No notes, books, electronic devices, or outside materials of any kind.

Read each problem carefully and simplify your answers.

Unless otherwise indicated, supporting work will be required on every problem worth more than 2 points.

You are expected to know the values of all trig functions at multiples of $\pi/4$ and of $\pi/6$.

$$\int \sin^n x\,dx = -\frac{1}{n}\sin^{n-1} x\cos x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$$
$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$
$$\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\,dx$$
$$\int \sec^n x\,dx = \frac{1}{n-1}\sec^{n-2} x\tan x + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx \qquad (n \neq 1)$$

1(20 pts). Find the minimum value of the function $Q(x, y, z) = x^2 + 2y^2 + 2z^2$ subject to the constraint $2x + y - z = 20$ and the point where this minimum occurs. ($Q(x, y, z)$ has no maximum subject to this constraint.)

2a(5 pts). Find cylindrical coordinates of the point $(x, y, z) = (\sqrt{2}, \sqrt{2}, -2\sqrt{3})$.

$r =$ ________ $\theta =$ ________ $z =$ ________

2b(5 pts). Find spherical coordinates of the point $(x, y, z) = (\sqrt{2}, \sqrt{2}, -2\sqrt{3})$.

$\rho =$ ________ $\phi =$ ________ $\theta =$ ________

3(15 pts). Rewrite the iterated integral to the order indicated, but **do not evaluate**.

a. $\int_0^2 \int_{3y}^6 dx\,dy = \int_?^? \int_?^? dy\,dx$ b. $\int_0^3 \int_0^{3-x} \int_0^{9-x^2} dz\,dy\,dx = \int_?^? \int_?^? \int_?^? dy\,dx\,dz$

4(9 pts). Evaluate the double integral $\iint_{[-1,1]\times[0,1]} (x + y)\,dA$.

5(10 pts). Find the surface area of the graph of $z = x^2 + 3y$ above the triangle in the $xy$-plane with vertices $(0, 0)$, $(6, 0)$ and $(6, 2)$. Express your answer as an iterated integral, but **do not evaluate**.

6(16 pts). Find the volume of the region inside both the cylinder $x^2 + y^2 = 9$ and the ellipsoid $4x^2 + 4y^2 + z^2 = 64$.

7(20 pts). Evaluate the triple integral $\iiint_D \sqrt{x^2 + y^2 + z^2}\,dV$ where $D$ is the region above the cone $z = \sqrt{x^2 + y^2}$ and inside the sphere $x^2 + y^2 + z^2 = 4$.

1(21 pts).(Source: 14.8.7) By the method of Lagrange multipliers, the minimum of $Q$ subject to $G(x,y,z) = 2x+y-z = 20$ must occur at the points on $G = 20$ at which $\nabla Q \times \nabla G = \mathbf{0}$:

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2x & 4y & 4z \\ 2 & 1 & -1 \end{vmatrix} = 2\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & 2y & 2z \\ 2 & 1 & -1 \end{vmatrix} = 2\langle -2y-2z, x+4z, x-4y\rangle = \mathbf{0}$$

By setting the first and second components equal to 0, find $y = -z$ and $x = -4z$. Substitute these into the constraint:

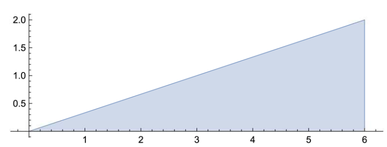
$$2x + y - z = -8z - z - z = 20 \implies z = -2,$$

which implies that $y = 2$ and $x = 8$. Since this is the only critical point, the minimum must be $Q(8, 2, -2) = 8^2 + 2 \cdot 2^2 + 2 \cdot (-2)^2 = 80$.

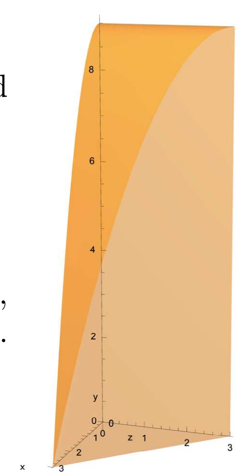
2a(3 pts).(Source: 15.7.3,4) $r = \sqrt{x^2+y^2} = 2$. The ray from the origin to the point $(\sqrt{2}, \sqrt{2})$ in the $xy$-plane make an angle $\frac{\pi}{4}$ with the positive $x$-axis, so $\theta = \frac{\pi}{4}$. $z$ in cylindrical coordinates is the same as $z$ in rectangular coordinates, so $z = -2\sqrt{3}$.

