Name: ______________________

No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Unless otherwise indicated, supporting work will be required on every problem worth more than 3 points.

You are expected to know the values of all trig functions at multiples of $\pi/4$ and of $\pi/6$.

1a(16 pts). Find a function $f(x, y, z)$ for which $\nabla f = \langle y + 1, x - z, -y\rangle$ or show that none exists.

1b(4 pts). Suppose C is curve that begins and ends at the point $(1, 1, 0)$. What, if anything, can you say about the value of the line integral $\int_C (y + 1)\, dx + (x - z)\, dy - y\, dz$, and why?

2a(6 pts). Find the vector projection of $\mathbf{u} = \langle -1, 1, 1\rangle$ onto $\mathbf{v} = \langle 2, 2, -1\rangle$.

2b(6 pts). Find the area of the parallelogram with vertices $(0, 0, 0)$, $(-1, 1, 1)$, $(2, 2, -1)$, $(1, 3, 0)$.

3(6 pts). Find the point (x, y, z) of intersection of the plane given by $x + 2y + 3z = 13$ with the line parametrized by $x = 2 - 3t$, $y = 2t$, $z = 1 + t$.

4(16 pts). Find the curvature κ along the curve parametrized by $\mathbf{r} = \langle 2t, \sin t, \cos t\rangle$.

5. Let $q(x, y) = \dfrac{x + 2y}{x - y}$.

a(12 pts). Find $q_x(x, y)$ and $q_y(x, y)$.

b(8 pts). At $(4, 3)$, in which direction does the greatest directional derivative of q occur? State your answer as a unit vector.

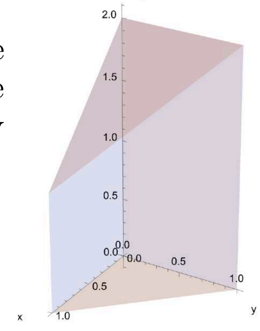
c(8 pts). Suppose that x and y are functions of t and that, at $t = 0$, $x = 4$, $x_t = -2$, $y = 3$, and $y_t = 5$. Find q_t at $t = 0$.

6a(20 pts). Find the maximum and minimum values of $w(x, y) = x^2 + y^2 - 8x - 6y$ subject to the constraint $x^2 + y^2 = 1$ and the points (x, y) where these occur.

6b(6 pts). I need to find the absolute maximum and minimum values of $w(x, y)$ from 6a on the set $D = \{\,(x, y) \mid x^2 + y^2 \le 1\,\}$, that is, the unit circle and its interior. Explain why the max and min of w must occur at points on $x^2 + y^2 = 1$ and not in $x^2 + y^2 < 1$.

7(20 pts). Find the flux of the curl of $\mathbf{F} = xy\,\mathbf{i} + yz\,\mathbf{j} + ze^{x+y}\,\mathbf{k}$ over the part of the sphere $x^2 + y^2 + z^2 = 5$ above $z = 1$, oriented upwards, by first rewriting the flux as a line integral.

8(20 pts). Let D be the region in $\mathbb{R}^3$ above the triangle in the xy-plane with verticies $(0, 0)$, $(1, 0)$, $(0, 1)$ and below the plane $z = 2 - x$ (see figure). Find the flux of $\mathbf{H} = (x + y)\,\mathbf{i} + (y + z)\,\mathbf{j} + (x - z)\,\mathbf{k}$ out of D by first rewriting the flux as a triple integral.

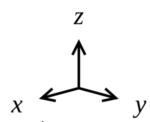


9a(3 pts). Find cylindrical coordinates for the point $(x, y, z) = (1, -1, -\sqrt{2})$.

9b(3 pts). Find spherical coordinates for the point $(x, y, z) = (1, -1, -\sqrt{2})$.

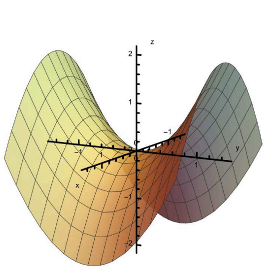
(There are multiple correct answers to 9a and b.)

