A Flat Extension Theorem via Ideal Projection

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Abstract

Any linear functional on the space of polynomials Π is determined by its values at the monomials, or its **moments**, and the **flat extension problem** has to do with whether a functional with finitely many prescribed moments can be extended to Π without increasing the rank of the moment matrix. We obtain a generalization of the the flat extension theorems of Curto and Fialkow and Laurent and Mourrain by looking at the problem as one of ideal projection.

Definitions

Let $\Pi := \mathbb{K}[x_1, x_2, \ldots, x_k]$, the set of polynomials in k variables over the field $\mathbb{K}$, and Π_n be the set of polynomials of total degree $\le n$.
The typical element p of Π is

$$p = \sum_\alpha \hat{p}(\alpha) x^\alpha$$

For any $a \in \mathbb{K}^k$, let $\delta_a \in \Pi'$ denote the functional: $\delta_a p = p(a)$.
A functional $L \in \Pi'$ given by the rule

$$Lp := \sum_\alpha \hat{p}(\alpha) y_\alpha$$

is completely determined by its **moments** $\{y_\alpha\}_{\alpha \in \mathbb{Z}_+^k}$.
The **moment matrix** of L

$$M := [y_{\alpha+\beta}]_{\alpha,\beta \in \mathbb{Z}_+^k}$$

satisfies $L(pq) = \hat{p} M \hat{q}$.

Moment matrices of finite rank

Curto and Fialkow [2] and later Laurent and Mourrain [4] address the (multivariate)

Flat Extension Problem:

When can a truncated moment matrix

$$M_B := [y_{\alpha+\beta}]_{\alpha,\beta \in B}$$

(for some finite $B \subset \mathbb{Z}_+^k$) be extended to M without increasing its rank?

For corresponding univariate results, see Iokhvidov [3].

If M and its submatrix M_B have the same rank, we write

$$M_B \overset{\flat}{\to} M.$$

When $\mathbb{K} = \mathbb{C}$, $\operatorname{rank} M < \infty$ iff L is finitely atomic:

$$L = \sum_{a \in U} \delta_a q_a(D)$$

for some finite set U and polynomials $q_a(x)$.

Flat Extension & Ideal projection

In order that the moment matrix M be a flat extension of M_B,

$$M = P' M_B P \quad \text{and} \quad L = LP$$

for a projector P onto B.
The dependence relations among the columns of M are simultaneously recurrence relations on the moments of L:

$$\ker M = \{p \in \Pi : L(p\Pi) = 0\} = \{p \in \Pi : p(\Delta)y = 0\}$$

If M_B is invertible, then $\ker M = \ker P$, and so P is an **ideal projector**, a finite-rank projector whose kernel is a ideal.
For an introduction to ideal projection see de Boor [1]. A fundamental property of any ideal projector P is that

$$Pp = p(X)(P1) \quad \forall p \in \Pi$$

where $X^\alpha := \prod_{i=1}^k X_i^{\alpha(i)}$, a product of the commuting operators $X_i : B \to B : b \mapsto P(()^{e_i} b)$.
Consequently, an ideal projector is determined by its restriction N to

$$\Pi_0 + \Pi_1 \cdot B =: B^\star,$$

and any flat extension $M_B \overset{\flat}{\to} M$ is uniquely determined by the extension $M_B \overset{\flat}{\to} M_{B^\star}$.

Examples

Because the moment matrix must be a Hankel* matrix, flat extension sometimes isn't possible:

$$\begin{array}{c} \\ 1 \\ x \end{array}\!\!\begin{array}{c} \begin{array}{cc} 1 & x \end{array} \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{array} \overset{\flat}{\not\to} \begin{array}{c} \\ 1 \\ x \\ x^2 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 1 & x & x^2 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & \blacksquare \\ 1 & \blacksquare & \blacksquare \end{pmatrix} \end{array}.$$

rank = 1 rank = 3

* $M_{\alpha,\beta} = M_{\alpha',\beta'}$ if $\alpha + \beta = \alpha' + \beta'$.

