MATH 221–02 (Kunkle), Quiz 6 | Name: ____________________
10 pts, 10 minutes | Mar 14, 2024

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1 (10 pts). Evaluate $\iint_D x^2 y\, dA$ where D is quarter disk $x^2+y^2 \le 1$ in the first quadrant $x \ge 0$, $y \ge 0$.

1.(Source: 15.2.20,15.3.7) In rectangular coordinates:

The quarter circle in the first quadrant can be written either $y = \sqrt{1-x^2}$ or $x = \sqrt{1-y^2}$. Here are solutions in both orders of integration.

Solution one: the integral equals

$$\int_0^1 \int_0^{\sqrt{1-x^2}} x^2 y\, dy\, dx = \int_0^1 \tfrac{1}{2} x^2 y^2 \Big|_0^{\sqrt{1-x^2}} dx = \int_0^1 \tfrac{1}{2} x^2 (1-x^2)\, dx$$
$$= \tfrac{1}{2} \int_0^1 (x^2 - x^4)\, dx = \tfrac{1}{2}(\tfrac{1}{3}x^3 - \tfrac{1}{5}x^5)\Big|_0^1 = \tfrac{1}{2}(\tfrac{1}{3} - \tfrac{1}{5}) = \tfrac{1}{15}.$$

(done)

Solution two: the integral equals

$$\int_0^1 \int_0^{\sqrt{1-y^2}} x^2 y\, dx\, dy = \int_0^1 \tfrac{1}{3} x^3 y \Big|_0^{\sqrt{1-y^2}} dy = \tfrac{1}{3} \int_0^1 (1-y^2)^{3/2} y\, dy$$

At this point, you could let $u = 1-y^2$, so that $-\frac{1}{2}\, du = y\, dy$. Change limits as well: $u = 1$ when $y = 0$, and $u = 0$ when $y = 1$. Integral becomes

$$-\tfrac{1}{2} \cdot \tfrac{1}{3} \int_1^0 (u)^{3/2}\, du = -\tfrac{1}{2} \cdot \tfrac{1}{3} \cdot \tfrac{2}{5} (u)^{5/2} \Big|_1^0 = -\tfrac{1}{15}(0-1) = \tfrac{1}{15}.$$

(done)

We can also calculate this integral using polar coordinates:

The quarter disk in the first quadrant is described in polar coordinates by $0 \le \theta \le \frac{\pi}{2}$, $0 \le r \le 1$. Rewrite the integrand using $x = r\cos\theta$ and $y = r\sin\theta$:

$$\int_0^{\pi/2} \int_0^1 (r\cos\theta)^2 (r\sin\theta) r\, dr\, d\theta = \int_0^{\pi/2} \int_0^1 r^4 \cos^2\theta \sin\theta\, dr\, d\theta$$
$$= \int_0^{\pi/2} \tfrac{1}{5} r^5 \Big|_0^1 \cos^2\theta \sin\theta\, dr\, d\theta = \tfrac{1}{5} \int_0^{\pi/2} \cos^2\theta \sin\theta\, dr\, d\theta$$

Now substitue $u = \cos\theta$, $du = -\sin\theta\, d\theta$ and change the limits:

$$= \tfrac{1}{5} \int_1^0 u^2\, (-du) = \tfrac{1}{5} \int_0^1 u^2\, du = \tfrac{1}{5} \cdot \tfrac{1}{3} u^3 \Big|_0^1 = \tfrac{1}{15}.$$

(done)