10(6 pts). Find the graphs of the given equations. Each graph is seen from the same point in the first octant with the axes oriented like this:



a. $-x^2 + y^2 = z^2$ b. $x^2 - x + y^2 + y + z^2 = \frac{1}{2}$ c. $x^2 + y^2 - z^2 = -1$

d. $-x^2 + y^2 = z$ e. $z + y^2 = 1$ f. $x^2 + y^2 - z^2 = 1$



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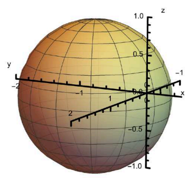

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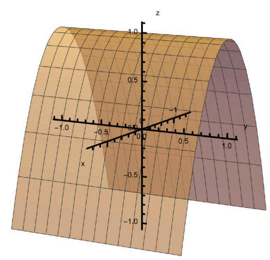

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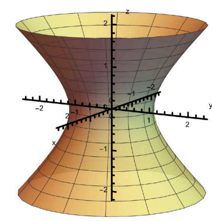

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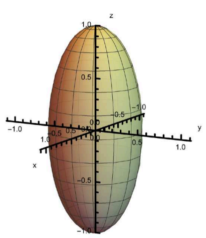

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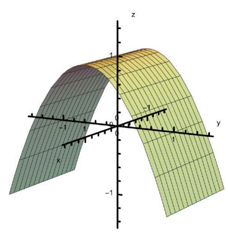

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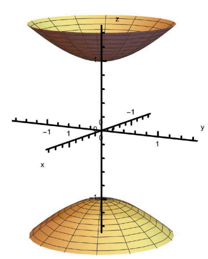

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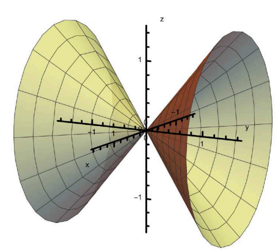


8.

1a(16 pts).(Source: 16.3.15-18) Look for a function $f(x, y, z)$ satisfying

$$(0) \qquad f_x = y + 1 \qquad f_y = x - z \qquad f_z = -y$$

Here's a complete solution, but at any point, you might find a correct potential function. As long as you demonstrate that the function you find satisfies (0), you don't need to complete all the following steps.

$$\begin{aligned} f_x = y + 1 \;\Rightarrow\; & f = xy + x + C(y, z) \\ \Rightarrow\; & f_y = x + C_y(y, z) = x - z \\ & \Rightarrow\; C_y(y, z) = -z \;\Rightarrow\; C(y, z) = -yz + D(z) \end{aligned}$$

Now differentiate $f = xy + x + -yz + D(z)$ with respect to z:

$$f_z = -y + D_z(z) = -y \;\Rightarrow\; D_z(z) = 0 \;\Rightarrow\; D = \text{constant}$$

Since you were required to find only one potential function, you can take $D = 0$ making $f = xy + x + -yz$.

1b(4 pts).(Source: 16.3.15-18) Since C is a closed curve and the vector field $\langle y + 1, x - z, -y\rangle$ is conservative, the Fundamental Theorem for line integrals implies that $\int_C (y + 1)\, dx + (x - z)\, dy - y\, dz = 0$. (In particular, see $\boxed{3}$ on page 1089 of our text.)

2a(6 pts).(Source: 12.3.0) $\text{proj}_{\mathbf{v}}\, \mathbf{u} = \frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} = \frac{-1}{9}\langle 2, 2, -1$, or $\langle -\frac{2}{9}, -\frac{2}{9}, \frac{1}{9}\rangle$.

2b(6 pts).(Source: 12.4.28) Calculate the cross product.

$$\begin{aligned} \mathbf{u} \times \mathbf{v} = \langle -1, 1, 1\rangle \times \langle 2, 2, -1\rangle &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 1 \\ 2 & 2 & -1 \end{vmatrix} \\ &= \left(\mathbf{i} \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} -1 & 1 \\ 2 & -1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} -1 & 1 \\ 2 & 2 \end{vmatrix} \right) = \langle -3, 1, -4\rangle. \end{aligned}$$

The area of the parallelogram is the magnitude $|\mathbf{u} \times \mathbf{v}| = \sqrt{9 + 1 + 16} = \sqrt{26}$.

3(6 pts).(Source: 12.5.45-47) Substitute the parametrization of the line into the equation of the plane:

$$(2 - 3t) + 2(2t) + 3(1 + t) = 13 \quad \Rightarrow \quad 4t + 5 = 13 \quad \Rightarrow \quad t = 2$$

so the point of intersection is $(2 - 3t, 2t, 1 + t) = (-4, 4, 3)$.

