MATH 221–02 (Kunkle), Quiz 3 Name: ______________________

10 pts, Take-home Due by 5:00pm Friday Feb 9, 2024

You have my permission to work with whoever and use whatever resources you want on this quiz, but you must write complete solutions. Answers only will not receive full credit. You do not need to turn in a printout of this quiz.

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1 (10 pts). Let the $\mathbf{r}(t) = \langle t, t^2, \frac{2}{3}t^3 \rangle$ be the position of a particle at time t.

a. Express the particle's velocity, acceleration, and speed as functions of t. Simplify to show that speed is a polynomial function of t.

b. Find curvature κ of the particle's path at time t.

c. Find the tangential and normal components of acceleration at time $t = -1$.

d. At what point(s) (x, y, z) is the particle's normal plane parallel to $x + 4y + 8z = 3$?

e. At what time t is the particle's osculating plane parallel to $x + 4y + 8z = 3$?

Solution: 1a.(Source: 13.4.9) $\mathbf{v} = \frac{d\mathbf{r}}{dt} = \langle 1, 2t, 2t^2 \rangle$. $\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = \langle 0, 2, 4t \rangle$.
Speed $= \frac{ds}{dt} = |\mathbf{v}| = \sqrt{1 + (2t)^2 + (2t^2)^2} = \sqrt{1 + 4t^2 + 4t^4} = \sqrt{(1 + 2t^2)^2} = 1 + 2t^2$.

b.(Source: 13.3.23) $\mathbf{v} \times \mathbf{a} = \langle 1, 2t, 2t^2 \rangle \times \langle 0, 2, 4t \rangle = 2\langle 2t^2, -2t, 1 \rangle$. Then

$$\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3} = \frac{2|\langle 2t^2, -2t, 1 \rangle|}{|\langle 1, 2t, 2t^2 \rangle|^3} = \frac{2}{|\mathbf{v}|^2} = \frac{2}{(1 + 2t^2)^2}$$

1c.(Source: 13.4.40) Solution one. $a_T = \frac{d^2s}{dt^2} = 4t$, which equals -4 at $t = -1$. $a_N = \sqrt{|\mathbf{a}|^2 - a_t^2} = \sqrt{2^2 + (4t)^2 - (4t)^2} = 2$ for all t.
Solution two. Using the above, at $t = -1$

$$\mathbf{v} = \langle 1, -2, 2 \rangle \qquad \mathbf{a} = \langle 0, 2, -4 \rangle \qquad \mathbf{v} \times \mathbf{a} = 2\langle 2, 2, 1 \rangle$$

Then $a_T = \frac{\mathbf{v} \cdot \mathbf{a}}{|\mathbf{v}|} = \frac{-12}{3} = -4$, and $a_N = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|} = \frac{2|\langle 2,2,1 \rangle|}{|\langle 1,-2,2 \rangle|} = 2$

1d.(Source: 13.3.53) The normal vector to the normal plane is $\mathbf{v}$, so seek times t at which $\mathbf{v}$ is parallel the normal vector of the given plane, that is, a scalar multiple of $\langle 1, 4, 8 \rangle$:

$$\langle 1, 2t, 2t^2 \rangle = \langle c, 4c, 8c \rangle \Rightarrow 1 = c, \Rightarrow t = 2.$$

At this time $(x, y, z) = \mathbf{r}(2) = (2, 4, \frac{16}{3})$.

e.(Source: 13.3.53) The normal vector to the osculating plane is $\mathbf{v} \times \mathbf{a}$, so seek times t at which $\mathbf{v} \times \mathbf{a}$ (or more simply, $\frac{1}{2}\mathbf{v} \times \mathbf{a}$) is parallel the normal vector of the given plane:

$$\langle 2t^2, -2t, 1 \rangle = \langle c, 4c, 8c \rangle \Rightarrow c = \frac{1}{8}, \Rightarrow t = -\frac{1}{4}.$$