Box-like Splines With Nonuniform Stepsize

Thomas Kunkle

Abstract. We construct a polynomial spline similar to the box spline, except that we relax the condition that its knots are spaced uniformly in each direction. The recurrence relation for this spline is a generalization of the recurrence relations for the box spline and univariate B-spline. The construction generalizes to exponential-polynomial splines [5].

§1 Introduction

The box spline [1] is a well-known generalization of the univariate cardinal B-spline. In this note, we present a spline that likewise generalizes the univariate B-spline with arbitrarily spaced (and possibly multiple) knots. Its recurrence relation is an extension of the formulas for both the box spline and univariate B-spline. This work is the necessary first step towards a generalization of the exponential box spline [7]; in an upcoming paper [5], we investigate a space of such exponential box-like splines, its approximation order, and the linear independence of a spanning set.

The spline discussed here arises naturally in the study of multivariate divided differences [4]. For a thorough introduction to the box spline and its history, see the recent book [2].

After introducing some notation in Section 2, we define the generalized box spline and prove its derivative and recurrence formulas in Section 3.

§2 Notation

To borrow notation from the box spline literature [2], we shall use the letter N to mean (1) a set of distinct vectors in $\mathbb{R}^d$, (2) a matrix in $\mathbb{R}^{d\times\#\mathrm{N}}$, and (3) the map

$$\mathrm{N} : \mathbb{R}^{\mathrm{N}} := \{x : \mathrm{N} \to \mathbb{R}\} \to \mathbb{R} : x \mapsto \mathrm{N}x := \sum_{\mathrm{N}} x(\nu)\nu.$$

Approximation Theory VIII
Charles K. Chui and Larry L. Schumaker (eds.), pp. 1–6.

ISBN 0-12-xxxxxx-x

We assume throughout that no two elements of N are parallel, and that 0 is not in the convex hull of N. The jth coordinate, $j \in \{1, \ldots, d\}$, of an element, say ν, of $\mathbb{R}^d$ shall be written $\nu(j)$.

Let α be in $\mathbf{Z}_+^{\mathrm{N}}$; that is, $\forall \nu \in \mathrm{N}$, $\alpha(\nu)$ is a nonnegative integer. By N^α, we mean the multiset of cardinality $|\alpha|$ (or the $d \times |\alpha|$ matrix, or the map from $\mathbb{R}^{\mathrm{N}^\alpha}$ into $\mathbb{R}^d$) containing $\alpha(\nu)$ copies of each element ν of N. Let D_ν denote differentiation in the direction ν, and $D_{\mathrm{N}}^\alpha := \prod_{\mathrm{N}} D_\nu$. For each ν in N, let e_ν in $\mathbf{Z}_+^{\mathrm{N}}$ be given by

$$e_\nu(\mu) = \begin{cases} 1 & \text{if } \nu = \mu, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

That 0 is not is not in the convex hull of N is enough to guarantee that the multivariate truncated power $T(\mathrm{N}^\alpha)$ [3] is well defined by the rule that, for any test function ϕ,

$$\langle T(\mathrm{N}^\alpha), \phi \rangle := \int_{[0,\infty)^{\mathbf{N}^\alpha}} \phi(\mathrm{N}^\alpha t)\, dt.$$

For completeness, $T(\emptyset)$ is defined to be the Dirac δ. Two consequences of this definition are that $T(\mathrm{N}^\beta) * T(\mathrm{N}^\alpha) = T(\mathrm{N}^{\beta+\alpha})$ and $D_{\mathrm{N}}^\beta T(\mathrm{N}^\alpha) = T(\mathrm{N}^{\alpha-\beta})$ for $\beta \le \alpha$. In particular, $D_{\mathrm{N}}^\alpha T(\mathrm{N}^\alpha) = T(\emptyset)$, so that $T(\mathrm{N}^\alpha)$ is a Green's function for the operator D_{N}^α.