4(16 pts).(Source: 13.3.17) $\mathbf{v} = \langle 2, \cos t, -\sin t\rangle$ and $\mathbf{a} = \langle 0, -\sin t, -\cos t\rangle$. Calculate their cross product:

$$\begin{aligned} \mathbf{v} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & \cos t & -\sin t \\ 0 & -\sin t & -\cos t \end{vmatrix} &= \mathbf{i} \begin{vmatrix} \cos t & -\sin t \\ -\sin t & -\cos t \end{vmatrix} - \mathbf{j} \begin{vmatrix} 2 & -\sin t \\ 0 & -\cos t \end{vmatrix} + \mathbf{k} \begin{vmatrix} 2 & \cos t \\ 0 & -\sin t \end{vmatrix} \\ &= \langle -1, 2\cos t, -2\sin t\rangle \end{aligned}$$

and then

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \frac{\sqrt{1 + 4(\cos^2 t + \sin^2 t)}}{\sqrt{4 + \cos^2 t + \sin^2 t}^3} = \frac{1}{5}.$$

5a(12 pts).(Source: 14.3.23) To find q_x, treat y as a constant and differentiate $\frac{x+2y}{x-y}$ with respect to x using the rules of differentiation from MATH 120.

$$\begin{aligned} \left(\frac{x+2y}{x-y}\right)_x &= \frac{(x+2y)_x(x-y) - (x-y)_x(x+2y)}{(x-y)^2} \\ &= \frac{1(x-y) - 1(x+2y)}{(x-y)^2} = \frac{-3y}{(x-y)^2}. \end{aligned}$$

To find q_y, hold x constant and differentiate with respect to y:

$$\begin{aligned} \left(\frac{x+2y}{x-y}\right)_y &= \frac{(x+2y)_y(x-y) - (x-y)_y(x+2y)}{(x-y)^2} \\ &= \frac{2(x-y) - (-1)(x+2y)}{(x-y)^2} = \frac{3x}{(x-y)^2} \end{aligned}$$

5b(8 pts).(Source: 14.6.21-23) The directional derivative is greatest in the direction of $\nabla q = \langle q_x, q_y \rangle$, which, at $(4, 3)$, equals $\langle -9, 12 \rangle$. Normalize this, or, what's easier, $\frac{1}{3}\langle -9, 12 \rangle = \langle -3, 4 \rangle$, the norm of which is 5. The desired unit vector is $\frac{1}{5}\langle -3, 4 \rangle$, or $\langle -\frac{3}{5}, \frac{4}{5} \rangle$.

5c(8 pts).(Source: 14.5.13-16) By the chain rule, $q_t = q_x x_t + q_y y_t = (-9)(-2) + (12)(5) = 78$.

6a(20 pts).(Source: 14.8.4) Let $g(x, y) = x^2 + y^2$. The desired max and min can only occur at those points on $g(x, y) = 1$ at which

$$\nabla w(x, y) = \lambda \nabla g(x, y)$$

for some scalar λ or one of the gradients is zero, or, equivalently,

$$\nabla w \times \nabla g = \mathbf{0}$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2x - 8 & 2y - 6 & 0 \\ 2x & 2y & 0 \end{vmatrix} = 4 \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x - 4 & y - 3 & 0 \\ x & y & 0 \end{vmatrix} = 4\langle 0, 0, -4y + 3x \rangle = \langle 0, 0, 0 \rangle,$$

and so $y = \frac{3}{4}x$. Substitute this into the constraint:

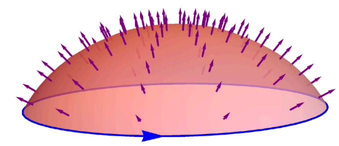
$$x^2 + \frac{9}{16}x^2 = 1 \quad \Longrightarrow \quad x^2 = \frac{16}{25} \quad \Longrightarrow \quad x = \pm\frac{4}{5} \quad y = \frac{3}{4}x = \pm\frac{3}{5}$$

Now compare the values of w at these critical points:

critical point	$w(x,y)$	conclusion
$(-\frac{4}{5},-\frac{3}{5})$	$1+\frac{32}{5}+\frac{18}{5}=11$	maximum
$(\frac{4}{5},\frac{3}{5})$	$1-\frac{32}{5}-\frac{18}{5}=-9$	minimum

6b(6 pts).(Source: 14.7.37) See [9] p. 966. The absolute max and min for w can occur only on the boundary $x^2+y^2=1$ or at critical points interior to D. But the only critical point of w, that is, where both $w_x=2x-8$ and $w_y=2y-6$ equal zero, is at $(x,y)=(4,3)$. Since this is not in D, the max and min of w must occur on the boundary.

