MATH 221–02 (Kunkle), Quiz 9 Name: ____________________

10 pts, 10 minutes April 18, 2024

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1 (10 pts). Calculate $\iint_G x\,dS$, where G be the surface parametrized by $\langle u\sin v, u\cos v, v\rangle$ for $0 \le u \le 1$ and $0 \le v \le \frac{\pi}{2}$.

1a.(Source: 16.7.7) Let $\mathbf{r} = \langle u\sin v, u\cos v, v\rangle$. Then

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sin v & \cos v & 0 \\ u\cos v & -u\sin v & 1 \end{vmatrix} = \langle \cos v, -\sin v, -u\sin^2 v - u\cos^2 v\rangle$$
$$= \langle \cos v, -\sin v, -u\rangle$$

and

$$dS = |\mathbf{r}_u \times \mathbf{r}_v|\,du\,dv = \sqrt{\cos^2 v + \sin^2 v + u^2}\,du\,dv$$
$$= \sqrt{1+u^2}\,du\,dv$$

and so the integral is

$$\int_0^{\pi/2}\int_0^1 x\sqrt{1+u^2}\,du\,dv = \int_0^{\pi/2}\int_0^1 u\sin v\sqrt{1+u^2}\,du\,dv$$
$$= \int_0^{\pi/2}\sin v\,dv \int_0^1 u(1+u^2)^{1/2}\,du$$
$$= \left(-\cos v\,\Big|_0^{\pi/2}\right)\left(\frac{1}{3}(1+u^2)^{3/2}\,\Big|_0^1\right)$$
$$= \frac{1}{3}(2^{3/2}-1)$$

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1 (10 pts). Let $\mathbf{F} = \langle x, y, e^z \rangle$ and G be the surface parametrized by $\langle u \sin v, u \cos v, v \rangle$ for $0 \le u \le 1$ and $0 \le v \le \frac{\pi}{2}$, oriented upwards. Calculate the flux of $\mathbf{F}$ across G:

$$\iint_G \mathbf{F} \cdot \mathbf{n} \, dS$$

1a.(Source: 16.7.7)

Let $\mathbf{r} = \langle u \sin v, u \cos v, v \rangle$. Then $\mathbf{n} \, dS = \pm \mathbf{r}_u \times \mathbf{r}_v \, du \, dv$, whichever has a nonnegative third component.

$$\begin{aligned} \mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sin v & \cos v & 0 \\ u \cos v & -u \sin v & 1 \end{vmatrix} &= \langle \cos v, -\sin v, -u \sin^2 v - u \cos^2 v \rangle \\ &= \langle \cos v, -\sin v, -u \rangle \end{aligned}$$

and observe that the $\mathbf{k} \cdot (\mathbf{r}_u \times \mathbf{r}_v) = -u \le 0$ for the given range of u. Therefore

$$\mathbf{n} \, dS = -\mathbf{r}_u \times \mathbf{r}_v \, du \, dv$$

and the flux is is

$$\begin{aligned} &\int_0^{\pi/2} \int_0^1 \langle x, y, e^z \rangle \cdot \langle -\cos v, \sin v, u \rangle \, du \, dv \\ &= \int_0^{\pi/2} \int_0^1 \langle u \sin v, u \cos v, e^v \rangle \cdot \langle -\cos v, \sin v, u \rangle \, du \, dv \\ &= \int_0^{\pi/2} \int_0^1 u e^v \, du \, dv \\ &= \int_0^{\pi/2} e^v \, dv \int_0^1 u \, du \\ &= \left(e^v \Big|_0^{\pi/2} \right) \left(\frac{1}{2} u^2 \Big|_0^1 \right) \\ &= \frac{1}{2} (e^{\pi/2} - 1) \end{aligned}$$