Let $[t_0, \ldots, t_n]$ denote the divided difference functional at the (not necessarily distinct) real numbers $\{t_0, \ldots, t_n\}$. We define the univariate B-spline by

$$B(x \mid t_0, \ldots, t_n) = [t_0, \ldots, t_n] \frac{(x-t)_+^{n-1}}{(n-1)!} \tag{2.1}$$

where the difference is applied in the variable t. Up to a constant factor of sign $(-1)^n$, this is the standard B-spline of degree $n-1$ with knots at $\{t_0, \ldots, t_n\}$. When it is convenient to do so, we'll suppress the variable: $B(t_0, \ldots, t_n)$ is the function whose value at x is $B(x \mid t_0, \ldots, t_n)$.

§3 A box-like spline and its recurrence

Given ν in N and the real numbers $t_0 \le t_1 \le \cdots \le t_n$, define a multivariate distribution $B_\nu(t_0, \cdots, t_n)$ by the rule that, for every test function ϕ,

$$\langle B_\nu(t_0, \cdots, t_n), \phi \rangle := \int_{\mathbb{R}} B(t \mid t_0, \cdots, t_n) \phi(\nu t)\, dt.$$

The support of B_ν is the line segment in $\mathbb{R}^d$ joining $t_0\nu$ and $t_n\nu$.

Assume that for each $\nu \in \mathrm{N}$, t_ν is the *knot sequence*

$$\cdots \le t_\nu(-1) \le t_\nu(0) \le t_\nu(1) \le \cdots$$

and that $z \in \mathbf{Z}^{\mathrm{N}}$. We then define the multivariate distribution $B(z, \alpha)$ to be the convolution of the distributions

$$\Big\{ B_\nu \Big(t_\nu\big(z(\nu)\big), t_\nu\big(z(\nu)+1\big), \ldots, t_\nu\big(z(\nu)+\alpha(\nu)\big) \Big) : \nu \in \mathrm{N} \Big\}. \qquad (3.1)$$

As such, $B(z, \alpha)$ has sign $(-1)^{|\alpha|}$ and is supported on the sum of the line segments supporting the B_νs; that is,

$$\operatorname{supp} B(z, \alpha) = \{ \mathrm{N}x : \forall \nu \in \mathrm{N}, t_\nu(z(\nu)) \le x(\nu) \le t_\nu(z(\nu) + \alpha(\nu)) \}.$$

The value at $x \in \mathbb{R}^d$ of $B(z, \alpha)$ is written $B(x \mid z, \alpha)$.

For smooth d-variate functions f, define

$$\nabla_z^{ne_\nu} : f \to [t_\nu(z(\nu)), \ldots, t_\nu(z(\nu)+n)] f(\cdot - \nu t)$$

where the divided difference is applied in the scalar variable t, and

$$\nabla_z^\alpha = \prod_{\mathrm{N}} \nabla_z^{\alpha(\nu) e_\nu}.$$

From (2.1) it follows that

$$B_\nu \Big(t_\nu\big(z(\nu)\big), t_\nu\big(z(\nu)+1\big), \ldots, t_\nu\big(z(\nu)+n\big) \Big) = \nabla_z^{ne_\nu} T(\nu^n),$$

(where ν^n is the multiset consisting of n copies of ν) and therefore

$$B(z, \alpha) = \nabla_z^\alpha T(\mathrm{N}^\alpha).$$

In the case that each knot sequence t_ν is strictly increasing, $B(z, \alpha)$ arose earlier [4] as the representer of a multivariate difference. Below, we briefly sketch the connection between $B(z, \alpha)$ and differences in this more general setting.

Let $S \subset \mathbb{R}^d$, let $c(s)$ be a scalar and p_s a d-variate polynomial for each $s \in S$, and let δ_z^α be the functional acting on d-variate smooth functions by the rule

$$\delta_z^\alpha : f \mapsto \sum_S c(s) p_s(D) f(s)$$

with the property that

$$\nabla_z^\alpha f = \delta_z^\alpha * f := \sum_S c(s) p_s(D) f(\cdot - s).$$

In case N is the $d \times d$ identity matrix, δ_z^α is the tensor product of the divided differences

$$\Big\{[t_\nu\big(z(\nu)\big), t_\nu\big(z(\nu)+1\big), \ldots, t_\nu\big(z(\nu)+\alpha(\nu)\big)] : \nu \in \mathrm{N}\Big\}. \tag{3.2}$$

More generally, δ_z^α acts on a d-variate function f by applying the tensor product of (3.2) to the function $f(\mathrm{N}\cdot) : \mathbb{R}^{\mathrm{N}} \to \mathbb{R}$.