2b(3 pts).(Source: 15.8.3,4) $\rho = \sqrt{x^2+y^2+z^2} = 4$. The right triangle with sides $\rho = 4$, $r = 2$, and $z = -2\sqrt{3}$ is a $\frac{\pi}{6}$-$\frac{\pi}{3}$-$\frac{\pi}{2}$ with the $\frac{\pi}{3}$ angle between sides 2 and 4. $\phi = \frac{\pi}{2} + \frac{\pi}{3} = \frac{5\pi}{6}$.

3a(5 pts).(Source: 15.2.51) See figure for this region in the plane. Integral equals $\int_0^6 \int_0^{x/3} dy\, dx$




3b(8 pts).(Source: 15.6.28,31) See figure for this region in space. Integral is $\int_0^9 \int_0^{\sqrt{9-z}} \int_0^{3-x} dy\, dx\, dz$.




4(7 pts).(Source: 15.1.16) Evaluate the double integral by iterated integration: $\int_{-1}^1 \int_0^1 (x+y)\, dy\, dx = \int_{-1}^1 (xy + \frac{1}{2}y^2)\Big|_0^1 dx$

$$= \int_{-1}^1 (x + \tfrac{1}{2})\, dx = (\tfrac{1}{2}x^2 + \tfrac{1}{2}x)\Big|_{-1}^1 = 1.$$

5(11 pts).(Source: 15.5.3,4) The sides of the triangle are $y = 0$, $x = 6$, and $y = 3x$, the same as the region of integration in problem 3a. The surface area is the integral of

$$dS = \sqrt{1 + z_x^2 + z_y^2}\, dx\, dy = \sqrt{1 + (2x)^2 + 3^2}\, dx\, dy$$

over the triangle. This can be written either of two ways:

$$\int_0^2 \int_{3y}^6 \sqrt{4x^2+10}\, dx\, dy = \int_0^6 \int_0^{x/3} \sqrt{4x^2+10}\, dy\, dx.$$

6(15 pts).(Source: 15.3.27,15.7.24) The ellipsoid is symmetric across the $xy$-plane, so the volume is double the volume above that plane. You can write the volume as either a triple integral in cylindrical coordinates ($dV = r\,dz\,dr\,d\theta$):

$$2\int_0^{2\pi}\int_0^3\int_0^{\sqrt{64-4r^2}} dz\,r\,dr\,d\theta$$

or a double integral in polar coordinates ($dA = r\,dr\,d\theta$):

$$2\int_0^{2\pi}\int_0^3 r(\sqrt{64-4r^2})\,dr\,d\theta.$$

After one integration, the triple integral becomes exactly this double integral. Proceeding,

$$2\int_0^{2\pi}\left(-\frac{1}{8}\cdot\frac{2}{3}(64-4r^2)^{3/2}\Big|_0^3\right)d\theta = \frac{\pi}{3}(64^{3/2}-28^{3/2}),$$

or, $\frac{\pi}{3}(512-28^{3/2}) = \frac{8\pi}{3}(64-t^{3/2})$.

7(19 pts).(Source: 15.8.25,26) In spherical coordinates, the cone is $\phi = \frac{\pi}{4}$, the sphere is $\rho = 2$, the integrand is $\rho$, and $dV = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$, and so the triple integral is

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^2 \rho^3\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}d\theta\int_0^{\pi/4}\sin\phi\,d\phi\int_0^2\rho^3\,d\rho$$
$$= 2\pi\cdot\left(-\cos\phi\Big|_0^{\pi/4}\right)\cdot\left(\frac{1}{4}\rho^4\Big|_0^2\right) = 2\pi\left(1-\frac{1}{\sqrt{2}}\right)\frac{1}{4}2^4,$$

or $4\pi(2-\sqrt{2})$.

The integral is prohibitively difficult using cylindrical coordinates. Since the cone and sphere intersect at $x^2+y^2=2$, the integral starts as

$$\int_0^{2\pi}\int_0^{\sqrt{2}}\int_r^{\sqrt{4-r^2}} r\sqrt{r^2+z^2}\,dz\,dr\,d\theta$$