=0$. Substituting these into $\phi_x = x^2-4+y^2=0$ gives four critical points: $(0,2)$, $(0,-2)$, $(2,0)$, and $(-2,0)$.

Now apply the Second Derivative Test at these points.

$$D = \begin{vmatrix} \phi_{xx} & \phi_{xy} \\ \phi_{xy} & \phi_{yy} \end{vmatrix} = \begin{vmatrix} 2x & 2y \\ 2y & 2x \end{vmatrix} = 4(x^2-y^2).$$

critical point	D	ϕ_{xx}	conclusion
$(2,0)$	16	4	local minimum
$(-2,0)$	16	-4	local maximum
$(0,2)$	-16	irrelevant	saddle point
$(0,-2)$	-16	irrelevant	saddle point

8(4 pts).(Source: 16.3.3,12,21) Since $(x+y)_x = 1$ is not equal to $(x-y)_y = -1$, this integral is not path independent, so it is doubtful that the integral can be calculated without knowing the path. Tell your friend that you can't help him, but say it in a nice way.
(In fact, Green's theorem tells use that the line integral around the closed path in the positive direction equals the double integral of $Q_x - P_y = 1 - (-1) = 2$ over the region enclosed by the curve; that is, your friend's line integral equals twice the area of the region enclosed by the path, which is not the same for all paths beginning and ending at $(1, 0)$.)

9(16 pts).(Source: 15.3.25,15.7.23,15.8.30) Solution one (cylindrical coordinates):
The cone and sphere intersect when $x^2 + y^2 + 3(x^2 + y^2) = 16$, or $x^2 + y^2 = 4$. Limits are $0 \le \theta \le 2\pi$, $0 \le r \le 2$. Using $\sqrt{x^2 + y^2} = r$, the volume can be written

$$\begin{aligned}\int_0^{2\pi}\int_0^2\int_{r\sqrt{3}}^{\sqrt{16-r^2}} dz\, r\, dr\, d\theta &= \int_0^{2\pi}\int_0^2 (\sqrt{16-r^2} - r\sqrt{3})r\, dr\, d\theta \\ &= \int_0^{2\pi} (-\frac{1}{3}(16-r^2)^{3/2} - \frac{\sqrt{3}}{3}r^3)\Big|_0^2 d\theta \\ &= 2\pi(-\frac{1}{3}12^{3/2} + \frac{1}{3}16^{3/2} - \frac{\sqrt{3}}{8}),\end{aligned}$$

Solution two (spherical coordinates):
$\frac{r}{z} = \tan\theta$ and $z = r\sqrt{3}$ together tell us that $\tan\phi = \frac{1}{\sqrt{3}}$. Therefore $\phi = \frac{\pi}{6}$ along the cone. The volume can be written

$$\int_0^{2\pi}\int_0^{\pi/6}\int_0^4 \rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \int_0^{2\pi} d\theta \int_0^{\pi/6}\sin\phi\, d\phi \int_0^4 \rho^2\, d\rho$$
$$2\pi(-\cos(\pi/6) + \cos 0)\frac{1}{3}4^3 = 2\pi(1 - \frac{\sqrt{3}}{2})\frac{1}{3}4^3$$

With some effort, you can show that the answers in solutions one and two are the same. In rectangular coordinates, the volume is

$$\int_{-2}^2\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{3}\sqrt{x^2+y^2}}^{\sqrt{16-x^2-y^2}} dz\, dy\, dx,$$

but evaluating this would be a challenge.

10a(14 pts).(Source: 16.9.5-6, 15.6.2) The divergence of $\mathbf{F} = \langle x^2 - y + z, x + y^2 - z, -x + y + z^2\rangle$ is $\nabla\cdot\mathbf{F} = (x^2 - y + z)_x + (x + y^2 - z)_y + (-x + y + z^2)_z = 2x + 2y + 2z$. By the divergence theorem, the flux of $\mathbf{F}$ out of the parallelogram is the triple integral of $\operatorname{div}\mathbf{F}$:

$$\int_0^1\int_0^2\int_0^3 (2x + 2y + 2z)\, dz\, dy\, dx.$$

Remembering that the integral of a constant is that constant times the length of the integral, the triple integral is

$$\int_0^1 \int_0^2 (6x + 6y + 9)\, dy\, dx = \int_0^1 (12x + 12 + 18)\, dx = (6 + 12 + 18) = 36.$$

10b(22 pts).(Source: 16.8.7, 16.7.11, 15.2.19,22) Stokes's theorem says that the circulation around the triangular path is the same as the integral of $\operatorname{curl} \mathbf{F} \cdot \mathbf{n}\, dS$ over any surface whose boundary is that triangle. The simplest surface to use is the interior of the triangle. First, we need an equation of the plane passing through the three points. You can use the method of problem 5a, but, instead, let's observe that $2x + 2y + z = 2$ is satisfied by the coordinates of each of the three given points, so this linear equation must be the equation of the plane.
Since z is a function of x and y along this plane, we can parametrize the plane using $\mathbf{r}(x, y) = \langle x, y, 2 - 2x - 2y\rangle$. Choosing limits in x and y that describe the shadow of the triangle in the xy-plane gives $0 \le x \le 1$, $0 \le y \le 1 - x$.

$$\mathbf{n}\, dS = \pm\, \mathbf{r}_x \times \mathbf{r}_y\, dx\, dy = \pm \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -2 \\ 0 & 1 & -2 \end{vmatrix} dx\, dy = \pm\langle 2, 2, 1\rangle\, dx\, dy.$$

In order for the positive direction around the triangle to be the one indicated in the statement of the problem, choose $\mathbf{n}\, dS = +\langle 2, 2, 1\rangle\, dx\, dy$.
The curl of $\mathbf{F}$ is

$$\begin{aligned} \nabla \times \mathbf{F} &= \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\rangle \times \langle x^2 - y + z, x + y^2 - z, -x + y + z^2\rangle \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y + z & x + y^2 - z & -x + y + z^2 \end{vmatrix} = \langle 2, 2, 2\rangle \end{aligned}$$

.
The circulation equals the surface integral

$$\int_0^1 \int_0^{1-x} \langle 2, 2, 2\rangle \cdot \langle 2, 2, 1\rangle\, dy\, dx = \int_0^1 \int_0^{1-x} 9\, dy\, dx = \int_0^1 9(1 - x)\, dx = \frac{9}{2}.$$