The distribution $B(z,\alpha)$ is the representer of $(-1)^{|\alpha|}\delta_z^\alpha$ in the sense that, for f sufficiently smooth,

$$(-1)^{|\alpha|}\delta_z^\alpha f = \langle B(z,\alpha), D_N^\alpha f\rangle. \tag{3.3}$$

The proof [4,5,6] comes from justifying that, since both $T(-N^\alpha) * D_{-N}^\alpha f$ and f have the same N^αth derivative,

$$\delta_z^\alpha f = \delta_z^\alpha(T(-N^\alpha) * D_{-N}^\alpha f) = \langle B(z,\alpha), D_N^\alpha f\rangle(-1)^{|\alpha|}.$$

Applying the familiar derivative formula (in case $t_0 \neq t_n$)

$$\frac{d}{dx}B(x \mid t_0, \ldots t_n) = \frac{B(x \mid t_1, \ldots t_n) - B(x \mid t_0, \ldots t_{n-1})}{t_n - t_0}$$

to the convolution of the splines (3.1) immediately yields the following result.

Lemma 3.4 *If $\nu \in \mathrm{N}$ and if $t_\nu(z(\nu)) < t_\nu(z(\nu)+\alpha(\nu))$, then*

$$D_\nu B(z,\alpha) = \frac{B(z+e_\nu, \alpha - e_\nu) - B(z, \alpha - e_\nu)}{t_\nu(z(\nu)+\alpha(\nu)) - t_\nu(z(\nu))}.$$

We end with a recurrence relation for $B(z,\alpha)$.

Theorem 3.5 *If $t_\nu(z(\nu)) < t_\nu(z(\nu)+\alpha(\nu))$ for all $\nu \in \mathrm{N}$, and if $x = \mathrm{N}\xi$ for some $\xi \in \mathbb{R}^{\mathrm{N}}$, then*

$$(|\alpha| - d)B(x \mid z, \alpha) = \sum_{\mathrm{N}} (w_{z\nu}(\xi(\nu)) - 1)B(x \mid z + e_\nu, \alpha - e_\nu) - w_{z\nu}(\xi(\nu))B(x \mid z, \alpha - e_\nu)$$

where

$$w_{z\nu}(s) = \frac{s - t_\nu(z(\nu))}{t_\nu(z(\nu)+\alpha(\nu)) - t_\nu(z(\nu))}.$$

Proof: The proof is a straightforward modification of the proof for the box spline recurrence [2].

For ϕ a function on $\mathbb{R}^d$, define $\widehat{\phi}(s) := \int_{\mathbb{R}^d} \phi(x) e^{-is\cdot x}\,dx$. Define the function $DB : x \mapsto D_x B(x \mid z, \alpha)$. For $j \in \{1, \ldots, d\}$, define the function $\mathrm{id}_j : \mathbb{R}^d \to \mathbb{R} : y \mapsto y(j)$, the jth coordinate of y, and let D_j denote differentiation in the jth direction. Then

$$DB = \sum_{j=1}^{d} \mathrm{id}_j D_j B$$

where $B = B(z, \alpha)$. Since $\widehat{\mathrm{id}_j \phi} = i D_j \widehat{\phi}$ and $-i\widehat{D_j \phi} = \mathrm{id}_j \widehat{\phi}$,

$$\begin{aligned} \widehat{DB}(s) &= -\sum_{j=1}^{d} D_j \mathrm{id}_j \widehat{B}(s) \\ &= -d\widehat{B}(s) - \sum_{j=1}^{d} s(j) D_j \widehat{B}(s). \end{aligned} \tag{3.6}$$

