No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Unless otherwise indicated, supporting work will be required on every problem worth more than 3 points.

You are expected to know the values of all trig functions at multiples of $\pi/4$ and of $\pi/6$.

1(26 pts). Find the following if $\mathbf{f} = \langle 1, -2, 2\rangle$ and $\mathbf{g} = \langle 2, -3, 6\rangle$.

a. $\mathbf{f} - 2\mathbf{g}$ b. $|-5\mathbf{f}|$ c. The angle between $\mathbf{f}$ and $\mathbf{g}$.

d. A unit vector in the same direction as $\mathbf{f}$.

e. A nonzero vector perpendicular to both $\mathbf{f}$ and $\mathbf{g}$.

f. The vector projection of $\mathbf{g}$ onto $\mathbf{f}$.

2. Find an equation (or equations) of the given line or plane.

a(7 pts). The plane passing through the points $(-1, 0, 1)$, $(0, -2, 3)$, $(1, -3, 7)$.

b(6 pts). The line passing through the point $(-1, 0, 1)$ and parallel to the line $x = 4t-1 \quad y = 2-t \quad z = 3+t$.

3(14 pts). Find the intersection point (x, y, z) or show that it does not exist.

a. The intersection point of the line $x = 4t - 1 \quad y = 2 - t \quad z = 3 + t$ and the plane $x + 3y = 8 + z$.

b. The intersection point of the lines

$$\begin{aligned} x &= -3 - 2t & y &= 7 + t & z &= -7 - 3t \\ x &= -5 + 4s & y &= 8 - 2s & z &= -2 + 2s \end{aligned}$$

4(12 pts). Sketch the surface in (x, y, z)-space given by each equation. Label your axes x, y, z and use arrows to indicate the positive direction along each. Label your answers so I can tell which is which.

a. $x^2 = y$ b. $x^2 - y^2 = z^2$

5(13 pts). Find parametric equations of the line tangent to the curve given by

$$x = t\sin(\pi t) \qquad y = \ln(t+1) \qquad z = \frac{t}{t+1}$$

at the point $(0, \ln 2, \frac{1}{2})$.

6(8 pts). Evaluate the indefinite integral $\displaystyle\int \left(\csc^2 t\,\mathbf{i} + (t-1)(3t+1)\mathbf{j} + \frac{e^t}{e^t+2}\mathbf{k}\right) dt$.

7(8 pts). Find the center and radius of the sphere $x^2 + y^2 + z^2 - 4x + 8z = -4$.

8(6 pts). Find the graph of given vector-valued function.

a. $\langle 2-3t, 2t+1, 4t\rangle$

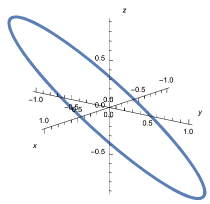
b. $\langle \sin(10t), \cos(10t), t\rangle$

c. $\langle 2-t^2, t, t^2-1\rangle$

d. $\langle \sin t, \cos t, \frac{1}{2}\sin t - \frac{3}{4}\cos t\rangle$

e. $\langle \sqrt{t}\sin t, \sqrt{t}\cos t, \sqrt{t}\rangle$

f. $\langle \sin t, \cos t, \sin(4t)\rangle$



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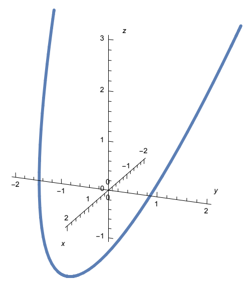

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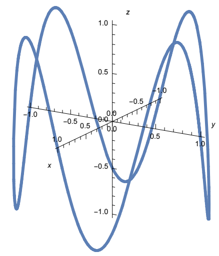

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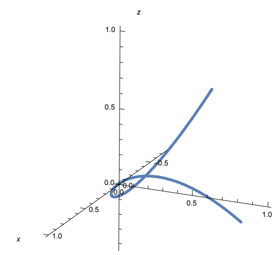

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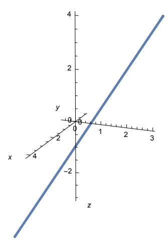

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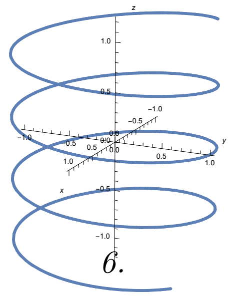

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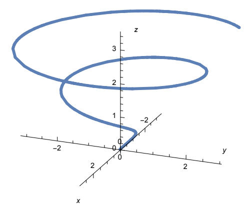

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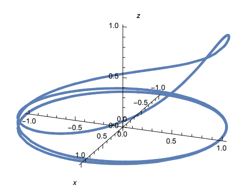

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1a(2 pts).(Source: 12.2.6,19-22) $\langle 1,-2,2\rangle - 2\langle 2,-3,6\rangle = \langle 1,-2,2\rangle - \langle 4,-6,12\rangle = \langle -3,4,-10\rangle$.

