

1 (10 pts). Evaluate $\iint_D x^2 y \, dA$ where D is quarter disk $x^2 + y^2 \leq 1$ in the first quadrant $x \geq 0, y \geq 0$.

1.(Source: 15.2.20,15.3.7) In rectangular coordinates:

The quarter circle in the first quadrant can be written either $y = \sqrt{1-x^2}$ or $x = \sqrt{1-y^2}$. Here are solutions in both orders of integration.

Solution one: the integral equals

$$\begin{aligned} \int_0^1 \int_0^{\sqrt{1-x^2}} x^2 y \, dy \, dx &= \int_0^1 \left. \frac{1}{2} x^2 y^2 \right|_0^{\sqrt{1-x^2}} dx = \int_0^1 \frac{1}{2} x^2 (1-x^2) \, dx \\ &= \frac{1}{2} \int_0^1 (x^2 - x^4) \, dx = \frac{1}{2} \left(\frac{1}{3} x^3 - \frac{1}{5} x^5 \right) \Big|_0^1 = \frac{1}{2} \left(\frac{1}{3} - \frac{1}{5} \right) = \frac{1}{15}. \end{aligned}$$

(done)

Solution two: the integral equals

$$\int_0^1 \int_0^{\sqrt{1-y^2}} x^2 y \, dx \, dy = \int_0^1 \left. \frac{1}{3} x^3 y \right|_0^{\sqrt{1-y^2}} dy = \frac{1}{3} \int_0^1 (1-y^2)^{3/2} y \, dy$$

At this point, you could let $u = 1 - y^2$, so that $-\frac{1}{2} du = y \, dy$. Change limits as well: $u = 1$ when $y = 0$, and $u = 0$ when $y = 1$. Integral becomes

$$-\frac{1}{2} \cdot \frac{1}{3} \int_1^0 (u)^{3/2} \, du = -\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{2}{5} (u)^{5/2} \Big|_1^0 = -\frac{1}{15} (0 - 1) = \frac{1}{15}.$$

(done)

We can also calculate this integral using polar coordinates:

The quarter disk in the first quadrant is described in polar coordinates by $0 \leq \theta \leq \frac{\pi}{2}$, $0 \leq r \leq 1$. Rewrite the integrand using $x = r \cos \theta$ and $y = r \sin \theta$:

$$\begin{aligned} \int_0^{\pi/2} \int_0^1 (r \cos \theta)^2 (r \sin \theta) r \, dr \, d\theta &= \int_0^{\pi/2} \int_0^1 r^4 \cos^2 \theta \sin \theta \, dr \, d\theta \\ &= \int_0^{\pi/2} \left. \frac{1}{5} r^5 \right|_0^1 \cos^2 \theta \sin \theta \, d\theta = \frac{1}{5} \int_0^{\pi/2} \cos^2 \theta \sin \theta \, d\theta \end{aligned}$$

Now substitute $u = \cos \theta$, $du = -\sin \theta \, d\theta$ and change the limits:

$$= \frac{1}{5} \int_1^0 u^2 (-du) = \frac{1}{5} \int_0^1 u^2 \, du = \frac{1}{5} \cdot \frac{1}{3} u^3 \Big|_0^1 = \frac{1}{15}.$$

(done)