By equation (3.3), we can obtain the Fourier transform of $B(z, \alpha)$ by applying δ_z^α to a function whose N^αth derivative is $(-1)^{|\alpha|} e^{-is\cdot}$:

$$\widehat{B}(s) = \frac{\delta_z^\alpha e^{-is\cdot r}}{\prod_{N^\alpha} (is \cdot \nu)}$$

where the functional δ_z^α is applied in the variable r. Therefore

$$D_j \widehat{B}(s) = \frac{-i\delta_z^\alpha r(j) e^{-is\cdot r}}{\prod_{N^\alpha} (is \cdot \nu)} - \frac{\delta_z^\alpha e^{-is\cdot r}}{\prod_{N^\alpha} (is \cdot \nu)^2} \sum_{\mathrm{N}^\alpha} i\nu(j) \prod_{\mu \in \mathrm{N}^{\alpha - e_\nu}} (is \cdot \mu).$$

Using this in (3.6) gives

$$\begin{aligned} \widehat{DB}(s) &= -d\widehat{B}(s) + \frac{\delta_z^\alpha (is \cdot r) e^{-is\cdot r}}{\prod_{N^\alpha} (is \cdot \nu)} + |\alpha| \frac{\delta_z^\alpha e^{-is\cdot r}}{\prod_{N^\alpha} (is \cdot \nu)} \\ &= (|\alpha| - d)\widehat{B}(s) + \frac{\delta_z^\alpha (is \cdot r) e^{-is\cdot r}}{\prod_{N^\alpha} (is \cdot \nu)}. \end{aligned} \tag{3.7}$$

In the last quotient, we apply Leibniz's rule for the tensor product divided difference of a product:

$$\delta_z^\alpha fg = \sum_{0 \le \beta \le \alpha} \delta_z^\beta f \delta_{z+\beta}^{\alpha - \beta} g,$$

and note that $\delta_z^\beta (s \cdot r) = 0$ if $|\beta| > 1$. The quotient then becomes

$$is \cdot \mathrm{N}t(z)\widehat{B}(s) + \sum_{\nu \in \mathrm{N}} \frac{\delta_z^{e_\nu} (is \cdot r)}{(is \cdot \nu)} \frac{\delta_{z+e_\nu}^{\alpha - e_\nu} e^{-is\cdot r}}{\prod_{\mu \in \mathrm{N}^{\alpha - e_\nu}} (is \cdot \mu)},$$

where $\mathrm{N}t(z) = \sum_{\mathrm{N}} \nu t_\nu(z(\nu))$. Since $\widehat{D_y f}(s) = is \cdot y\widehat{f}(s)$, the expression above is the transform of

$$D_{\mathrm{N}t(z)}B + \sum_{\mathrm{N}} B(z + e_\nu, \alpha - e_\nu),$$

and therefore, by inverting (3.7), we obtain

$$D_{x-\mathrm{N}t(z)}B(x \mid z, \alpha) = (|\alpha| - d)B(x \mid z, \alpha) + \sum_{\mathrm{N}} B(x \mid z + e_\nu, \alpha - e_\nu).$$

By Lemma 3.1, the left hand side is

$$\sum_{\mathrm{N}} w_{z\nu}(\xi(\nu))\big(B(x \mid z + e_\nu, \alpha - e_\nu) - B(x \mid z, \alpha - e_\nu)\big),$$

which gives the desired conclusion. ■

References

1. de Boor, C., and K. Höllig, B-splines from parallelepipeds, J. Analyse Math. **42** (1982), 99–115.
2. de Boor, C., K. Höllig, and S. Riemenschneider, *Box Splines*, Applied Mathematical Sciences, v. 98, Springer-Verlag, New York, 1993.
3. Dahmen, W., On multivariate B-splines, SIAM J. Numer. Anal. **17** (1980), 179–191.
4. Kunkle, T., Multivariate differences, polynomials, and splines, J. Approx. Theory, to appear.
5. Kunkle, T., Exponential box-like splines on nonuniform grids, in preparation.
6. Kunkle, T., Characterizations of multivariate differences and associated exponential splines, in preparation.
7. Ron, A., Exponential box splines, Constr. Approx. **4** (1988), 357–378.

Thomas Kunkle
Mathematics Dept.
College of Charleston
Charleston, SC 29424
kunkle@math.cofc.edu
http://math.cofc.edu:8080/~kunkle/home.html