1b(3 pts).(Source: 12.2.6,19-22) $|-5\mathbf{f}| = |-5||\mathbf{f}| = 5\sqrt{1^2+(-2)^2+2^2} = 5\sqrt{9} = 15$.

Alternate solution: $|-5\mathbf{f}| = |\langle 5,-10,10\rangle| = \sqrt{5^2+(-10)^2+10^2} = \sqrt{225} = 15$.

1c(9 pts).(Source: 12.3.17-18) If we call the angle in question θ, then $\mathbf{f}\cdot\mathbf{g} = |\mathbf{f}||\mathbf{g}|\cos\theta$. Calculate the dot product and the two magnitudes:

$$\mathbf{f}\cdot\mathbf{g} = 1\cdot 2 + (-2)(-3) + 2\cdot 6 = 20, \quad |\mathbf{g}| = \sqrt{2^2+(-3)^2+6^2} = 7$$

and $|\mathbf{f}| = 3$, as found in b. Therefore $\cos\theta = 20/(3\cdot 7)$, and $\theta = \cos^{-1}(\frac{20}{21})$.

1d(4 pts).(Source: 12.2.33-35) Normalize $\mathbf{f}$ to obtain $\frac{1}{|\mathbf{f}|}\mathbf{f} = \frac{1}{3}\langle 1,-2,2\rangle$, or $\langle \frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\rangle$.

1e(13 pts).(Source: 12.4.1-7) $\mathbf{f}$ and $\mathbf{g}$ are both orthogonal to their cross product:

$$\begin{aligned}
\mathbf{f}\times\mathbf{g} = \langle 1,-2,2\rangle \times \langle 2,-3,6\rangle &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 2 \\ 2 & -3 & 6 \end{vmatrix} \\
&= \mathbf{i}\begin{vmatrix} -2 & 2 \\ -3 & 6 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 & 2 \\ 2 & 6 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & -2 \\ 2 & -3 \end{vmatrix} \\
&= \big((-2)6 - (-3)2\big)\mathbf{i} - \text{etc.} \\
&= -6\mathbf{i} - 2\mathbf{j} + 1\mathbf{k}.
\end{aligned}$$

1f(6 pts).(Source: 12.3.39-44) The projection is

$$\left(\frac{\mathbf{g}\cdot\mathbf{f}}{\mathbf{g}\cdot\mathbf{f}}\right)\mathbf{f} = \frac{20}{9}\langle 1,-2,2\rangle, \text{ or } \left\langle \frac{20}{9}, -\frac{40}{9}, \frac{40}{9}\right\rangle.$$

2a(7 pts).(Source: 12.5.31-34) The vectors $\langle -1,0,1\rangle - \langle 0,-2,3\rangle = \langle 1,-2,2\rangle$ and $\langle -1,0,1\rangle - \langle 1,-3,7\rangle = \langle 2,-3,6\rangle$ must lie in the plane, so their cross product (found in 1) is orthogonal to the plane. See 7 p. 827. The plane is given by the equation $-6(x+1) - 2y1(z-1) = 0$, or $-6x - 2y + z = 7$.

2b(6 pts).(Source: 12.5.4) See 2 p. 824. The given line is parallel to the vector $\langle 4,-1,1\rangle$, so the line in question is givein parametrically by $x = 4t-1 \quad y = -t \quad z = 1+t$, in vector form by $\langle -1,0,1\rangle + t\langle 4,-1,1\rangle$, and in symmetric form by

$$\frac{x+1}{4} = -y = z-1.$$

Any of these three forms is sufficient for full credit.

3a(5 pts).(Source: 12.5.45-47) Substitute $x = 4t-1$, $y = 2-t$, $z = 3+t$ into the equation of the plane to obtain $4t - 1 + 3(2-t) = 8 + 3 + t$, or $t + 5 = 11 + t$. This equation has no solution, so the line and plane do not intersect.