7(20 pts).(Source: 16.8.2-6) Solution one: if we denote the surface by R (shown here), then its boundary ∂R is the curve $x^2+y^2+1=5$ at altitude $z=1$. By Stokes's Theorem, the flux of the curl of $\mathbf{F}$ over R equals



$$\iint_R \operatorname{curl}\mathbf{F}\cdot\mathbf{n}\,dS=\int_{\partial R}\mathbf{F}\cdot d\mathbf{r}=\int_{\partial R} xy\,dx+yz\,dy+ze^{x+y}\,dz,\quad (1)$$

where ∂R is traversed in the positive direction when viewed from above. Since we'll be calculating the line integral in (1), it's not necessary to calculate $\operatorname{curl}\mathbf{F}$.
Parametrize the circle ∂R by

$$\begin{array}{lll} x=2\cos\theta & y=2\sin\theta & z=1 \\ dx=-2\sin\theta\,d\theta & dy=2\cos\theta\,d\theta & dz=0 \end{array}$$

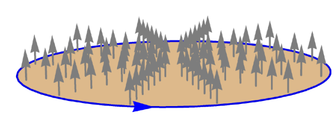
and the line integral in (1) equals

$$\int_0^{2\pi}(-8\sin^2\theta\cos\theta\,d\theta+4\sin\theta\cos\theta\,d\theta).$$

Substitute $\sigma=\sin\theta$ and $d\sigma=\cos\theta\,d\theta$ to transform the integral to $\int_{\sigma=0}^{\sigma=0}(-8\sigma^2+4\sigma)\,d\sigma=0$.
(done)

Solution two: rewrite the line integral (1) by Stokes again as the flux of

$$\operatorname{curl}\mathbf{F}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ xy & yz & ze^{x+y}\end{vmatrix}=\langle ze^{x+y}-y,\,ze^{x+y},\,-x\rangle.$$



over the horizontal surface $x^2+y^2=4$ at $z=1$, which we'll denote H (shown here), oriented upward. Then, in polar coordinates, $dS=r\,dr\,d\theta$ and $\mathbf{n}$ is $\mathbf{k}$ (which means we needed only to calculate the third component of $\operatorname{curl}\mathbf{F}$ above). Then the flux of $\operatorname{curl}\mathbf{F}$ across H is

$$\int_0^{2\pi}\int_0^2 -xr\,dr\,d\theta=\int_0^{2\pi}\int_0^2 -r^2\cos\theta\,dr\,d\theta=-\int_0^2 r^2\,dr\int_0^{2\pi}\cos\theta\,d\theta=0.$$

(done)

If you ignored the instructions and calculated the flux of the curl over R directly, you'll need a parametrization of R such as

$$\mathbf{r}(r,\theta) = \langle r\cos\theta, r\sin\theta, \sqrt{5-r^2}\rangle \qquad 0 \le r \le 2,\ 0 \le \theta \le 2\pi \tag{2}$$

Don't confuse this parametrization with $\mathbf{F}$, which is unrelated to the surface R and its parametrization. Using this $\mathbf{r}$, calculate $\mathbf{n}\,dS$ as

$$\begin{aligned} \pm\,\mathbf{r}_r \times \mathbf{r}_\theta\,dr\,d\theta &= \pm \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & -r(5-r^2)^{-1/2} \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} dr\,d\theta \\ &= \pm\,\langle -r^2(5-r^2)^{-1/2}\cos\theta, r^2(5-r^2)^{-1/2}\sin\theta, r\rangle\,dr\,d\theta \end{aligned}$$