When it is, as in

$$M_B = \begin{array}{c} \\ 1 \end{array}\!\!\begin{array}{c} 1 \\ \begin{pmatrix} 1 \end{pmatrix} \end{array} \overset{\flat}{\to} \begin{array}{c} \\ 1 \\ x \\ x^2 \\ x^3 \\ \vdots \end{array}\!\!\begin{array}{c} \begin{array}{ccccc} 1 & x & x^2 & x^3 & \cdots \end{array} \\ \begin{pmatrix} 1 & a & a^2 & a^3 & \cdots \\ a & a^2 & a^3 & a^4 & \cdots \\ a^2 & a^3 & a^4 & a^5 & \cdots \\ a^3 & a^4 & a^5 & a^6 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \end{array} = M$$

(in which case $L = \delta_a$) it's determined entirely by the initial extension

$$M_B \overset{\flat}{\to} M_{B^\star} = \begin{array}{c} \\ 1 \\ x \end{array}\!\!\begin{array}{c} \begin{array}{cc} 1 & x \end{array} \\ \begin{pmatrix} 1 & a \\ a & a^2 \end{pmatrix} \end{array}.$$

One of the main results of Curto and Fialkow and later Laurent and Mourrain has to do with whether a flat extension $M_C \overset{\flat}{\to} M_{C^\star}$ is the restriction of a flat extension $M_C \overset{\flat}{\to} M$. Our main result has been to replace earlier sufficient conditions with a necessary and sufficient condition.

Flat Extension Theorems:

In the field $\mathbb{F}$, if $M_B \overset{\flat}{\to} M_C \overset{\flat}{\to} M_{C^\star}$, where M_B is invertible, then $M_{C^\star}$ has a flat extension M …

[2] **if** $C = \Pi_n$ for some n and $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$.
[4] **if** C is a monomial space connected to 1.
[K] **if and only if** $Gp = p(X)N1$ for all $p \in C^\star$

where $N : B^\star \to B$ and $G : C^\star \to B$ are the projectors associated to the flat extensions $M_B \overset{\flat}{\to} M_{B^\star}$ and $M_B \overset{\flat}{\to} M_{C^\star}$.

Interpretation in terms of recurrences

To give a simple example, suppose that

$$C = B = \operatorname{span}\{x^{1,0}, x^{0,1}\},$$

so that

$$B^\star = \operatorname{span}\{x^{0,0}, x^{1,0}, x^{0,1}, x^{2,0}, x^{1,1}, x^{0,2}\}$$

and suppose $M_B \overset{\flat}{\to} M_{B^\star}$. The dependence relations of the columns of $M_{B^\star}$ mean that

$$\begin{pmatrix} y_{0,0} \\ y_{2,0} \\ y_{1,1} \\ y_{0,2} \end{pmatrix} = A \begin{pmatrix} y_{1,0} \\ y_{0,1} \end{pmatrix}$$

for some matrix A. If we can extend $M_{B^\star} \overset{\flat}{\to} M$, it can only be by the same relations,

$$\begin{pmatrix} y_{i,j} \\ y_{i+2,j} \\ y_{i+1,j+1} \\ y_{i,j+2} \end{pmatrix} = A \begin{pmatrix} y_{i+1,j} \\ y_{i,j+1} \end{pmatrix}$$

for all i, j. But are these recurrence relations consistent? The flat extension theorem promises that they are consistent for all $y_{i,j}$ if and only if they are consistent in the values they prescribe for

$$y_{0,0}, y_{1,0}, y_{0,1}, y_{2,0}, y_{1,1}, y_{0,2}$$

References

[1] C. de Boor. Ideal interpolation. In C.K. Chui, M. Neamtu, and L. Schumaker, editors, *Approximation Theory XI: Gatlinburg 2004*, pages 59–91, Brentwood TN, 2005. Nashboro Press.

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