3b(9 pts).(Source: 12.5.19-20) Set the coordinates equal and solve for s and t:

$$(1)\qquad \begin{aligned} -3-2t &= -5+4s \\ 7+t &= 8-2s \\ -7-3t &= -2+2s \end{aligned} \quad \Rightarrow \quad \begin{aligned} 4s+2t &= 2 \\ 2s+t &= 1 \\ 2s+3t &= -5 \end{aligned}$$

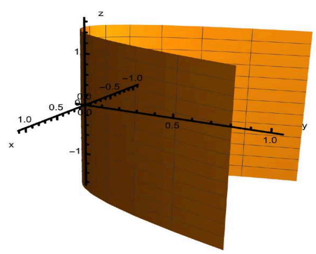
The 1st and 2nd equations of (1) are **equivalent**: any s and t that satisfies one of these equations satisfies both. From the second, we learn that $t = 1-2s$. Substitute this into the third to obtain

$$2s+3(1-2s) = -5 \quad\Longrightarrow\quad -4s = -8 \quad\Longrightarrow\quad s = 2 \quad\Longrightarrow\quad t = 1-2\cdot 2 = -3.$$

Substituting either of these into the equations of its corresponding line gives the point $(x, y, z) = (3, 4, 2)$.

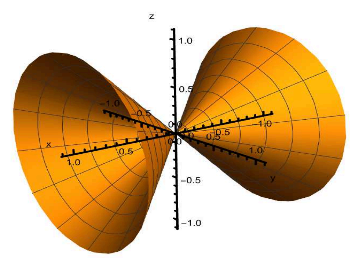
4a(4 pts).(Source: 12.6.5-6) $y = x^2$ is a cylinder obtained by taking the parabola $y = x^2$ in the xy-plane and dragging backwards and forwards in the z-direction.

Here's a nice drawing by Mathematica. The positive z, x, and y directions in this figure are up, down-and-left, and (slightly) down-and-right. There's an interactive graph at https://www.desmos.com/3d/81009b1adc



4b(8 pts).(Source: 12.6.13,18) The graph of $x^2 = y^2 + z^2$ is a cone. Cross sections at constant x-values are circles centered at $(x, 0, 0)$, so the surface is symmetric about the x-axis.

Here it is in Mathematica. The positive z, x, and y directions in this figure are up, down-and-left, and down-and-right. See the interactive graph at https://www.desmos.com/3d/907f6f5a3a



5(13 pts).(Source: 13.2.23-26) To find an equation of the line, we need a point on the line and a parallel vector. Set $\mathbf{r} = \langle t\sin(\pi t), \ln(t+1), \frac{t}{t+1}\rangle$, which equals $\langle 0, \ln 2, \frac{1}{2}\rangle$ at $t = 1$. For the parallel vector, use $\frac{d\mathbf{r}}{dt}$ at $t = 1$.
By the product, chain, and quotient rules, $\frac{d\mathbf{r}}{dt} = \langle \sin(\pi t) + \pi t\cos(\pi t), \frac{1}{t+1}, \frac{1}{(t+1)^2}\rangle$. When $t = 1$, $\frac{d\mathbf{r}}{dt} = \langle -\pi, \frac{1}{2}, \frac{1}{4}\rangle$. Therefore, the tangent line is parametrized by $\langle 0, \ln 2, \frac{1}{2}\rangle + t\langle -\pi, \frac{1}{2}, \frac{1}{4}\rangle$, or, if you prefer,

$$x = -\pi t, \qquad y = \ln 2 + \frac{1}{2}t, \qquad z = \frac{1}{2} + \frac{1}{4}t.$$

6(8 pts).(Source: 13.3.35-40) $\int \csc^2 t\, dt = -\cot t + C_1$. To calculate the second integral, integration by parts is unnecessary. Expand the polynomial and integrate:
$\int (t-1)(3t+1)\, dt = \int (3t^2 - 2t - 1)\, dt = t^3 - t^2 - t + C_2$.
If we substitute $u = e^t + 2$, then $\int \frac{e^t}{e^t+2} = \int \frac{du}{u} = \ln|u| + C_3 = \ln(e^t + 2) + C_3$.
Therefore, the indefinite integral is $-(\cot t)\mathbf{i} + (t^3 - t^2 - t)\mathbf{j} + \ln(e^t + 2)\mathbf{k} + \mathbf{C}$ (where $\mathbf{C}$ is a constant (vector) of integration.

7(8 pts).(Source: 12.1.17-20) Complete the square:

$$x^2 - 4x + 4 + y^2 + z^2 + 8z + 16 = -4 + 4 + 16$$
$$(x-2)^2 + y^2 + (z+4)^2 = 16$$

The center is $(2, 0, -4)$, and the radius is $\sqrt{16} = 4$.

8(6 pts).(Source: 13.1.21-26) a5, b6, c2, d1, e7, f3.