Since $r \ge 0$, choose $+$ to that $\mathbf{n}\,S$ points upward. The surface integral is

$$\begin{aligned} &\int_0^{2\pi}\int_0^2 \langle ze^{x+y}-y, ze^{x+y}, -x\rangle \cdot \langle -r^2(5-r^2)^{-1/2}\cos\theta, r^2(5-r^2)^{-1/2}\sin\theta, r\rangle\,dr\,d\theta \\ = &\int_0^{2\pi}\int_0^2 \big(-(ze^{x+y}-y)r^2(5-r^2)^{-1/2}\cos\theta + ze^{x+y}r^2(5-r^2)^{-1/2}\sin\theta, -xr\big)\,dr\,d\theta \end{aligned}$$

Use (2) to rewrite x, y, z in terms of r, θ and with some effort, you can show that this double integral is zero.

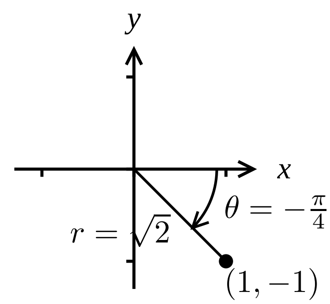
8(20 pts).(Source: 16.9.10) Calculate the divergence of $\mathbf{H}$:

$$\operatorname{div}\mathbf{H} = \nabla \cdot \langle x+y, y+z, x-z\rangle = (x+y)_x + (y+z)_y + (x-z)_z = 1+1-1 = 1.$$

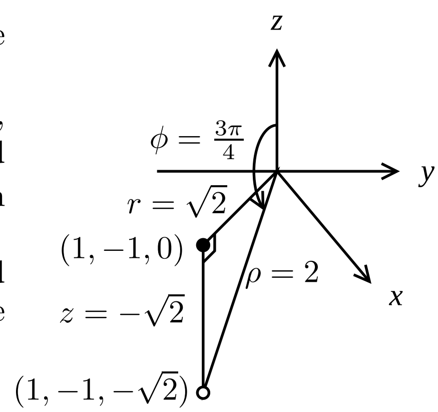
By the Divergence Theorem, the flux of $\mathbf{H}$ out of D equals the triple integral

$$\begin{aligned} \iiint_D \operatorname{div}\mathbf{H}\,dV &= \int_0^1\int_0^{1-x}\int_0^{2-x} 1\,dz\,dy\,dx \\ &= \int_0^1\int_0^{1-x} (2-x)\,dy\,dx \\ &= \int_0^1 (1-x)(2-x)\,dx \\ &= \int_0^1 (2-3x+x^2)\,dx = \left(2x - \tfrac{3}{2}x^2 + \tfrac{1}{3}x^3\right)\Big|_0^1 = 2 - \tfrac{3}{2} + \tfrac{1}{3} = \tfrac{5}{6}. \end{aligned}$$

9a(3 pts).(Source: 15.7.3) The cylindrical coordinates for (x, y, z) are (r, θ, z), where z is the same for both and r and θ are the polar coordinates for the point $(1, -1)$ in the xy-plane. $r = \sqrt{x^2 + y^2} = \sqrt{2}$. The ray from $(0, 0)$ to $(1, -1)$ makes an angle of $\frac{\pi}{4}$ radians with the positive x axis, and since $(1, -1)$ is in quadrant IV, take $\theta = -\frac{\pi}{4}$.
That is, the cylindrical coordinates are $r = \sqrt{2}$, $\theta = -\frac{\pi}{4}$, and $z = -\sqrt{2}$. (For other correct answers, add $n\pi$ to θ and multiply r by $(-1)^n$.)



9b(3 pts).(Source: 15.8.3) The spherical coordinates are (ρ, ϕ, θ), where θ is the same as in 9a.
$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{1 + 1 + 2} = 2$. Since $r = |z|$, both acute angles in the ρ-r-z right triangle equal $\frac{\pi}{4}$, and therefore the ray from $(0, 0, 0)$ to $(1, -1, \sqrt{2})$ makes an angle of $\phi = \frac{3\pi}{4}$ with the positive z-axis.
That is, the spherical coordinates are $\rho = 2$, $\phi = \frac{3\pi}{4}$ and $\theta = -\frac{\pi}{4}$. (For other correct answers, add any even multiple of π to θ.)



10(6 pts).(Source: 11.11.11) a8. b2. c7. d1. e6. f4.